TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 163, January 1972

A CHARACTERIZATION OF ODD ORDER EXTENSIONS OF THE FINITE PROJECTIVE SYMPLECTIC GROUPS PSp (4, q)

BY
MORTON E. HARRIS(1)

Abstract. In a recent paper, W. J. Wong characterized the finite projective symplectic groups PSp (4, q) where q is a power of an odd prime integer by the structure of the centralizer of an involution in the center of a Sylow 2-subgroup of PSp (4, q). In the present paper, finite groups which contain an involution in the center of a Sylow 2-subgroup whose centralizer has a more general structure than in the PSp (4, q) case are classified by showing them to be odd ordered extensions of PSp (4, q).

Introduction and statement. Let p denote an odd prime integer and let $q=p^f$ where f is a positive integer. Let PSp (4, q) denote the projective symplectic group in dimension 4 over a field F_q of q elements. The group PSp (4, q) is described in [8, §1], is simple of order $\frac{1}{2}q^4(q^2+1)(q^2-1)^2$ and has a Sylow 2-subgroup with center of order 2 so that involutions which lie in the centers of Sylow 2-subgroups form a single conjugacy class.

Let $\bar{\sigma}$ denote an automorphism of F_q. Then $\bar{\sigma}$ induces, in the natural way, an automorphism of PSp (4, q) (cf. [8, §1]) and centralizes an involution in the center of a Sylow 2-subgroup of PSp (4, q). In fact $\langle\bar{\sigma}\rangle$, the cyclic subgroup of Aut (F_q) generated by $\bar{\sigma}$, acts faithfully on PSp (4, q) and one may form the natural semidirect product $\langle\bar{\sigma}\rangle$ PSp (4, q). If $\bar{\sigma}$ is an odd ordered automorphism of F_q, then $\langle\bar{\sigma}\rangle$ PSp (4, q) is an odd ordered extension of PSp (4, q) with trivial 2-core. In fact, any odd ordered extension of PSp (4, q) with trivial 2-core is of this form (cf. [1]).

Let t be an involution in the center of a Sylow 2-subgroup of PSp (4, q) such that t is centralized by $\bar{\sigma}$. Then the centralizer $C(t)$ of t in $\langle\bar{\sigma}\rangle$ PSp (4, q) is a semidirect product $\langle\bar{\sigma}\rangle\mathscr{C}$ where $\mathscr{C}$ denotes the centralizer of t in PSp (4, q) and $C(t)$ has trivial 2-core.

Finite groups G containing an involution t such that $C_G(t)\simeq\mathscr{C}$ have been classified in [8]. However, for example in classifying groups by their Sylow 2-subgroups, one may arrive at a situation in which the centralizer $C_G(t)$ of an involution t in a group

Received by the editors August 19, 1970 and, in revised form, February 15, 1971.

AMS 1970 *subject classifications*. Primary 20D05.

Key words and phrases. Centralizer of an involution, projective symplectic groups, simple group, odd ordered extension.

(1) This research was partially supported by National Science Foundation Grant GP-9584 at the University of Illinois at Chicago Circle.

G is an odd ordered extension of $\mathscr{C}$ with trivial 2-core. This is the case in the group $\langle\bar{\sigma}\rangle$ PSp $(4, q)$ of the above paragraph.

To handle this situation, we prove the following more general result:

THEOREM. *Let G be a finite group with an involution t such that*

(a) $O(C_G(t))=\{1\}$ *and*

(b) $C_G(t)$ *contains a normal subgroup* $\mathfrak{H}$ *of index* 2ρ *with* ρ *odd and an involution* $u\in C_G(t)-\mathfrak{H}$ *such that* $\mathfrak{H}$ *contains a subgroup* $L_1\cong \mathrm{SL}(2,q)$ *where* $q=p^f$, p *is an odd prime integer and* f *a positive integer such that if* $L_2=L_1^u$, *then* $\mathfrak{H}=L_1L_2$, $[L_1, L_2]=\{1\}$ *and* $L_1\cap L_2=\langle t\rangle$.

Then, either (i) $G=C_G(t)O(G)$ *or* (ii) $G\cong\langle\bar{\sigma}\rangle$ PSp $(4, q)$ *where* $\bar{\sigma}\in \mathrm{Aut}\,(F_q)$ *and is of order* ρ.

