

THE SIGN OF LOMMEL'S FUNCTION

BY
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Abstract. Lommel's function $s_{\mu,\nu}(x)$ is a particular solution of the differential equation $x^2y'' + xy' + (x^2 - \nu^2)y = x^{\mu+1}$. It is shown here that $s_{\mu,\nu}(x) > 0$ for $x > 0$, if $\mu = \frac{1}{2}$ and $|\nu| < \frac{1}{2}$, or if $\mu > \frac{1}{2}$ and $|\nu| \leq \mu$. This includes earlier results of R. G. Cooke's. The sign of $s_{\mu,\nu}(x)$ for other values of μ and ν is also discussed.

1. Introduction. In 1876, Lommel [6] considered the inhomogeneous differential equation

$$(1) \quad z^2y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1},$$

where μ and ν are complex parameters. He obtained two particular solutions: the Lommel functions of the first kind, $s_{\mu,\nu}(z)$, and of the second kind, $S_{\mu,\nu}(z)$. The homogeneous equation associated with (1) is

$$(2) \quad z^2y'' + zy' + (z^2 - \nu^2)y = 0,$$

Bessel's equation.

The function $s_{\mu,\nu}(z)$ is defined for all pairs μ, ν such that neither $\mu - \nu$ nor $\mu + \nu$ is an odd negative integer, and for all z with $-\pi < \arg z \leq \pi$, by the series

$$(3) \quad s_{\mu,\nu}(z) = \frac{1}{4}z^{\mu+1} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}z)^{2n} \Gamma((\mu - \nu + 1)/2) \Gamma((\mu + \nu + 1)/2)}{\Gamma((\mu - \nu + n + 3)/2) \Gamma((\mu + \nu + n + 3)/2)}.$$

The symmetry property

$$(4) \quad s_{\mu,\nu}(z) = s_{\mu,-\nu}(z)$$

is obvious from (3).

We shall consider $s_{\mu,\nu}(z)$ for μ and ν real, and positive z . Its importance arises from the formula [1, §3.20], [9, §10.74]

$$(5) \quad \int x^{\mu} C_{\nu}(x) dx = x[C_{\nu}(x)s'_{\mu,\nu}(x) - s_{\mu,\nu}(x)C'_{\nu}(x)],$$

in which $C_{\nu}(x)$ denotes any real solution of equation (2), that is,

$$C_{\nu}(x) = \alpha J_{\nu}(x) + \beta Y_{\nu}(x),$$

where ν, α and β are real, $x > 0$, and $J_{\nu}(x)$ and $Y_{\nu}(x)$ denote the usual Bessel functions.

Presented to the Society, November 28, 1970; received by the editors December 22, 1970.
AMS 1970 subject classifications. Primary 33A70, 33A40; Secondary 26A33, 34C10, 44A20.

Key words and phrases. Lommel functions, Bessel functions, changes of sign, oscillation theorems, inhomogeneous Bessel equation.

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A case of particular interest is that in which $\mu = \nu$. Then, unless ν is half of an odd negative integer, we have

$$(6) \quad s_{\nu,\nu}(x) = 2^{\nu-1} \Gamma(\tfrac{1}{2}) \Gamma(\nu + \tfrac{1}{2}) H_{\nu}(x),$$

where $H_{\nu}(x)$ is Struve's function of order ν [1, (3.127)]. Now it is known [9, §10.45] that $H_{\nu}(x) > 0$ for all $x > 0$ if $\nu > \frac{1}{2}$, that $H_{1/2}(x) = (2/\pi x)^{1/2}(1 - \cos x)$, and that $H_{\nu}(x)$ has an infinity of changes of sign if $\nu < \frac{1}{2}$. The corresponding problem for $s_{\mu,\nu}(x)$ is more difficult. R. G. Cooke [4] found conditions on μ and ν sufficient to ensure that $s_{\mu,\nu}(x) > 0$ for all $x > 0$. We may state his results as

THEOREM A. *If $\nu \geq 0$ and $\mu \geq \nu + 1$, then $s_{\mu,\nu}(x) > 0$ for $x > 0$. If $\nu \geq \frac{1}{2}$ and $\mu \geq \nu$, then $s_{\mu,\nu}(x) > 0$ for $x > 0$, unless $\mu = \nu = \frac{1}{2}$, when $s_{\mu,\nu}(x) \geq 0$ for $x > 0$.*

The symmetry relation (4) gives the corresponding results for $\nu < 0$.

