EQUIVARIANT BORDISM AND SMITH THEORY. II

BY

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Abstract. This paper analyzes the homomorphism from equivariant bordism to Smith homology for spaces with an action of a finite group G.

1. **Introduction.** Let G be a finite group, and let (X, A, ψ) be a pair with G action. One then has defined the G-equivariant bordism group $\mathfrak{N}_*^G(X, A, \psi)$ and the Smith homology group $H_*^G(X, A, \psi; Z_2)$. These define equivariant homology theories on the category of G pairs and G-equivariant maps, and the object of this paper is to explore the relationship between these theories.

Briefly, being given an equivariant bordism element $f: (M, \partial M, \varphi) \to (X, A, \psi)$, the image of the fundamental Smith theory class of $(M, \partial M, \varphi)$ gives a natural transformation

$$\bar{\mu}: \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}^G} Z_2 \to H_*^G(X, A, \psi; Z_2).$$

It was shown in [9] that $\bar{\mu}$ is an isomorphism if $G = Z_2$.

The main results of this paper are

THEOREM 1. *$\bar{\mu}$ is always epic*

and

THEOREM 2. *$\bar{\mu}$ is an isomorphism for all G pairs (X, A, ψ) if and only if G is 2-nilpotent and has Sylow 2 subgroup a Z_2 vector space.*

(G is called 2-nilpotent if the elements of odd order in G form a subgroup.)

2. **The representation theorem.** Let G be a finite group, X a simplicial complex, $\psi: G \times X \to X$ a simplicial G action and $A \subset X$ a subcomplex invariant under G. It will be assumed that X is "finely" triangulated so that the fixed set of any subgroup H is a subcomplex and the projection $\pi: X \to X/H$ is simplicial (E. E. Floyd [4] shows that this may be accomplished by taking the second barycentric subdivision).

Let $C(X) \otimes Z_2$ denote the chains of X with Z_2 coefficients and let $g_\#: C(X) \otimes Z_2 \to C(X) \otimes Z_2$ be the chain map induced by $\psi(g, \ ): X \to X: x \to \psi(g, x)$. One then lets $C^0(X) \subset C(X) \otimes Z_2$ denote the subgroup consisting of chains σ so that

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$g_\# \sigma = \sigma$ for all $g \in G$. Since $g_\#$ commutes with the boundary, one has an induced boundary $\partial : C^0(X) \to C^0(X)$, and if $C^0(X, A) = C^0(X)/C^0(A)$ one has induced a homomorphism ∂ making this a chain complex. The *Smith homology groups* of (X, A, ψ), $H_*^G(X, A, \psi; Z_2)$, are then defined to be the homology groups of the complex $(C^0(X, A), \partial)$.

By using Čech [7], [5] or singular [3] methods to obtain a complex, this may be extended to all topological G pairs.

Being given a compact differentiable manifold M^n with differentiable G action φ, one may triangulate M "finely" so that G acts simplicially. Clearly the fundamental cycle $\mu = \sum \Delta^i$, the sum of all n-simplices, is then an invariant chain, defining a fundamental class $[M, \partial M, \varphi] \in H_n^G(M, \partial M, \varphi; Z_2)$. This lifts the ordinary fundamental class back to Smith theory.

One then has a natural transformation

$$\mu : \mathfrak{N}_*^G(X, A, \psi) \to H_*^G(X, A, \psi; Z_2)$$

assigning to the equivariant bordism element $f : (M, \partial M, \varphi) \to (X, A, \psi)$ the class $f_*[M, \partial M, \varphi]$.

Letting $\varepsilon : \mathfrak{N}_*^G \to Z_2$ be the augmentation to $\mathfrak{N}_0 \cong Z_2$ given by ignoring G action and the positive dimensional part, one has $\mu(\alpha \cdot \beta) = \varepsilon(\alpha)\mu(\beta)$ for $\alpha \in \mathfrak{N}_*^G$, $\beta \in \mathfrak{N}_*^G(X, A, \psi)$ as in [9] (*Note*: $H_i^G(M^n, \partial M^n, \varphi; Z_2) = 0$ if $i > n$) and thus μ induces a natural transformation

$$\bar\mu : \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}^G} Z_2 \to H_*^G(X, A, \psi; Z_2).$$

One has the analogue of [9, Proposition 2.1]:

LEMMA 2.1. *If G is a 2 group then $\bar\mu$ is epic.*

Proof. The result is known for $G = \{1\}$ or $G = Z_2$ and so one may induct on the order of G. Let $T = \{1, t\}$ be a central subgroup of G of order 2.

