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CONFORMALITY AND ISOMETRY OF RIEMANNIAN MANIFOLDS TO SPHERES(1)

BY
CHUAN-CHIH HSIUNG(2) AND LOUIS W. STERN

Abstract. Suppose that a compact Riemannian manifold M^n of dimension $n>2$ admits an infinitesimal nonisometric conformal transformation v. Some curvature conditions are given for M^n to be conformal or isometric to an n-sphere under the initial assumption that $L_vR=0$, where L_v is the operator of the infinitesimal transformation v and R is the scalar curvature of M^n. For some special cases, these conditions were given by Yano [10] and Hsiung [2].

1. **Introduction.** Let M^n be a Riemannian manifold of dimension $n \geqq 2$ and class C^3, (g_{ij}) the symmetric matrix of the positive definite metric of M^n, and (g^{ij}) the inverse matrix of (g_{ij}), and denote by ∇_i, R_{hijk}, $R_{ij}=R^k{}_{ijk}$ and $R=g^{ij}R_{ij}$ the operator of covariant differentiation with respect to g_{ij}, the Riemann tensor, the Ricci tensor and the scalar curvature of M^n respectively. Let d be the operator of exterior derivation, δ the operator of coderivation, and $\Delta = d\delta + \delta d$ the Laplace-Beltrami operator. Throughout this paper all Latin indices take the values $1, \ldots, n$ unless stated otherwise. We shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively, and that repeated indices imply summation.

Let v be a vector field defining an infinitesimal conformal transformation on M^n. Denote by the same symbol v the 1-form corresponding to the vector field v by the duality defined by the metric of M^n, and by L_v the operator of the infinitesimal transformation v. Then we have

$$L_v g_{ij} = \nabla_i v_j + \nabla_j v_i = 2\rho g_{ij}. \tag{1.1}$$

The infinitesimal transformation v is said to be homothetic or an infinitesimal isometry according as the scalar function ρ is constant or zero. On a compact orientable Riemannian manifold, an infinitesimal homothetic transformation is necessarily an infinitesimal isometry; see [9]. We also denote by $L_{d\rho}$ the operator of the infinitesimal transformation generated by the vector field ρ^i defined by

$$\rho^i = g^{ij}\rho_j, \qquad \rho_j = \nabla_j \rho. \tag{1.2}$$

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Let $\xi_{I(p)}$ and $\eta_{I(p)}$ be two tensor fields of the same order $p \leqq n$ on a compact orientable manifold M^n, where $I(p)$ denotes an ordered subset $\{i_1, \ldots, i_p\}$ of the set $\{1, \ldots, n\}$ of positive integers less than or equal to n. Then the local and global scalar products $\langle \xi, \eta \rangle$ and (ξ, η) of the tensor fields ξ and η are defined by

$$\langle \xi, \eta \rangle = (1/p!)\xi^{I(p)}\eta_{I(p)}, \tag{1.3}$$

$$(\xi, \eta) = \int_{M^n} \langle \xi, \eta \rangle \, dV, \tag{1.4}$$

where dV is the element of volume of the manifold M^n at a point. We also define

$$\|\xi\| = p!\langle \xi, \xi \rangle. \tag{1.5}$$

From (1.3) and (1.4) it follows that (ξ, ξ) is nonnegative, and that $(\xi, \xi)=0$ implies that $\xi=0$ on the whole manifold M^n.

In the last decade or so various authors have studied the conditions for a Riemannian manifold M^n of dimension $n>2$ with constant scalar curvature R to be either conformal or isometric to an n-sphere. Very recently Yano, Obata, Hsiung and Mugridge (see [13], [10], [4]) have been able to extend some of these results by replacing the constancy of R by $L_u R=0$, where u is a certain vector field on M^n. The purpose of this paper is to continue their work, in particular Yano's [10], by establishing the following theorems.

To begin we denote by (C) the following condition:

(C) A compact Riemannian manifold M^n of dimension $n>2$ admits an infinitesimal nonisometric conformal transformation v satisfying (1.1) with $\rho \neq 0$ such that $L_v R=0$.