Note that when $\rho=1$, then $C_G(t)$ is isomorphic to the centralizer of an involution in the center of a Sylow 2-subgroup in PSp $(4, q)$ and the theorem follows from [8].

Thus for the rest of the paper, we assume that G is a finite group with an involution t such that $C_G(t)$ satisfies the hypotheses of the theorem with $\rho>1$. Let $q\equiv\delta \pmod 4$ where $\delta=\pm 1$ and let $q-\delta=2^ne$ with e an odd integer and $n\geqq 2$.

The paper is organized as follows. In §1, we study the structure of $C_G(t)$ and in §2, we show that if case (i) of the conclusion of the theorem does not hold, then G has exactly two conjugacy classes of involutions. In §3, we determine the structure of the centralizer of an involution in the second conjugacy class, and we show that the situation is exactly as in the group $\langle\bar{\sigma}\rangle$ PSp $(4, q)$. Moreover, we show, using a lemma of Thompson, that $|G|=\rho|\mathrm{PSp}\,(4, q)|$. Then in §4, we find the structure of the normalizer of a Sylow p-subgroup of G. In §5, we construct a normal subgroup G_0 of G such that $|G:G_0|=\rho$ and then we show that G_0 is isomorphic to PSp $(4, q)$ by applying [8]. The rest of the theorem now follows easily.

Our notation is fairly standard and tends to follow that of [5]. We use $O(X)$ to denote the 2-core of X, i.e., the largest normal subgroup of odd order in the finite group X. We write $[x, y]=x^{-1}x^y=x^{-1}y^{-1}xy$ and, if $x^y=z$, we write $y\colon x\to z$. If $y\colon x\to x^{-1}$, we say that y inverts x and, if $y\colon x\to z$ and $y\colon z\to x$, we write $y\colon x\leftrightarrow z$.

1. The structure of $C_G(t)$.

Since $C_G(t)/\mathfrak{H}$ is a group of order 2ρ we have

LEMMA 1.1. $C_G(t)$ *contains a subgroup* $\mathscr{A}$ *of index* 2 *such that* $\mathscr{A}\geqq\mathfrak{H}$.

LEMMA 1.2. $\{L_1, L_2\}$ *is invariant in* $C_G(t)$.

Proof. This follows easily from the Krull-Schmidt Theorem applied to the group $\mathfrak{H}/\langle t\rangle\cong \mathrm{PSL}\,(2, q)\times \mathrm{PSL}\,(2, q)$.

COROLLARY 1.2.1. $N_G(L_1)=N_G(L_2)=\mathscr{A}$.

Proof. Because $Z(L_1)=\langle t\rangle$, we have $N_G(L_1)\leqq C_G(t)$. Also $\mathfrak{H}\leqq N_G(L_1)\cap N_G(L_1)$ and $|\mathscr{A}/\mathfrak{H}|=\rho$ is odd. Now we can conclude that $\mathscr{A}\leqq N_G(L_1)$ from Lemma 1.2. Hence $N_G(L_1)=\mathscr{A}$. Similarly $N_G(L_2)=\mathscr{A}$.

LEMMA 1.3. $C_G(\mathfrak{H})=\langle t\rangle$.

Proof. Since $C_G(\mathfrak{H})\leqq\mathscr{A}$ and $Z(\mathfrak{H})=\langle t\rangle$, $C_G(\mathfrak{H})$ has order $2\cdot r$ where $r|\rho$. But $C_G(\mathfrak{H})\lhd C_G(t)$ and $O(C_G(t))=\{1\}$; hence $r=1$.

LEMMA 1.4. *There exists a subgroup A of $\mathscr{A}$ of order ρ such that $\mathscr{A}=\mathfrak{H}A$ and $A\cap\mathfrak{H}=\{1\}$ and such that A centralizes a Sylow 2-subgroup of $\mathfrak{H}$. Moreover if, $f=2^a$, $a\geqq 0$, then $\rho=1$ and the result follows from* [8].