Being given a G complex (X, A, ψ), any element of $C^0(X)$ decomposes uniquely into a sum of invariant chains $\sigma_1 + \sigma_2$, where σ_1 is a sum of simplices Δ with $t_\# \Delta = \Delta$ and σ_2 is a sum of terms $\Delta + t_\# \Delta$ with $t_\# \Delta \neq \Delta$, Δ a simplex. This gives a natural decomposition

$$H_*^G(X, A, \psi) \cong H_*^G(X, F_T \cup A, \psi) \oplus H_*^G(F_T, F_T \cap A, \psi)$$

and

$$H_*^G(X, F_T \cup A, \psi) \cong H_*^{G/T}(X/T, A/T \cup F_T, \psi'),$$
$$H_*^G(F_T, F_T \cap A, \psi) \cong H_*^{G/T}(F_T, F_T \cap A, \psi')$$

where F_T is the fixed set of T and ψ' denotes the induced action (see [9, Theorem 2.1]).

Now $\mathfrak{N}_*^{G/T}(F_T, F_T \cap A)$ maps onto $H_*^{G/T}(F_T, F_T \cap A)$ by induction, and if $f : (M, \partial M, \varphi') \to (F_T, F_T \cap A, \psi')$ is a G/T bordism element representing α, f may

be considered a G bordism element with T acting trivially, to represent α as an element of $H_*^G(X, A, \psi; Z_2)$.

Also $\mathfrak{N}_*^{G/T}(X/T, A/T \cup F_T)$ maps onto $H_*^{G/T}(X/T, A/T \cup F_T)$ by induction, and by excision arguments as in [9, Proposition 2.1], a bordism element $f: (M, \partial M, \varphi') \to (X/T, A/T \cup F_T)$ may be lifted to

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & X \\
\pi \downarrow & & \downarrow \pi \\
M & \xrightarrow{f} & X/T
\end{array}
$$

with $\tilde{f}$ being G equivariant and representing the class in $H_*^G(X, A, \psi)$ corresponding to $\bar{\mu}([f]) \in H_*^{G/T}(X/T, A/T \cup F_T)$.　□

If $G = (Z_2)^k$ one also has an analog of [9, Proposition 2.2].

LEMMA 2.2. *If* $G = (Z_2)^k$, *then* $\bar{\mu}$ *is an isomorphism.*

Proof. This is known for $k = 0, 1$ and hence one may apply induction. Let $t_1, \ldots, t_k$ with $t_i^2 = 1$, $t_i t_j = t_j t_i$ be generators of G, with $T_i = \{1, t_i\} \subset G$.

Then for any (X, A, ψ) one has an exact sequence of $\mathfrak{N}_*^G$ modules, split as $\mathfrak{N}_*$ modules

$$
0 \to \mathfrak{N}_*^G(F_{T_1}, A \cap F_{T_1}) \to \mathfrak{N}_*^G(X, A) \to \mathfrak{N}_*^G(X, F_{T_1} \cup A) \to 0
$$

and hence a commutative diagram

$$
\begin{array}{ccccccccc}
Q & \longrightarrow & \mathfrak{N}_*^G(F_{T_1}, F_{T_1} \cap A) \otimes Z_2 & \longrightarrow & \mathfrak{N}_*^G(X, A) \otimes Z_2 & \longrightarrow & \mathfrak{N}_*^G(X, F_{T_1} \cup A) \otimes Z_2 & \to & 0 \\
& & \downarrow \bar{\mu}_0 & & \downarrow \bar{\mu}_1 & & \downarrow \bar{\mu}_2 & & \\
0 & \longrightarrow & H_*^G(F_{T_1}, F_{T_1} \cap A) & \longrightarrow & H_*^G(X, A) & \longrightarrow & H_*^G(X, F_{T_1} \cup A) & \longrightarrow & 0 \\
& & \wr \| & & & & \wr \| & & \\
& & H_*^{G/T_1}(F_{T_1}, F_{T_1} \cap A) & & & & H_*^{G/T_1}(X/T_1, F_{T_1} \cup A/T_1) & &
\end{array}
$$

with Q a "Tor"-term.