THEOREM I. *An orientable M^n is conformal to an n-sphere if it satisfies condition* (C) *and*

$$\left(\rho_i \rho^i - \frac{1}{n-1} R\rho^2, R\right) \geqq 0, \tag{1.6}$$

$$L_v\left(a^2 A + \frac{c-4a^2}{n-2} B\right) = 0, \tag{1.7}$$

where A and B are defined by

$$A = R^{hijk}R_{hijk}, \qquad B = R^{ij}R_{ij}, \tag{1.8}$$

and a, c are constant such that

$$\begin{aligned} c \equiv 4a^2 + (n-2)\Bigg[2a \sum_{i=1}^{4} b_i + \left(\sum_{i=1}^{6} (-1)^{i-1} b_i\right)^2 \\ -2(b_1 b_3 + b_2 b_4 - b_5 b_6) + (n-1) \sum_{i=1}^{6} b_i^2\Bigg] > 0, \end{aligned} \tag{1.9}$$

b's being any constants(3).

(3) An elementary calculation shows that $c \geqq 0$, where equality holds if and only if $b_1 = \cdots = b_4$, $b_5 = b_6 = 0$, $a = -(n-2)b_1$.

For the case $a \neq 0$, $c-4a^2=0$ and the case $a=0$, $c-4a^2 \neq 0$, Theorem I is due to Yano [10].

THEOREM II. *A manifold M^n is conformal to an n-sphere if it satisfies condition* (C) *and any one of the following three sets of conditions*:

$$\nabla_i \nabla_j (Rf) = R\rho g_{ij} \qquad (f \text{ is a scalar function}), \tag{1.10}$$

$$Q d\rho = (2/n)\, d(R\rho), \qquad \nabla_i \nabla_j (R\rho) = R \nabla_i \nabla_j \rho, \tag{1.11}$$

$$L_v R_{ij} = \alpha g_{ij} \qquad (\alpha \text{ is a scalar function}), \tag{1.12}$$

where Q is the operator of Ricci defined by, for any vector field u on M^n,

$$Q: u_i \to 2R_{ij}u^j. \tag{1.13}$$

For constant R, conditions (1.11) and (1.12) in Theorem II will lead to the conclusion that M^n is isometric to an n-sphere of radius $(n(n-1)/R)^{1/2}$; for this see [12].

THEOREM III. *A manifold M^n with constant R is isometric to an n-sphere of radius $(n(n-1)/R)^{1/2}$ if it satisfies conditions* (C) *and* (1.10).

Theorem III is due to Lichnerowicz [6] when condition (1.10) is replaced by the following one:

$$v \text{ is the gradient of a scalar function } f, \text{ i.e., } v_i = \nabla_i f. \tag{1.14}$$

For constant R, it is easily seen that condition (1.14) is a special case of condition (1.10). In fact, in this case by using (1.1) condition (1.10) becomes $\nabla_i v_j + \nabla_j v_i = 2\nabla_i \nabla_j f$, which is satisfied by $v_i = \nabla_i f + u_i$ where u_i is any vector field generating an infinitesimal isometry.

THEOREM IV. *A manifold M^n is isometric to an n-sphere if it satisfies condition* (C), $L_{d\rho}R=0$, *and*

$$A^a B^b = c = \text{const}, \tag{1.15}$$

$$c\left(\frac{2a}{A} + \frac{(n-1)b}{B}\right) = \frac{2^a (a+b) R^{2(a+b-1)}}{n^{a+b-1}(n-1)^{a-1}}, \tag{1.16}$$

where A, B are given by (1.8), *and a, b are nonnegative integers and not both zero.*

For constant R, Theorem IV is due to Lichnerowicz [6] for $a=0$, $b=1$ and due to Hsiung [2] for general a and b.

The following known theorems will be needed in the proofs of our Theorems I-IV.

THEOREM A (YANO AND NAGANO [11]). *If a complete Einstein space M^n of dimension $n>2$ admits an infinitesimal nonisometric conformal transformation, then M^n is isometric to an n-sphere.*

THEOREM B (OBATA [7]). *If a complete Riemannian manifold M^n of dimension $n \geqq 2$ admits a nonconstant function ρ such that $\nabla_i \nabla_j \rho = -c^2 \rho g_{ij}$, where c is a positive constant, then M^n is isometric to an n-sphere of radius $1/c$.*

THEOREM C (TASHIRO [8]). *If a complete Riemannian manifold M^n of dimension $n > 2$ admits a nonconstant function ρ such that*

$$\nabla_i \nabla_j \rho = -(1/n) g_{ij} \Delta \rho, \tag{1.17}$$

then M^n is conformal to an n-sphere.