Proof. Since $C_{\mathfrak{H}}(L_1)=L_2\lhd C_{\mathscr{A}}(L_1)\lhd\mathscr{A}$ and $|\mathscr{A}:L_1C_{\mathscr{A}}(L_1)|$ divides ρ, it follows from the structure of Aut (L_1) that there exists a subgroup A_1 of $\mathscr{A}$ such that $A_1\geqq C_{\mathscr{A}}(L_1)$, $\mathscr{A}=L_1A_1$, and $L_1\cap A_1\leqq C_{\mathscr{A}}(L_1)$. Since $|A_1:C_{\mathscr{A}}(L_1)|$ is odd, A_1 centralizes a Sylow 2-subgroup S_1 of L_1. Thus $\langle t\rangle\leqq L_1\cap A_1\leqq C_{\mathscr{A}}(L_1)\cap L_1=\langle t\rangle$ and $\mathfrak{H}\cap A_1=L_2\lhd A_1$. Hence $|A_1/L_2|$ is odd. Again, it follows that there exists a subgroup A_2 of A_1 such that $A_2\geqq C_{A_1}(L_2)$, $A_1=L_2A_2$ and $L_2\cap A_2\leqq C_{A_1}(L_2)$ and such that A_2 centralizes a Sylow 2-subgroup S_2 of L_2. Thus $\langle t\rangle\leqq A_2\cap\mathfrak{H}=A_2\cap A_1\cap\mathfrak{H}=A_2\cap L_2\leqq L_2\cap C_{A_1}(L_2)=\langle t\rangle$. Hence $\langle t\rangle$ is the Sylow 2-subgroup of A_2, and A_2 has a subgroup A of index 2. Then $\mathfrak{H}A=L_1L_2A=L_1L_2A_2=L_1A_1=\mathscr{A}$, $\mathfrak{H}\cap A=A\cap\mathfrak{H}\cap A_2=A\cap\langle t\rangle=\{1\}$, $|A|=\rho$ and A centralizes S_1S_2 which is a Sylow 2-subgroup of $\mathfrak{H}$. Finally, if $f=2^a$, $a\geqq 0$ then

$$|\mathscr{A}:L_1C_{\mathscr{A}}(L_1)|\quad\text{and}\quad|A_1/L_2C_{A_1}(L_2)|$$

are powers of 2. Thus $A_1=C_{\mathscr{A}}(L_1)$, $A_2=C_{A_1}(L_2)$ and $A_2\leqq C_G(\mathfrak{H})=\langle t\rangle$ and, therefore, $|A|=1=\rho$.

Note that we may now assume that $q>3$ and, hence, that $L_i/\langle t\rangle$ is simple for $i=1, 2$. We may also assume that $e>1$, for if $e=1$ then $q-\delta=2^n$ and [6, Lemma 19.3] yields that either q is a prime or $q=3^2$ both of which have been excluded.

LEMMA 1.5. *A is Abelian with at most* 2 *generators.*

Proof. Consider the natural homomorphism

$$\theta: A\to(\text{Aut}\,(L_1)/\text{Inn}\,(L_1))\times(\text{Aut}\,(L_2)/\text{Inn}\,(L_2)).$$

Suppose that $a\in\text{Ker}\,(\theta)$. Then there is an $l_i\in L_i$ such that $al_i\in C_G(L_i)$ for $i=1, 2$. Hence $al_1l_2\in C_G(\mathfrak{H})=\langle t\rangle$ and $a\in A\cap\mathfrak{H}=\{1\}$. Thus θ is one-to-one and since $|A|=\rho$ is odd, the lemma follows from the structures of Aut (L_1) and Aut (L_2).

LEMMA 1.6. *The subgroup A may be chosen so that it is normalized by* u. *If A is so chosen, there exists a Sylow 2-subgroup S_1 of L_1 centralized by A and hence S_1^u is a Sylow 2-subgroup of L_2 centralized by A.*

Proof. Let S_1 be a Sylow 2-subgroup of L_1 centralized by A. Since A centralizes a Sylow 2-subgroup of L_2 also, by conjugating A by an element of L_2, we may assume that A centralizes $S_1S_1^u$. Since $C_G(S_1S_1^u)=\langle t\rangle\times A$ and $u\in N_G(S_1S_1^u)$, u normalizes $O(C_G(S_1S_1^u))=A$.

Henceforth, we assume that A is normalized by u.