Cooke's proof requires an expression for $s_{\mu,\nu}(x)$ as a fractional integral involving $J_{\nu}(x)$, and a previous result of his ([2] and [3]) which implies that if $\mu = 0$ and $\nu > -1$, or $\mu = 1 - \nu$ and $\nu > \frac{1}{2}$, then

$$\int_0^{\xi} t^{\mu} J_{\nu}(t) dt > 0, \quad \xi > 0.$$

We shall here consider the same problem. It will often be convenient to refer to a (μ, ν) -plane, and to associate with each pair μ, ν the point with those coordinates.

We shall see that $s_{\mu,\nu}(x) > 0$ for $x > 0$ if $\mu \geq \frac{1}{2}$ and $|\nu| \leq \mu$, except when $\mu = |\nu| = \frac{1}{2}$, in which case $s_{\mu,\nu}(x) \geq 0$. These inequalities define a larger region in the (μ, ν) -plane than Cooke's. If $\mu < \frac{1}{2}$, or if $\mu = \frac{1}{2}$ and $|\nu| > \frac{1}{2}$, then $s_{\mu,\nu}(x)$ changes sign infinitely often on $(0, \infty)$. If $\mu > \frac{1}{2}$ and $\mu < |\nu| - 1$, $s_{\mu,\nu}(x)$ has an odd number of changes of sign on $(0, \infty)$. Finally, if $\mu > \frac{1}{2}$ and $|\nu| - 1 < \mu < |\nu|$, $s_{\mu,\nu}(x)$ has an even number of changes of sign on $(0, \infty)$; but I have not been able to decide whether this number is always positive. I shall show, however, that there are points (μ, ν) in this region such that the corresponding $s_{\mu,\nu}(x)$ has changes of sign (an arbitrarily large number of them, in fact).

Our main tool is an oscillation theorem of Makai's [7] for second order differential equations. With it, we can determine the sign of the function

$$h(\xi) = \int_0^{\xi} t^{\mu} C_{\nu}(t) dt, \quad \xi > 0,$$

for certain pairs μ, ν . Then, by using (5), we can deduce results on the sign of $s_{\mu,\nu}(x)$.

2. Oscillation theorems. Our starting point is

THEOREM B. *Let $y = y(x)$ be a solution of the differential equation $y'' + \varphi(x)y = 0$, where φ is continuous and increasing on (x_0, x_2) . Let x_1 be the only zero of $y(x)$ on (x_0, x_2) . Further, assume that*

$$(7) \quad \lim_{x \rightarrow x_0 + 0} y(x) = \lim_{x \rightarrow x_2 - 0} y(x) = 0.$$

Then,

$$(8) \quad \int_{x_0}^{x_1} |y(x)| dx \geq \int_{x_1}^{x_2} |y(x)| dx,$$

with strict inequality if φ is strictly increasing.

If φ is decreasing on (x_0, x_2) , then (8) is reversed.

Theorem B is due essentially to E. Makai [7], [8, §1.82], but the endpoint conditions (7) have been introduced to avoid difficulties which may arise when φ is discontinuous at x_0 , or at x_2 . The monotonicity condition on φ could be relaxed somewhat [7, §2(d)], but we will not need this here.

From Theorem B we deduce

THEOREM 1. Let $y=y(x)$ be a solution of the differential equation

$$(9) \quad (r(x)y')' + p(x)y = 0,$$

where r and p are continuous, pr is increasing and r is positive, on (x_0, x_2) . Assume further that $\int_{x_0}^x (r(u))^{-1} du$ converges. Let

$$(10) \quad \lim_{x \rightarrow x_0+0} y(x) = \lim_{x \rightarrow x_2-0} y(x) = 0,$$

and let x_1 be the only zero of y on (x_0, x_2) . Then,

$$(11) \quad \int_{x_0}^{x_1} \frac{|y(x)|}{r(x)} dx \geq \int_{x_1}^{x_2} \frac{|y(x)|}{r(x)} dx,$$

with strict inequality if pr is strictly increasing. And if pr is decreasing (11) is reversed.

Proof. We transform the independent variable in (9) by setting

$$t = f(x) = \int_{x_0}^x \frac{du}{r(u)}.$$

Equation (9) can then be written as

$$(12) \quad d^2y/dt^2 + r(g(t))p(g(t))y = 0,$$

where g denotes the inverse of the (increasing) function f [5, p. 235]. Inequality (11) now follows by applying Theorem B to (12), and then changing the independent variable back to x .