THEOREM D (YANO [10]). *An orientable manifold M^n is conformal to an n-sphere if it satisfies condition* (C) *and*

$$(R_{ij}\rho^i\rho^j - (1/n(n-1))R^2\rho^2, 1) \geqq 0. \tag{1.18}$$

2. **Notation and formulas.** In this section we shall list some known formulas (for the details of their derivations see Lichnerowicz' book [5, pp. 124–134] or Hsiung's paper [1]) which will be needed in the proofs to follow.

Let v be a vector field defining an infinitesimal conformal transformation on a Riemannian manifold M^n so that (1.1) holds. Then we have

$$\rho = -\delta v/n, \tag{2.1}$$

$$L_v R^h{}_{ijk} = -\varepsilon^h_k \nabla_i \rho_j + \varepsilon^h_j \nabla_i \rho_k - g_{ij} \nabla_k \nabla^h \rho + g_{ik} \nabla_j \nabla^h \rho, \tag{2.2}$$

where $\nabla^h = g^{ih} \nabla_i$, and $\varepsilon^h_k = 1$ for $h = k$ and $= 0$ for $h \neq k$. From (1.1) and (2.2) it follows immediately that

$$L_v R_{hijk} = 2\rho R_{hijk} - g_{hk} \nabla_i \rho_j + g_{hj} \nabla_i \rho_k - g_{ij} \nabla_h \rho_k + g_{ik} \nabla_h \rho_j, \tag{2.3}$$

$$L_v R_{ij} = g_{ij} \Delta \rho - (n-2) \nabla_i \rho_j, \tag{2.4}$$

$$L_v R = 2(n-1) \Delta \rho - 2R\rho. \tag{2.5}$$

For any scalar field f and vector field u on M^n, we have

$$\Delta f = -\nabla^i \nabla_i f, \tag{2.6}$$

$$(\Delta u)_i = -\nabla^j \nabla_j u_i + \tfrac{1}{2}(Qu)_i, \tag{2.7}$$

where Q is the operator of Ricci defined by (1.13).

A necessary and sufficient condition for a vector field v to define an infinitesimal conformal transformation on a compact manifold M^n is that it satisfy

$$\Delta v + (1 - 2/n)\, d\delta v = Qv. \tag{2.8}$$

For an infinitesimal transformation v on a manifold M^n, we have

$$\Delta \delta v = (1/(n-1)) R \delta v - (n/2(n-1)) L_v R. \tag{2.9}$$

For any 1-form ξ on a compact orientable manifold M^n we have

$$(\Delta \xi + (1 - 2/n)\, d\delta \xi - Q\xi, \xi) \geqq 0, \tag{2.10}$$

where the equality holds when and only when ξ defines an infinitesimal conformal transformation on M^n.

On the manifold M^n consider the following tensors:

$$T_{ij} = R_{ij}-(1/n)Rg_{ij}, \tag{2.11}$$

$$T_{hijk} = R_{hijk}-(1/n(n-1))R(g_{ij}g_{hk}-g_{ik}g_{hj}), \tag{2.12}$$

$$\begin{aligned} W_{hijk} = {} & aT_{hijk}+b_1 g_{hk}T_{ij}-b_2 g_{hj}T_{ik}+b_3 g_{ij}T_{hk} \\ & -b_4 g_{ik}T_{hj}+b_5 g_{hi}T_{jk}-b_6 g_{jk}T_{hi}, \end{aligned} \tag{2.13}$$

where a and b are constants. It is easily seen that

$$g^{ij}T_{ij} = 0, \qquad g^{hk}T_{hijk} = T_{ij}. \tag{2.14}$$

Moreover, by (1.3), (1.5) and (2.13) we have

$$\|W\| = a^2A+\frac{c-4a^2}{n-2}\,B-\frac{1}{n}\left(\frac{2a^2}{n-1}+\frac{c-4a^2}{n-2}\right)R^2, \tag{2.15}$$

where c is defined by (1.9).

3. **Lemmas.** Throughout this section M^n will always denote a compact orientable Riemannian manifold of dimension $n \geqq 2$.