COROLLARY 1.6.1. *$A = C_A(u) \times [A, u]$ where $[A, u]$ is the subgroup of A formed by the elements of A inverted by u together with the identity element.*

Let

$$B = \{x_1 x_1^u \mid x_1 \in L_1\}. \tag{1}$$

Then B is a subgroup of $C_G(t, u)$ and B is isomorphic to PSL $(2, q)$ where the isomorphism is induced by the epimorphism $\theta: L_1 \to B$ defined by $\theta(x_1) = x_1 x_1^u$ for $x_1 \in L_1$ since Ker $(\theta) = \langle t \rangle$.

An easy calculation gives

LEMMA 1.7. *$C_G(u, t) = (B \times \langle t, u \rangle) C_A(u)$ where $\mathfrak{H} \cap C_G(u, t) = (B \times \langle t \rangle) \lhd C_G(u, t)$.*

COROLLARY 1.7.1. *B is characteristic in $C_G(u, t)$.*

Proof. Since $q > 3$, B is simple and non-Abelian, thus $C_G(u, t)'' = B$.

2. **Classes of involutions in G.** From the structure of $\mathscr{H}$, we have

LEMMA 2.1. *All involutions of $\mathfrak{H} - \langle t \rangle = L_1 L_2 - \langle t \rangle$ are conjugate in $\mathfrak{H} = L_1 L_2$.*

An easy consequence of the fact that u normalizes A is

LEMMA 2.2. *All involutions of $C_G(t) - \mathfrak{H}$ are conjugate in $C_G(t)$ to u or tu.*

The proof of [2, (2B)] yields

LEMMA 2.3. *If H is a subgroup of G and if T is a Sylow 2-subgroup of $H \cap C_G(t)$ such that $\langle t \rangle$ is characteristic in T, then T is a Sylow 2-subgroup of H. In particular, a Sylow 2-subgroup of $C_G(t)$ is a Sylow 2-subgroup of G.*

Let $S_1 = \langle a_1, b_1 \rangle$ where $a_1^{2^{n-1}} = b_1^2 = t$ and $a_1 b_1 = b_1 a_1^{-1}$ be a Sylow 2-subgroup of L_1 centralized by A. Set $b_2 = b_1^u$, $a_2 = a_1^u$. Then $v = (a_1 a_2)^{2^{n-2}}$ and $w = b_1 b_2$ are two involutions in $\mathfrak{H} - \langle t \rangle$ centralized by u. Set $S = S_1 S_1^u \langle u \rangle = \langle a_1, b_1, a_2, b_2, u \rangle$. Then S is a Sylow 2-subgroup of G. Note that $Z(S) = \langle t \rangle$ so that the conjugacy class of t in G consists exactly of those involutions which are in the center of some Sylow 2-subgroup of G.

LEMMA 2.4. *The involutions of $\mathfrak{H} - \langle t \rangle$ are not conjugate in G to t.*

Proof. It suffices to show that v is not conjugate in G to t. Since $C_{L_1}(a_1^{2^{n-2}}) = \langle d_1 \rangle$ which is cyclic of order $q - \delta$, we may assume that $d_1^e = a_1$. Setting $d_2 = d_1^u$, we have

$$C_G(t, v) = \langle d_1, d_2, w \rangle A \cdot \langle u \rangle \tag{2}$$

which is of order $2(q-\delta)^2 \rho$. Also A normalizes $\langle d_1 \rangle$ and $\langle d_2 \rangle$ and $d_i^w = d_i^{-1}$ for $i = 1, 2$. A Sylow 2-subgroup of $C_G(t, v)$ is $T = \langle a_1, a_2, w, u \rangle$ which is of order 2^{2n+1}. The argument of [8, Lemma 2.2] applies with only a slight modification required if $n = 2$.

We now assume

$$G \neq C_G(t)O(G). \tag{3}$$

The proof of [8, Lemma 2.3] yields

LEMMA 2.5. *Either t or tu is conjugate in G to t.*

Since $C_G(t)$ has an automorphism which interchanges u and tu and fixes $\mathfrak{H}A$, we may assume

$$tu \text{ is conjugate in } G \text{ to } t. \tag{4}$$

LEMMA 2.6. *G has exactly two conjugacy classes of involutions K_1 and K_2 such that $K_1 \cap C_G(t)$ consists of the conjugacy classes in $C_G(t)$ represented by t and tu and $K_2 \cap C_G(t)$ consists of the conjugacy classes in $C_G(t)$ represented by v and u. There exists an element $z \in N_G(C_S(tu))$ such that $z^2 \in C_S(tu)$ and*

$$z\colon t \to uv, \qquad u \to tv, \qquad v \to v. \tag{5}$$

Also z normalizes $C_A(u)$.