In the sequel, we shall consider the equation

$$(13) \quad (x^{1-2\mu}y')' + x^{-2\mu-1}(x^2 + \mu^2 - \nu^2)y = 0,$$

which is of the form (9), with

$$(14) \quad (p(x)r(x))' = 2x^{-4\mu-1}[(1-2\mu)x^2 + 2\mu(\nu^2 - \mu^2)].$$

The integrand in (5), $x^\mu C_\nu(x)$, is a solution of equation (13) [9, §4.31, (19)–(20)]. We shall often choose $\mu = \frac{1}{2}$; (13) then becomes

$$(15) \quad y'' + (1 + (1 - 4\nu^2)/4x^2)y = 0.$$

3. **Lommel functions.** The two Lommel functions $s_{\mu,\nu}(x)$ and $S_{\mu,\nu}(x)$ are related by the identity

$$(16) \quad s_{\mu,\nu}(x) = S_{\mu,\nu}(x) + 2^{\mu-1} \Gamma\left(\frac{\mu-\nu+1}{2}\right) \Gamma\left(\frac{\mu+\nu+1}{2}\right) \cdot \left[\cos \frac{(\mu-\nu)\pi}{2} Y_\nu(x) - \sin \frac{(\mu-\nu)\pi}{2} J_\nu(x) \right],$$

whenever $s_{\mu,\nu}(x)$ is defined [9, §10.71].

Now it can be shown [9, §10.75] that $S_{\mu,\nu}(x) \sim x^{\mu-1}$, as $x \rightarrow +\infty$. On combining this with the relations

$$J_\nu(x) = (2/\pi x)^{1/2} [\cos(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + O(1/x)]$$

and

$$Y_\nu(x) = (2/\pi x)^{1/2} [\sin(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + O(1/x)]$$

[9, §7.22], we see that as $x \rightarrow +\infty$, the dominant term on the right-hand side of (16) is

$$2^{\mu-1} \left(\frac{2}{\pi x}\right)^{1/2} \Gamma\left(\frac{\mu-\nu+1}{2}\right) \Gamma\left(\frac{\mu+\nu+1}{2}\right) \sin(x - \frac{1}{2}\mu\pi - \frac{1}{4}\pi) \quad \text{or} \quad x^{\mu-1},$$

according as $\mu < \frac{1}{2}$ or $\mu > \frac{1}{2}$. It follows that $s_{\mu,\nu}(x)$ has an infinity of changes of sign if $\mu < \frac{1}{2}$, but is positive for all sufficiently large x , if $\mu > \frac{1}{2}$. We now proceed to make this simple observation more precise. We begin by establishing

THEOREM 2. *If $\mu < \frac{1}{2}$, or if $\mu = \frac{1}{2}$ and $|\nu| > \frac{1}{2}$, then $s_{\mu,\nu}(x)$ has an infinity of changes of sign on $(0, \infty)$.*

Proof. For $\mu < \frac{1}{2}$ and ν unrestricted, the result follows as above. For $\mu = \frac{1}{2}$ and $|\nu| > \frac{1}{2}$, we apply Theorem B to the particular solution $y(x) = x^{1/2} J_\nu(x)$ of equation (15). It is clear from (14) that pr is increasing on $(0, \infty)$ for such pairs μ, ν . Now let $j_{\nu,k}$ denote the k th positive zero of $J_\nu(x)$. In (7), we may take $x_0 = j_{\nu,k}$, for any k . And we may also take $x_0 = 0$, since $J_\nu(x) = O(x^\nu)$ as $x \rightarrow 0+$. Since $J_\nu(x) > 0$ for $0 < x < j_{\nu,1}$ if $\nu > -1$, it follows from (8) that

$$\int_0^\xi x^{1/2} J_\nu(x) dx > 0, \quad \xi > 0, \nu > \frac{1}{2}.$$

Together with (5), this shows that $J_\nu(\xi) s'_{1/2,\nu}(\xi) - s_{1/2,\nu}(\xi) J'_\nu(\xi) > 0$, for the right-hand side of (5) vanishes at $x=0$ if $\beta=0$ and $\mu+\nu+1 > 0$, by (3). In particular,

$$s_{1/2,\nu}(j_{\nu,k}) J'_\nu(j_{\nu,k}) < 0, \quad k = 1, 2, \dots, \nu > \frac{1}{2}.$$

But $\text{sgn } J'_\nu(j_{\nu,k}) = (-1)^k$ for $\nu \geq 0$; thus $s_{1/2,\nu}(x)$ must have an odd number of changes of sign between consecutive positive zeros of $J_\nu(x)$, if $\nu > \frac{1}{2}$. Because of (4), this is also true if $\nu < -\frac{1}{2}$, and the proof is complete.