LEMMA 3.1. *If f is a scalar field on M^n and $\Delta f=0$, then f is constant.*

Proof. From (2.6) and our assumption $\Delta f=0$, it follows that $\Delta(f^2)=-\nabla^i\nabla_i(f^2)$ $=-2(\nabla^i f)(\nabla_i f)$. By substituting $\nabla_i(f^2)$ for ξ_i in the well-known Green's formula

$$\int_{M^n} \nabla^i \xi_i \, dV = 0, \tag{3.1}$$

where ξ_i is any vector field on M^n, we therefore have

$$0 = \int_{M^n} \Delta(f^2)\, dV = -2\int_{M^n} (\nabla^i f)(\nabla_i f)\, dV, \tag{3.2}$$

which implies that $\nabla_i f=0$ since $(\nabla^i f)(\nabla_i f)$ is nonnegative.

LEMMA 3.2. *For an orientable M^n satisfying $L_{d\rho}R=0$ and* (C) *defined in* §1, *we have*

$$(R^{ij}\nabla_i\nabla_j\rho+R^2\rho/n(n-1),\ \rho) \geqq 0. \tag{3.3}$$

For constant R, Lemma 3.2 is due to Lichnerowicz [6].

Proof. By applying the integral formula (2.10) to the 1-form $d\rho$ we have

$$((2(n-1)/n)\Delta d\rho-Qd\rho,\ d\rho) \geqq 0. \tag{3.4}$$

On the other hand, covariant differentiation gives

$$\begin{aligned} & \nabla^i[\rho((2(n-1)/n)\Delta d\rho-Qd\rho)_i] \\ & \quad = \langle (2(n-1)/n)\Delta d\rho-Qd\rho,\ d\rho\rangle - \langle (2(n-1)/n)\Delta\Delta\rho-\delta Qd\rho,\ \rho\rangle. \end{aligned} \tag{3.5}$$

From (3.4), (3.5) and Green's formula (3.1) we thus obtain

$$((2(n-1)/n)\Delta\Delta\rho - \delta Q d\rho, \rho) \geqq 0. \tag{3.6}$$

Due to the assumption $L_v R=0$, (2.5) is reduced to

$$\Delta\rho = R\rho/(n-1). \tag{3.7}$$

Since $L_{d\rho}R=0$ implies

$$\rho^i \nabla_i R = 0, \tag{3.8}$$

substitution of $\rho^2 \nabla_i R$ for ξ_i in Green's formula (3.1) gives

$$(\rho\Delta R, \rho) = 0. \tag{3.9}$$

On the other hand, by the second Bianchi identity we have

$$\nabla^j R_{ij} = \tfrac{1}{2}\nabla_i R, \tag{3.10}$$

which together with (3.8) implies

$$\rho^i \nabla^j R_{ij} = 0. \tag{3.11}$$

From (1.13), (3.11) and (3.7) follow immediately

$$\delta Q\, d\rho = -2\nabla_i(R^{ij}\rho_j) = -2R^{ij}\nabla_i\rho_j, \tag{3.12}$$

$$\Delta\Delta\rho = (1/(n-1)^2)R^2\rho + (1/(n-1))\rho\Delta R. \tag{3.13}$$

Substituting (3.12), (3.13) in (3.6) and making use of (3.9), we hence obtain the required inequality (3.3).

4. Proofs of theorems.

Proof of Theorem I. From (2.13) and the condition $L_v R=0$ it follows that

$$L_v\|W\| = L_v\left(a^2 A + \frac{c-4a^2}{n-2} B\right). \tag{4.1}$$

By means of (2.13), (2.12), (1.1), (2.3), (2.4), (3.7) we can easily compute $L_v W_{hijk}$ (for the details see [3, p. 189]), and then multiplying both sides of the resulting expression by W^{hijk} and making use of (2.13), (2.12), (1.1), (2.15), (1.9) and $R^i{}_{ijk}=0$ an elementary but lengthy calculation yields

$$W^{hijk}L_v W_{hijk} = 2\rho\|W\| - cT^{ij}\nabla_i\rho_j. \tag{4.2}$$

Substitution of (4.2) in the well-known formula

$$L_v\|W\| = 2W^{hijk}L_v W_{hijk} - 8\rho\|W\| \tag{4.3}$$

thus gives

$$\rho L_v\|W\| = -4\rho^2\|W\| - 2c\rho T^{ij}\nabla_i\rho_j. \tag{4.4}$$

A straightforward computation and use of (2.6), (3.7), (2.11), (3.10) can easily show that