To see that $\bar{\mu}_2$ is monic, one notes that $(X, F_{T_1} \cup A)$ is relatively free as a T_1 pair, so $\mathfrak{N}_*^G(X, F_{T_1} \cup A) \cong \mathfrak{N}_*^{G/T_1}(X/T_1, F_{T_1} \cup A/T_1)$ by assigning to a T_1 free bordism element $f: M \to X$ the induced map $\bar{f}: M/T_1 \to X/T_1$. Further, this is a homomorphism of $\mathfrak{N}_*^{G/T_1}$ modules, where $\mathfrak{N}_*^{G/T_1} \to \mathfrak{N}_*^G$ by considering a G/T_1 manifold as a G manifold with trivial T_1 action. One then has a commutative

diagram

$$\mathfrak{N}_*^G(X, F_{T_1} \cup A) \otimes_{\mathfrak{N}_*^{G/T_1}} Z_2 \xrightarrow{\text{epic}} \mathfrak{N}_*^G(X, F_{T_1} \cup A) \otimes_{\mathfrak{N}_*^G} Z_2$$

$$\cong \Big\downarrow \qquad\qquad\qquad\qquad \bar{\mu}_2 \Big\downarrow \text{epic}$$

$$\mathfrak{N}_*^{G/T_1}(X/T_1, F_{T_1} \cup A/T_1) \otimes_{\mathfrak{N}_*^{G/T_1}} Z_2 \longrightarrow H_*^G(X, F_{T_1} \cup A)$$

$$\cong \searrow \qquad\qquad\qquad \swarrow \cong$$

$$H_*^{G/T_1}(X/T_1, F_{T_1} \cup A/T_1)$$

and hence $\bar{\mu}_2$ is monic.

By an elementary diagram chase, $\bar{\mu}_1$ will be monic provided $\bar{\mu}_0$ is monic. Thus, it suffices to prove the lemma for pairs (X, A, ψ) fixed by T_1. A similar analysis may then be applied to each T_i, and hence it suffices to prove the lemma for pairs (X, A, ψ) fixed by each T_i, hence by G.

If (X, A, ψ) is a trivial G space, one has $\mathfrak{N}_*^G(X, A, \psi) \cong \mathfrak{N}_*^G \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X, A)$, so $\mathfrak{N}_*^G(X, A, \psi) \otimes Z_2$ coincides with $H_*(X, A; Z_2)$ (the $G = \{1\}$ result) and the lemma is valid. $\square$

Now turning to the general case one has:

THEOREM 2.1. *For every G, $\bar{\mu}$ is always epic.*

Proof. Let G be a finite group and (X, A, ψ) a G pair. Let $S \subset G$ be a Sylow 2 subgroup and $\psi_S: S \times X \to X$ the S action given by restriction to $S \times X$ of ψ.

Considering a G invariant chain of X as being only S invariant defines a homomorphism

$$\theta: H_*^G(X, A, \psi; Z_2) \to H_*^S(X, A, \psi_S; Z_2).$$

Being given an S invariant chain $\sigma \in C(X) \otimes Z_2$ let $t\sigma = \sum g_\# \sigma$ where the sum is taken over a collection of g which represent the cosets G/S. (*Note.* if $g' \in gS$, $g'_\# \sigma = g_\# \sigma$ since σ is S invariant.) Clearly $t\sigma$ is G invariant and this induces a homomorphism

$$t: H_*^S(X, A, \psi_S; Z_2) \to H_*^G(X, A, \psi; Z_2).$$

If σ is G invariant, $g_\# \sigma = \sigma$, so $t\sigma$ is $[G:S]\sigma$ where $[G:S]$ is the index of S in G, and is odd, so $t\sigma = \sigma$. Thus $t\theta = 1$, or θ is monic and t is epic.