Proof. Let K_1 be the conjugacy class of t in G and let K_2 be the conjugacy class of v in G; clearly $K_1 \neq K_2$ by Lemma 2.4. Set

$$E = C_S(tu) = \langle a_1a_2, w\rangle \times \langle t, u\rangle. \tag{6}$$

Then $C_G(E)=Z(E)\times C_A(u)$ by Lemma 1.7 and the argument of [8, Lemma 2.4] applies with only a slight modification required if $n=2$.

The discussion at the end of [8, §2] yields

LEMMA 2.7. *The focal subgroup of S in G is S and hence G has no subgroup of index* 2.

3. Centralizers of involutions in K_2.

Clearly

$$C_G(t, v) = \langle d_1, d_2, w\rangle A\langle tu\rangle. \tag{7}$$

Transforming by the element z of Lemma 2.6, we obtain

$$C_G(u, v) = \langle d_1^z, d_2^z, w^z\rangle A^z\langle tu\rangle. \tag{8}$$

LEMMA 3.1. *$\langle d_1d_2\rangle^z=\langle d_1d_2\rangle$, $\langle a_1a_2\rangle^z=\langle a_1a_2\rangle$ and*

$$w = (w(d_1d_2)^m tu)^z \quad \textit{for some integer } m. \tag{9}$$

Proof. Since $C_G(t, u, v)=(\langle t, u\rangle \times \langle d_1d_2, w\rangle)C_A(u)$ by Lemma 1.7 and since $\langle d_1d_2, w\rangle$ is dihedral of order $q-\delta$, we have

$$O^{2'}(C_G(t, u, v)) = \langle t, u\rangle \times \langle d_1d_2, w\rangle. \tag{10}$$

Since $z \in N_G(\langle t, u, v\rangle)$, we obtain

$$O^{2'}(C_G(t, u, v)) = \langle t, u\rangle \times \langle (d_1d_2)^z, w^z\rangle. \tag{11}$$

Clearly the proof of [8, Lemma 3.1] applies with $C_G(t, u, v)$ replaced by

$$O^{2'}(C_G(t, u, v)).$$

Since $C_G(u, v) = \langle d_1^z, d_2^z, w^z \rangle A^z \langle tu \rangle$ by (8), $\langle tu, a_1^z, a_2^z \rangle$ is a subgroup of $C_G(u, v)$ of order 2^{2n}. Also w centralizes tu and, by (9), we have $(a_1^z)^w = (a_2^z)^{-1}$, $(a_2^z)^w = (a_1^z)^{-1}$. Thus $C_G(u, v)$ contains the subgroup $T = \langle tu, a_1^z, a_2^z, w \rangle$ of order 2^{2n+1}. Since u does not lie in the center of a Sylow 2-subgroup of G, T is a Sylow 2-subgroup of $C_G(u)$, $C_G(v)$ and $C_G(u, v)$.

LEMMA 3.2. *$C_G(u)$ has a normal subgroup K of index 2 with Sylow 2-subgroup $M = \langle a_1^z, a_2^z, w \rangle$. Also $C_G(u) = K\langle t \rangle$ where $K \cap \langle t \rangle = \{1\}$.*

Proof. The proof of [8, Lemma 3.2] applies to demonstrate the existence of K. Since $u = (tv)^z = (a_1^{2^{n-1}})^z((a_1a_2)^{2^{n-2}})^z \in M$ and $t \notin K$, the result follows.

LEMMA 3.3. *K has a normal subgroup L of index 2^n with Sylow 2-subgroup $J = \langle a_1a_2, w \rangle$.*

Proof. Clearly $M' = \langle (a_1a_2)^z \rangle = \langle a_1a_2 \rangle$ and so $N_K(M) \leqq C_K(v) \leqq C_G(u, v) = \langle d_1^z, d_2^z, w^z \rangle A^z \langle tu \rangle$ which has $\langle (d_1^{2^n})^z, (d_2^{2^n})^z \rangle A^z$ as a normal 2-complement. Hence $N_K(M)$ has a normal 2-complement so that $N_K(M)' \cap M = M'$. With only a slight modification, the proof of [8, Lemma 3.4] yields the result.