$$\nabla^i(R\rho\rho_i) = (\nabla_i R)\rho\rho^i + R\rho_i\rho^i - R^2\rho^2/(n-1), \tag{4.5}$$

$$\nabla^i(T_{ij}\rho\rho^j) = R_{ij}\rho^i\rho^j + \rho T^{ij}\nabla_i\rho_j + ((n-2)/2n)(\nabla_i R)\rho\rho^i - (1/n)R\rho_i\rho^i. \tag{4.6}$$

By substituting (4.5) for $(\nabla_i R)\rho\rho^i$ and (4.4) for $\rho T^{ij}\nabla_i\rho_j$ in (4.6), integrating over M^n and making use of (3.1), we thus obtain

$$\begin{aligned} &2c(R_{ij}\rho^i\rho^j - (1/n(n-1))R^2\rho^2, 1) \\ &\qquad = 4(\|W\|, \rho^2) + (L_v\|W\|, \rho) + c(\rho_i\rho^i - (1/(n-1))R\rho^2, R). \end{aligned} \tag{4.7}$$

Since $(\|W\|, \rho^2)$ is nonnegative, from (4.7), (4.1) and our assumption (1.6), (1.7) we obtain (1.18). Hence by Theorem D, M^n is conformal to an n-sphere.

Proof of Theorem II. First suppose (1.10) holds. Then from (1.10), (2.6) it follows that $\Delta(Rf) = -nR\rho$, implying, together with (3.7), that

$$\Delta(\rho + (1/n(n-1))Rf) = 0. \tag{4.8}$$

Thus by Lemma 3.1, $\rho + (1/n(n-1))Rf$ is a constant. Using (1.10), (3.7) we therefore obtain

$$\nabla_i\rho_j = -(1/n(n-1))\nabla_i\nabla_j(Rf) = -(1/n)g_{ij}\Delta\rho.$$

Hence by Theorem C, M^n is conformal to an n-sphere.

Next suppose (1.11) holds. From the definition of Δ it follows that

$$d\Delta\rho = \Delta d\rho, \tag{4.9}$$

which, together with (3.7) and the first equation of (1.11), implies

$$\Delta d\rho + (1-2/n)\, d\delta d\rho - Qd\rho = 0. \tag{4.10}$$

Thus by the necessary and sufficient condition (2.8) we see that $d\rho$ generates an infinitesimal conformal transformation on M^n so that $L_{d\rho}g_{ij} = 2\phi g_{ij}$, which shows

$$\nabla_i\rho_j = \phi g_{ij}, \tag{4.11}$$

where $\phi \neq 0$ in consequence of (3.7). From (4.11) and the second equation of (1.11) it follows that

$$\nabla_i\nabla_j(R\rho) = R\phi g_{ij}. \tag{4.12}$$

Thus the condition (1.10) is satisfied for $v = d\rho$ and $f = \rho$, and hence M^n is conformal to an n-sphere.

Finally suppose (1.12) holds. Then (2.4) becomes

$$\alpha g_{ij} = g_{ij}\Delta\rho - (n-2)\nabla_i\rho_j. \tag{4.13}$$

Multiplying (4.13) by g^{ij} and using (3.7) we obtain

$$\alpha = (2/n)R\rho. \tag{4.14}$$

Substitution of (4.14), (3.7) in (4.13) thus gives (1.17), and hence Theorem C completes the proof of our theorem.

Proof of Theorem III. It is exactly the same as that of Theorem II for condition (1.10) except that the application of Theorem C should be replaced by that of Theorem B.