Now consider the extension homomorphism

$$e_G^S: \mathfrak{N}_*^S(X, A, \psi_S) \to \mathfrak{N}_*^G(X, A, \psi)$$

defined in [8, §4]. If $f: (M, \partial M, \varphi) \to (X, A, \psi_S)$ is an S equivariant bordism element α, $e_G^S(\alpha)$ is represented by $\bar{f}: (\overline{M}, \partial \overline{M}, \bar{\varphi}) \to (X, A, \psi)$ where $\overline{M} = G \times M/(gs^{-1}, \varphi(s, m)) \sim (g, m)$, $\bar{\varphi}(g', (g, m)) = (g'g, m)$ and $\bar{f}(g, m) = \psi(g, f(m))$. If one considers

$i: M \to \overline{M}: m \to (1, m)$, $\bar{f} \circ i = f$ and the fundamental cycle of $\overline{M}$ is $\sum g_\#(i_\# \mu)$ where μ is the fundamental cycle of M. Thus $\bar{f}_*[\overline{M}, \partial \overline{M}, \bar{\varphi}] = t f_*[M, \partial M, \varphi]$. Thus the diagram

$$
\begin{array}{ccc}
\mathfrak{N}_*^S(X, A, \psi_S) & \xrightarrow{\ e_G^S\ } & \mathfrak{N}_*^G(X, A, \psi) \\
\Big\downarrow{\scriptstyle \mu_S} & & \Big\downarrow{\scriptstyle \mu} \\
H_*^S(X, A, \psi_S; Z_2) & \xrightarrow{\ t\ } & H_*^G(X, A, \psi; Z_2)
\end{array}
$$

commutes, with t and μ_S epic, so μ is epic. Hence also $\bar{\mu}$ is epic. $\quad\square$

LEMMA 2.3. *Let G be a finite group with Sylow 2 subgroup S and suppose the restriction $\rho_S^G: \mathfrak{N}_*^G \to \mathfrak{N}_*^S$ is epic. If (X, A, ψ) is a G pair with*

$$
\bar{\mu}_S: \mathfrak{N}_*^S(X, A, \psi_S) \otimes_{\mathfrak{N}_*^S} Z_2 \to H_*^S(X, A, \psi_S; Z_2)
$$

monic, then

$$
\bar{\mu}_G: \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}_*^G} Z_2 \to H_*^G(X, A, \psi; Z_2)
$$

is also monic.

Proof. Let $\rho_S^G: \mathfrak{N}_*^G(X, A, \psi) \to \mathfrak{N}_*^S(X, A, \psi_S)$ denote the restriction homomorphism which "ignores G equivariance". It is then immediate that the diagram

$$
\begin{array}{ccc}
\mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}_*^G} Z_2 & \xrightarrow{\ \bar{\rho}\ } & \mathfrak{N}_*^S(X, A, \psi_S) \otimes_{\mathfrak{N}_*^S} Z_2 \\
\Big\downarrow{\scriptstyle \bar{\mu}_G} & & \Big\downarrow{\scriptstyle \bar{\mu}_S} \\
H_*^G(X, A, \psi; Z_2) & \xrightarrow{\ \theta\ } & H_*^S(X, A, \psi_S; Z_2)
\end{array}
$$

with $\bar{\rho}$ induced by ρ_S^G commutes, with θ and $\bar{\mu}_S$ being monic.

Now consider the extension

$$
e_G^S: \mathfrak{N}_*^S(X, A, \psi_S) \to \mathfrak{N}_*^G(X, A, \psi).
$$

By [2, 6.3] e_G^S is an $\mathfrak{N}_*^G$ module homomorphism; i.e. if $\alpha \in \mathfrak{N}_*^S(X, A, \psi_S)$ and $\beta \in \mathfrak{N}_*^G$, then $e_G^S(\rho_S^G(\beta) \cdot \alpha) = \beta \cdot e_G^S(\alpha)$. In particular, if $\beta' \in \mathfrak{N}_*^S$, there is a $\beta \in \mathfrak{N}_*^G$ with $\rho_S^G(\beta) = \beta'$, so $e_G^S(\beta' \cdot \alpha) = \beta \cdot e_G^S(\alpha)$. Since $\varepsilon(\beta') = \varepsilon(\beta)$, e_G^S induces a homomorphism

$$
\bar{e}: \mathfrak{N}_*^S(X, A, \psi_S) \otimes_{\mathfrak{N}_*^S} Z_2 \to \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}_*^G} Z_2.
$$

(*Note.* This used the fact that $\rho_S^G: \mathfrak{N}_*^G \to \mathfrak{N}_*^S$ is epic. I cannot prove that $\bar{e}$ is meaningful without this, and in fact Theorem 2 of the Introduction would seem to imply that $\bar{e}$ cannot always exist.)