Next, we turn to the case $\mu \geq \frac{1}{2}$, and prove

THEOREM 3. *If $\mu \geq \frac{1}{2}$ and $|\nu| \leq \mu$, then $s_{\mu,\nu}(x) > 0$ for all $x > 0$, except if $\mu = |\nu| = \frac{1}{2}$, when $s_{\mu,\nu}(x) \geq 0$.*

Proof. For $\mu = |\nu| = \frac{1}{2}$ we have, from (6),

$$(17) \quad s_{1/2,1/2}(x) = s_{1/2,-1/2}(x) = x^{-1/2}(1 - \cos x).$$

For $\mu = \frac{1}{2}$ and $|\nu| < \frac{1}{2}$ we shall use Theorem B and (5). From (3), and familiar facts about the behavior of $J_\nu(x)$ and $Y_\nu(x)$ as $x \rightarrow 0+$, it is easily seen that we may take 0 as lower limit of integration in (5), and that the right-hand side of (5) vanishes at $x=0$, if $\mu > |\nu| - 1$. Hence, for $\mu > |\nu| - 1$ and $\xi > 0$, we have

$$\int_0^\xi x^\mu C_\nu(x) dx = \xi [C_\nu(\xi) s'_{\mu,\nu}(\xi) - s_{\mu,\nu}(\xi) C'_\nu(\xi)].$$

The particular choice $\xi = c_{\nu,k}$, the k th positive zero of $C_\nu(x)$, yields

$$(18) \quad \int_0^{c_{\nu,k}} x^\mu C_\nu(x) dx = -c_{\nu,k} s_{\mu,\nu}(c_{\nu,k}) C'_\nu(c_{\nu,k}).$$

Now $y(x) = x^\mu C_\nu(x)$ is a solution of (13). Since $\lim_{x \rightarrow 0+} x^\mu C_\nu(x) = 0$ if $\mu > |\nu|$, (10) is satisfied by $x_0 = 0$. Moreover, it is clear from (14) that pr is strictly decreasing on $(0, \infty)$, if $\mu = \frac{1}{2}$ and $|\nu| < \frac{1}{2}$.

Thus, the hypotheses of Theorem B are satisfied by $y(x) = x^{1/2} C_\nu(x)$ on each of the intervals $(0, c_{\nu,2})$ and $(c_{\nu,k}, c_{\nu,k+2})$, $k \geq 1$, if $|\nu| < \frac{1}{2}$. It follows that

$$(19) \quad \left\{ \int_0^{c_{\nu,k}} x^{1/2} C_\nu(x) dx \right\} \left\{ \int_0^{c_{\nu,k+1}} x^{1/2} C_\nu(x) dx \right\} < 0, \quad k \geq 1, |\nu| < \frac{1}{2}.$$

Since $C'_\nu(c_{\nu,k}) C'_\nu(c_{\nu,k+1}) < 0$ [9, §15.21], it follows from (18) and (19) that

$$(20) \quad s_{1/2,\nu}(c_{\nu,k}) > 0, \quad k \geq 1, |\nu| < \frac{1}{2}.$$

But $x^{1/2} C_\nu(x)$ is an *arbitrary* nonnull solution of (15). Therefore, (20) implies that $s_{1/2,\nu}(x) > 0$ for all $x > 0$ if $|\nu| < \frac{1}{2}$, since any $x > 0$ is a zero of some nonnull solution of (15). Together with (17), this proves Theorem 3 for $\mu = \frac{1}{2}$. And if we use the integrals

$$s_{\mu+\sigma,\nu+\sigma}(x) = 2x^\sigma \frac{\Gamma((\mu+\nu+2\sigma+1)/2)}{\Gamma(\sigma)\Gamma((\mu+\nu+1)/2)} \int_0^{\pi/2} s_{\mu,\nu}(x \sin \theta) \sin^{\nu+1} \theta \cos^{2\sigma-1} \theta d\theta,$$

$$s_{\mu+\sigma,\nu-\sigma}(x) = 2x^\sigma \frac{\Gamma((\mu+\nu+2\sigma+1)/2)}{\Gamma(\sigma)\Gamma((\mu-\nu+1)/2)} \int_0^{\pi/2} s_{\mu,\nu}(x \sin \theta) \sin^{1-\nu} \theta \cos^{2\sigma-1} \theta d\theta,$$

both valid for $\sigma > 0$ and $\mu \pm \nu > -1$ [1, §3.20], the theorem's truth for $\mu > \frac{1}{2}$ is an immediate consequence of its truth for $\mu = \frac{1}{2}$.