Proof of Theorem IV. Without loss of generality we may assume our manifold M^n to be orientable as otherwise we need only to take an orientable twofold covering space of M^n. On the manifold M^n consider the covariant tensor field T of order $2(2a+b)$:

$$T_{h_1 i_1 j_1 k_1 \cdots h_a i_a j_a k_a u_1 v_1 \cdots u_b v_b} = \prod_{r=1}^{a} R_{h_r i_r j_r k_r} \prod_{s=1}^{b} R_{u_s v_s} - \frac{R^{a+b}}{n^{a+b}(n-1)^a} \prod_{r=1}^{a} (g_{i_r j_r} g_{h_r k_r} - g_{i_r k_r} g_{h_r j_r}) \prod_{s=1}^{b} g_{u_s v_s}. \tag{4.15}$$

From (4.15) an elementary calculation gives the length of T:

$$[2(2a+b)]!\langle T, T\rangle = A^a B^b - 2^a R^{2(a+b)}/n^{a+b}(n-1)^a. \tag{4.16}$$

Thus by condition (1.15), $L_v R=0$ and the extension of formula (4.3) to the tensor T we immediately obtain

$$L_v\langle T, T\rangle = 2\langle L_v T, T\rangle - 4(2a+b)\rho\langle T, T\rangle = 0, \tag{4.17}$$

which implies

$$(\langle L_v T, T\rangle, \rho) = 2(2a+b)(\rho\langle T, T\rangle, \rho). \tag{4.18}$$

On the other hand, from (2.3), (2.4) we obtain

$$\begin{aligned} L_v T_{h_1 i_1 j_1 k_1 \cdots h_a i_a j_a k_a u_1 v_1 \cdots u_b v_b} &= 2a\rho \prod_{r=1}^{a} R_{h_r i_r j_r k_r} \prod_{s=1}^{b} R_{u_s v_s} \\ &\quad - \sum_{r=1}^{a} [R_{h_1 i_1 j_1 k_1} \cdots R_{h_{r-1} i_{r-1} j_{r-1} k_{r-1}} \\ &\qquad \cdot (g_{h_r k_r} \nabla_{i_r} \nabla_{j_r} \rho - g_{h_r j_r} \nabla_{i_r} \nabla_{k_r} \rho + g_{i_r j_r} \nabla_{k_r} \nabla_{h_r} \rho - g_{i_r k_r} \nabla_{j_r} \nabla_{h_r} \rho) \\ &\qquad \cdot R_{h_{r+1} i_{r+1} j_{r+1} k_{r+1}} \cdots R_{h_a i_a j_a k_a}] \prod_{s=1}^{b} R_{u_s v_s} \\ &\quad + \prod_{r=1}^{a} R_{h_r i_r j_r k_r} \sum_{s=1}^{b} \{R_{u_1 v_1} \cdots R_{u_{s-1} v_{s-1}} \\ &\qquad \cdot [g_{u_s v_s} \Delta\rho - (n-2)\nabla_{v_s} \nabla_{u_s} \rho] \cdot R_{u_{s+1} v_{s+1}} \cdots R_{u_b v_b}\} \\ &\quad - \frac{2(2a+b)R^{a+b}}{n^{a+b}(n-1)^a} \rho \prod_{r=1}^{a} (g_{i_r j_r} g_{h_r k_r} - g_{i_r k_r} g_{h_r j_r}) \prod_{s=1}^{b} g_{u_s v_s}. \end{aligned} \tag{4.19}$$

By means of (4.15), (4.19), (3.7), (1.16), (1.8), (4.16) an elementary calculation yields

$$\begin{aligned} (\langle L_v T, T\rangle, \rho) &= 2a(\rho\langle T, T\rangle, \rho) \\ &\quad - \frac{A^a B^b}{[2(2a+b)]!} \left(\frac{4a}{A} + \frac{(n-2)b}{B}\right) \left(R^{jk} \nabla_j \nabla_k \rho + \frac{R^2 \rho}{n(n-1)}, \rho\right). \end{aligned} \tag{4.20}$$

By comparing (4.18) and (4.20), noticing that $\rho \neq 0$, and making use of Lemma 3.2, we thus have

$$T_{h_1 i_1 j_1 k_1 \cdots h_a i_a j_a k_a u_1 v_1 \cdots u_b v_b} = 0. \tag{4.21}$$

Multiplying (4.21) by

$$g^{h_1 k_1} \prod_{r=2}^{a} g^{h_r k_r} g^{i_r j_r} \prod_{s=1}^{b} g^{u_s v_s},$$

and using (4.15) we obtain $R_{i_1 j_1} = R g_{i_1 j_1}/n$, which implies M^n is an Einstein space. Hence, by Theorem A, M^n is isometric to an n-sphere, and our theorem is proved.

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Lehigh University, Bethlehem, Pennsylvania 18015