Then $\bar{e}\bar{\rho}: \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}_*^G} Z_2 \to \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}_*^G} Z_2$ is induced by $e_G^S \circ \rho_S^G$. By [8, Proposition 13.2], $e_G^S \circ \rho_S^G$ is multiplication by the class of $[G/S, \mu] \in \mathfrak{N}_0^G$, so that $\bar{e}\bar{\rho}$ is multiplication by $\varepsilon[G/S, \mu] = 1$.

Thus $\bar{\rho}$ is monic, and so $\bar{\mu}_S \bar{\rho}$ is monic, which gives $\bar{\mu}_G$ monic. $\square$

One then has the first half of Theorem 2, given by

THEOREM 2.2. *Let G be 2-nilpotent with Sylow 2 subgroup a Z_2 vector space. Then $\bar{\mu}$ is an isomorphism for all G pairs.*

Proof. Let S be a Sylow 2 subgroup of G. Then $S = (Z_2)^k$ for some k, so by Lemma 2.2, $\bar{\mu}_S$ is always monic. Letting $K \subset G$ be the subgroup of elements of odd order, K is normal and $G/K \cong S$, giving a homomorphism $\varphi: G \to S$ with $\varphi|S = 1$. Thus $\rho_S^G: \mathfrak{N}_*^G \to \mathfrak{N}_*^S$ is epic, for if (M, ψ) is an S action, $(M, \psi \circ (\varphi \times 1))$ is a G action restricting to (M, ψ). Thus Lemma 2.3 applies to each G pair and $\bar{\mu}_G$ is always monic. By Theorem 2.1, $\bar{\mu}$ is then an isomorphism. $\square$

3. The isomorphism theorem. In order to simplify notation, temporarily say that the finite group G has the *isomorphism property* if for all G pairs (X, A, ψ), the natural transformation

$$\bar{\mu}: \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}_*^G} Z_2 \to H_*^G(X, A, \psi; Z_2)$$

is an isomorphism.

Following Bredon [1], one knows that two G equivariant homology theories agree for all spaces if and only if they agree for all of the coset spaces $(G/H, \mu)$, with H a subgroup of G.

Letting $H \subset G$ be a subgroup, consider the pair $(X, A, \psi) = (G/H, \phi, \mu)$.

Clearly $C(G/H) \otimes Z_2$ is the Z_2 vector space with base the points of G/H, and these are permuted by G, so $C^0(G/H) \cong Z_2$ with base the sum of all the points. Thus $H_*^G(G/H, \mu; Z_2) \cong Z_2$.

Now consider $\mathfrak{N}_*^G(G/H, \mu)$. If $f: (M, \varphi) \to (G/H, \mu)$ is a G bordism element, then $M_0 = f^{-1}(H)$, the inverse image of the coset H, is invariant under H, and hence $(M_0, \varphi|H \times M_0)$ is an H bordism element in $\mathfrak{N}_*^H$. It is immediate that $f: (M, \varphi) \to (G/H, \mu)$ is the extension to G of the H equivariant bordism element

$$f|M_0: (M_0, \varphi|H \times M_0) \to (G/H, \mu)$$

given by the point map. Thus, this correspondence defines an isomorphism $\mathfrak{N}_*^G(G/H, \mu) \cong \mathfrak{N}_*^H$. If $f: (M, \varphi) \to (G/H, \mu)$ and $(N, \psi) \in \mathfrak{N}_*^G$, the product is

$$f \circ \pi_M: (M \times N, \varphi \times \psi) \to (G/H, \mu)$$

so that $(f \circ \pi_M)^{-1}(H) = M_0 \times N$ with action $(\varphi|H \times M_0) \times (\psi|H \times N)$. Thus identifying $\mathfrak{N}_*^G(G/H, \mu)$ with $\mathfrak{N}_*^H$, $\mathfrak{N}_*^H$ is an $\mathfrak{N}_*^G$ module by $\alpha \cdot \beta = \rho_H^G(\alpha) \cdot \beta$ for $\alpha \in \mathfrak{N}_*^G$, $\beta \in \mathfrak{N}_*^H$ with $\rho_H^G: \mathfrak{N}_*^G \to \mathfrak{N}_*^H$ the restriction.