An intermediate situation between those of the last two theorems is described by

THEOREM 4. *Let $\mu > \frac{1}{2}$. If $\mu < |\nu| - 1$, then $s_{\mu,\nu}(x)$ has an odd number of changes of sign on $(0, \infty)$. If $|\nu| - 1 < \mu < |\nu|$, then $s_{\mu,\nu}(x)$ has an even number of changes of sign (perhaps none) on $(0, \infty)$.*

Proof. From (3) we have $s_{\mu, \nu}(x) \sim x^{\mu+1}/((\mu+1)^2 - \nu^2)$, as $x \rightarrow 0+$. On the other hand, it follows from our discussion of (16) that when $\mu > \frac{1}{2}$, $s_{\mu, \nu}(x) \sim x^{\mu-1}$, as $x \rightarrow +\infty$. The conclusion is now obvious.

I have not been able to decide whether the even number alluded to in Theorem 4 is in fact always positive. However, I can show that if the point (μ, ν) , with $\mu > \frac{1}{2}$ and $|\nu| - 1 < \mu < |\nu|$, is close enough to the line $\mu = \frac{1}{2}$, or to one of the lines $\mu = |\nu| - 1$, then the corresponding $s_{\mu, \nu}(x)$ changes sign. In fact, by choosing the point (μ, ν) close enough to the line $\mu = \frac{1}{2}$ in the region $\mu > \frac{1}{2}$, $|\nu| - 1 < \mu < |\nu|$, we can find Lommel functions with an arbitrarily large number of changes of sign.

This last statement can be verified as follows. We may assume $\nu > 0$. By applying Theorem 1 to (13), we see that if

$$(21) \quad (1 - 2\mu)x^2 + 2\mu(\nu^2 - \mu^2) > 0 \quad \text{for } 0 < x < \bar{x},$$

and if j is the largest zero of $J_\nu(x)$ in $(0, \bar{x}]$, then

$$\int_0^x t^{3\mu-1} J_\nu(t) dt > 0 \quad \text{for } 0 < x \leq j,$$

since (10) is satisfied by $y(x) = x^\mu J_\nu(x)$ and $x_0 = 0$, if $\mu + \nu > 0$. For $\mu \geq \frac{1}{2}$, this implies that

$$(22) \quad \int_0^x t^\mu J_\nu(t) dt > 0 \quad \text{for } 0 < x \leq j,$$

by the second mean value theorem.

Thus, if $(0, \bar{x}]$ contains at least two positive zeros of $J_\nu(t)$, then (22) holds with $j = j_{\nu, 2}$, and hence also with $j = j_{\nu, 3}$. As before, if $\mu > \nu - 1$, this will imply that $s_{\mu, \nu}(x)$ changes sign on $(j_{\nu, 1}, j_{\nu, 2})$ and on $(j_{\nu, 2}, j_{\nu, 3})$. Now for $\mu > \frac{1}{2}$, (21) holds if and only if $\mu < |\nu|$ and $|\bar{x}| \leq (2\mu(\nu^2 - \mu^2)/(2\mu - 1))^{1/2}$. Therefore, $s_{\mu, \nu}(x)$ will change sign if $\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$, and

$$(23) \quad j_{\nu, 2} \leq (2\mu(\nu^2 - \mu^2)/(2\mu - 1))^{1/2}.$$

Now if we choose some $\nu > \frac{1}{2}$, and keep it fixed, then the left-hand side of (23) is fixed. But the right-hand side tends to $+\infty$, as $\mu \rightarrow \frac{1}{2} + 0$. Hence for each ν , $\frac{1}{2} < \nu < \frac{3}{2}$, we can find a μ ($\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$) such that $s_{\mu, \nu}(x)$ has at least two changes of sign. In order to produce a Lommel function with at least $2n$ changes of sign, it suffices to satisfy the inequality obtained by writing $j_{\nu, 2n}$ for $j_{\nu, 2}$ in (23).

We now turn our attention to points near the line $\mu = \nu - 1$, in the region $\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$. Here, we use (16). We fix $\nu \geq \frac{3}{2}$, and choose $x^* > 0$ such that $J_\nu(x^*) < 0$. Then, we let $\mu \rightarrow (\nu - 1) + 0$. Now $S_{\mu, \nu}(x^*)$ tends to a limit, as $\mu \rightarrow \nu - 1$ [9, §10.73]. And

$$\lim_{\mu \rightarrow \nu - 1} \left[\cos \frac{(\mu - \nu)\pi}{2} Y_\nu(x^*) - \sin \frac{(\mu - \nu)\pi}{2} J_\nu(x^*) \right] = J_\nu(x^*).$$

Hence, for $x=x^*$, the right-hand side of (16) tends to $-\infty$ as $\mu \rightarrow (\nu-1)+0$, because of the factor $\Gamma((\mu-\nu+1)/2)$. Therefore, $s_{\mu,\nu}(x^*) < 0$ for an appropriate choice of μ .

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