Thus, one has

LEMMA 3.1. *If G has the isomorphism property, then for all $H \subset G$,*

$$\mathfrak{N}_*^H \otimes_{\mathfrak{N}_*^G} Z_2 \cong Z_2$$

where $\mathfrak{N}_*^H$ is an $\mathfrak{N}_*^G$ module via the restriction $\rho_H^G: \mathfrak{N}_*^G \to \mathfrak{N}_*^H$.

LEMMA 3.2. *If $H \subset G$ is a 2 group then*

$$\mathfrak{N}_*^H \otimes_{\mathfrak{N}^G} Z_2 \cong Z_2$$

if and only if $\rho_H^G: \mathfrak{N}_*^G \to \mathfrak{N}_*^H$ is epic.

Proof. Clearly if ρ_H^G is epic, the tensor product rule holds. Now suppose ρ_H^G is not epic. Since H is a 2 group $\mathfrak{N}_0^H \cong Z_2$ generated by a point with trivial action, which comes from a trivial G action, $(\rho_H^G)_0$ is epic (see [8, p. 67]). Suppose

$$(\rho_H^G)_i: \mathfrak{N}_i^G \to \mathfrak{N}_i^H$$

is epic for $i < k$ and is not epic for $i = k$. Let $K = \text{cokernel } (\rho_H^G)_k$ and let

$$\varphi: \mathfrak{N}_*^H \to Z_2 \oplus K$$

be the homomorphism $\varepsilon: \mathfrak{N}_0^H \xrightarrow{\cong} Z_2$ and quotient homomorphism $\varphi: \mathfrak{N}_k^H \to K$ and zero in all other degrees. This is clearly a vector space epimorphism.

If $\alpha \in \mathfrak{N}_*^G$, $\beta \in \mathfrak{N}_*^H$, consider $\varphi(\alpha \cdot \beta) = \varphi(\rho_H^G(\alpha)\beta) = x$. If $\dim \alpha > 0$, $x = 0$ unless $\dim \alpha + \dim \beta = k$, when $\dim \beta < k$. But then $\beta = \rho_H^G(\beta')$ for some $\beta' \in \mathfrak{N}_*^G$, so $x = \varphi(\rho_H^G(\alpha\beta'))$ and represents zero in the cokernel. Thus $x = \varepsilon(\alpha)\varphi(\beta)$ if $\dim \alpha > 0$. If $\dim \alpha = 0$, $x = \varphi(\varepsilon(\alpha) \cdot \beta) = \varepsilon(\alpha)\varphi(\beta)$ for $(\rho_H^G)_0$ and ε coincide as maps to Z_2 and φ is Z_2 linear.

Thus, φ induces an epimorphism $\tilde{\varphi}: \mathfrak{N}_*^H \otimes_{\mathfrak{N}^G} Z_2 \to Z_2 \oplus K$, so $\mathfrak{N}_*^H \otimes_{\mathfrak{N}^G} Z_2 \ncong Z_2$. $\qquad \square$
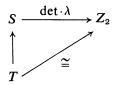
Now let G have the isomorphism property, let $S \subset G$ be a Sylow 2 subgroup, and let $T \subset S$ be a central subgroup of order 2, with $T = \{1, t\}$. By the lemmas, ρ_T^G is epic, but ρ_T^G is the composite

$$\mathfrak{N}_*^G \xrightarrow{\rho_S^G} \mathfrak{N}_*^S \xrightarrow{\rho_T^S} \mathfrak{N}_*^T$$

and hence ρ_T^S is epic.

Now let M^3 be the manifold obtained from the 3 disc D^3 by identifying antipodal points of S^2 (i.e. RP (3)) with the T action φ given by the involution $t \cdot x = -x$ on D^3. Since ρ_T^S is epic, there is an S action (N, ψ) cobordant to (M, φ) as T action. The fixed set of T in M is a point $(0 \in D^3)$ and RP (2) (image of S^2), and this is cobordant to the fixed set of T in N, $F_T(N)$. In particular, the zero dimensional part $F_T(N)^0$ is an odd number of points. Since T is normal in S, S acts on $F_T(N)$ and hence also on $F_T(N)^0$. Since S is a 2 group, each orbit of S on $F_T(N)^0$ consists of 2^r points, and since $F_T(N)^0$ is odd, there must be a point orbit. Thus, there is a point $p \in F_T(N)^0$ which is fixed by S. Giving N an S invariant Riemannian metric, S acts on the tangent space to N at p orthogonally, giving a homomorphism $\lambda: S \to O_3$ (dim $N = 3$) and $\lambda(t)$ is multiplication by -1 in R^3. Taking the determinant

det: $O_3 \rightarrow Z_2$, one has a commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\;\det \cdot \lambda\;} & Z_2 \\
\uparrow & \nearrow & \\
T & \cong &
\end{array}
$$

so that T splits out of S. Since this is true for all central subgroups of order 2 in S, S must be a Z_2 vector space. (If not, $S = A \times C$, C the central elements of order 2, but if A is nontrivial, its center is nontrivial, giving a central order 2 element of S not in C.) Thus one has

LEMMA 3.3. *If G has the isomorphism property, then the Sylow 2 subgroup of G is a Z_2 vector space.*

In order to show that G is 2-nilpotent requires a digression.

Let G be a finite group and $\alpha: G \rightarrow G$ an automorphism. One lets $\alpha_*: \mathfrak{N}_*^G \rightarrow \mathfrak{N}_*^G$ by $\alpha_*(M, \varphi) = (M, \varphi \circ (\alpha^{-1} \times 1))$, where

$$
G \times M \xrightarrow{\;\alpha^{-1} \times 1\;} G \times M \xrightarrow{\;\varphi\;} M
$$

defines a new G action on M. If β is another automorphism of G, $(\alpha\beta)^{-1} \times 1 = (\beta^{-1} \times 1)(\alpha^{-1} \times 1)$ so $(\alpha\beta)_* = \alpha_* \beta_*$, and thus one has a homomorphism

$$
*: \mathrm{Aut}\,(G) \rightarrow \mathrm{Aut}\,(\mathfrak{N}_*^G): \alpha \rightarrow \alpha_*.
$$

Now let $g \in G$ and $\alpha: G \rightarrow G: h \rightarrow ghg^{-1}$ the inner automorphism so that $\alpha^{-1}(h) = g^{-1}hg$. Then if (M, φ) is a G action, $\alpha_*(M, \varphi) = (M, \psi)$ where $\psi(h, m) = \varphi(g^{-1}hg, m)$. Letting $\rho: (M, \varphi) \rightarrow (M, \psi): m \rightarrow \varphi(g^{-1}, m)$ one has an equivariant diffeomorphism, so $\alpha_*(M, \varphi) = (M, \varphi)$. Thus one has induced a homomorphism

$$
*: \mathrm{Aut}\,(G)/\mathrm{Inn}\,(G) \rightarrow \mathrm{Aut}\,(\mathfrak{N}_*^G)
$$

where Inn (G) is the normal subgroup of inner automorphisms.

Similarly, if $\alpha \in \mathrm{Aut}\,(G)$, α acts on the set of irreducible (real) representations of G, IR (G), by sending $\theta: G \times V \rightarrow V$ to $\theta \circ (\alpha^{-1} \times 1): G \times V \rightarrow V$. This defines a homomorphism

$$
-: \mathrm{Aut}\,(G) \rightarrow \mathrm{Perm}\,(\mathrm{IR}\,(G)): \alpha \rightarrow \bar{\alpha}, \quad \text{where } \bar{\alpha}(V, \theta) = (V, \theta \circ (\alpha^{-1} \times 1)).
$$

Notice that Inn (G) acts trivially on IR (G), that $\bar{\alpha}$ preserves the dimension of the representation, and $\bar{\alpha}$ sends the trivial representation ($\theta(g, v) = v$ for all (g, v)) to itself.

Now let $\theta: G \times V \rightarrow V$ be an irreducible real representation of G. Let M be the manifold obtained from the disc in $V \oplus V$, $D(2V)$, by identifying antipodal points

of the sphere, with G action φ given by $\varphi(g, [v_1, v_2]) = [\theta(g, v_1), \theta(g, v_2)]$. Then the fixed set of G in (M, φ) is

$$F_G(M, \varphi) = \{0\} \qquad\qquad \text{if } \dim V > 1,$$
$$= \{0\} \cup \mathrm{RP}\,(2V) \quad \text{if } \dim V = 1,$$

and G acts in the normal bundle at 0 as two copies of the representation V. Thus letting

$$F_G \colon \mathfrak{N}^G_{2\dim V} \to \mathfrak{N}_0(F'_G(BO_{2\dim v}))$$

be the fixed point homomorphism, $F_G(M, \varphi)$ is given by the inclusion of a point in the component of $F'_G(BO_{2\dim v})$ over which G acts as $2V$. (See [8] for the definition of $F'_G(BO_n)$ and the fixed point homomorphism.)

It is immediate that if (M, φ) is defined by the representation (V, θ), then $\alpha_*(M, \varphi)$ is defined by $\bar\alpha(V, \theta)$. Then if $\alpha \in \mathrm{Aut}\,(G)$ with $\alpha_* = 1$, $F_G(M, \varphi) = F_G(\alpha_*(M, \varphi))$ so that (V, θ) and $\bar\alpha(V, \theta)$ are equivalent representations, or $\bar\alpha = 1$. Thus one has

LEMMA 3.4. *If $\alpha \in \mathrm{Aut}\,(G)$ and $\alpha_* \colon \mathfrak{N}^G_* \to \mathfrak{N}^G_*$ is trivial, then $\bar\alpha \colon \mathrm{IR}\,(G) \to \mathrm{IR}\,(G)$ is also trivial, or α acts trivially on the irreducible representations of G.*

Now consider a finite group G with the isomorphism property, and let S be a Sylow 2 subgroup of G, so that ρ^G_S is epic (Lemmas 3.1 and 3.2). If N is the normalizer of S in G, $S \subset N \subset G$, then $\rho^G_S = \rho^N_S \rho^G_N$, so $\rho^N_S \colon \mathfrak{N}^N_* \to \mathfrak{N}^S_*$ is epic.

If $n \in N$, $c_n \colon N \to N \colon g \to ngn^{-1}$ is an inner automorphism, so $(c_n)_* \in \mathrm{Aut}\,(\mathfrak{N}^N_*)$ is trivial. Since S is normal in N, $c_n(S) \subset S$ and c_n is an automorphism of S, so $(c_n)_* \in \mathrm{Aut}\,(\mathfrak{N}^S_*)$. Since ρ^N_S is epic, $(c_n)_*$ is trivial in $\mathrm{Aut}\,(\mathfrak{N}^S_*)$, and thus the homomorphism

$$N \to \mathrm{Aut}\,(\mathfrak{N}^S_*) \colon n \to (c_n)_*$$

is trivial, and by Lemma 3.4

$$N \to \mathrm{Perm}\,(\mathrm{IR}\,(S)) \colon n \to \bar c_n$$

is trivial.

By Lemma 3.3, $S = (Z_2)^k$, and every irreducible representation of S is of the form (R, θ) where $\theta(s, v) = \bar\theta(s) \cdot v$, with $\bar\theta \colon S \to Z_2 = \{+1, -1\}$. Thus $\mathrm{IR}\,(S) = \mathrm{Hom}\,(S, Z_2)$. If $\alpha \in \mathrm{Aut}\,(S)$, $\bar\alpha(R, \theta) = (R, \psi)$ with $\bar\psi(s) \cdot x = \bar\theta(\alpha^{-1}s) \cdot x$, so $\bar\alpha = 1$ implies $\bar\theta \circ \alpha^{-1} = \bar\theta$ for all $\bar\theta \in \mathrm{Hom}\,(S, Z_2)$ and hence $\alpha = 1$. Thus, the homomorphism

$$N \to \mathrm{Aut}\,(S) \colon n \to c_n$$

is trivial, or S is central in its normalizer.

One may now apply the theorem of Burnside [6, Theorem 14.3.1]: If a Sylow subgroup P of G is in the center of its normalizer, then G has a normal subgroup H which has the elements of P as its coset representatives.

Thus one has

THEOREM 3.1. *If G is a finite group with*

$$\bar{\mu}: \mathfrak{N}^G_*(X, A, \psi) \otimes_{\mathfrak{N}^G} Z_2 \to H^G_*(X, A, \psi; Z_2)$$

an isomorphism for all G pairs (X, A, ψ), *then G is 2-nilpotent and has Sylow* 2 *subgroup a* Z_2 *vector space.*

Combining this with Theorem 2.2 gives Theorem 2 of the Introduction.

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