Since $u \notin L$, we have

LEMMA 3.4. *If k is any integer and $s = a_1^z w(d_1d_2)^k$, then $s^{2^{n-1}} = u$, the order of s is 2^n, $s^t = s^{-1}$ and hence $\langle t, s \rangle$ is dihedral of order 2^{n+1}. Also $K = L\langle s \rangle$ and $L \cap \langle s \rangle = \{1\}$.*

LEMMA 3.5. (a) *For $i = 1$ or 2, if $x \in N_G(L_i) = \mathscr{A}$ and x acts trivially on $L_i/\langle t \rangle$ then x centralizes L_i.*

(b) *If $x \in C_G(B) \cap \mathscr{A}$, then $x \in \langle t \rangle$.*

Proof. Let x satisfy the hypotheses of (a) and let $y \in L_i \cong \mathrm{SL}(2, q)$, y of odd order. Clearly x centralizes y^2 and hence $[x, y] = 1$. But $L_i = \mathrm{SL}(2, q)$ is generated by its elements of odd order and (a) follows. Let x satisfy the hypothesis of (b) and let $y_1 \in L_1$. Then $(y_1 y_1^u)^x = y_1 y_1^u = y_1^x y_1^{ux}$, so that $y_1^{-1} y_1^x \in L_1 \cap L_2 = \langle t \rangle$. Hence x acts trivially on $L_1/\langle t \rangle$ and we conclude from (a) that x centralizes L_1. Similarly x centralizes L_2. Thus $x \in \langle t \rangle$ by Lemma 1.3.

We shall need the following well-known result:

LEMMA 3.6. *Let H be a group with $O(H) = \{1\}$ and such that H contains a normal subgroup $M \cong \mathrm{PSL}(2, r)$ with r a prime power. Suppose also that $4 \nmid |H:M|$. Then H/M is Abelian.*

LEMMA 3.7. (a) *u centralizes A and hence $A \leqq L$.*

(b) *B is characteristic in L.*

(c) *$L = (B \times O(L))A$ and $(B \times O(L)) \cap A = \{1\}$.*

Proof. $A^z \leqq L$ and $B \leqq L$ since B is simple. Also $B \cap O(L)=\{1\}$ and $z \in N_G(C_A(u))$. Moreover, $C_G(t, u)=(B\times\langle t, u\rangle)C_A(u)$ and $\mathfrak{H} \cap C_G(t, u)=B\times\langle t\rangle$ so that $C_A(u)$ normalizes B. If $a \in C_A(u)$ centralizes B, then $a \in \langle t\rangle$ by Lemma 3.5(b). Hence $a=1$ and $C_A(u)$ acts faithfully on B. But $[O(L) \cap C_A(u), B] \leqq B \cap O(L)=\{1\}$. Thus $O(L) \cap C_A(u)=\{1\}$. By a theorem of Gorenstein and Walter [4, Theorem 1], since J is dihedral and $B \cong BO(L)/O(L)$ is an odd indexed subgroup of $L/O(L)$, we have two cases:

(a) $L/O(L)$ is isomorphic to the alternating group $\mathscr{A}_7$. Hence B is isomorphic to a subgroup of $\mathscr{A}_7$ which implies that $q=5$, 7 or 9. These possibilities have been excluded.

(b) $L/O(L)$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}(2, r)$ containing $\mathrm{PSL}(2, r)$ with r odd. Clearly B must be isomorphic to a subgroup of $\mathrm{PSL}(2, r)$ so that $q=5$ (which is excluded) or r is a power of q. Let M be the subgroup of L such that $L \trianglerighteq M > O(L)$ and $M/O(L) \cong \mathrm{PSL}(2, r)$. Thus $BO(L) \leqq M$ and $|L:M|$ is odd. Now $C_G(u, v)=\langle d_1^z, d_2^z, w^z\rangle A^z\langle tu\rangle \geqq \langle s, t\rangle$. But $C_G(u)=L\langle s, t\rangle$; hence $C_G(u, v)=C_L(v)\langle s, t\rangle$. Clearly $C_L(v) \geqq \langle d_1^{2^n z}, d_1 d_2, w\rangle \cdot A^z$ which is a product of two groups. But $|C_G(t, v)| =|C_G(u, v)|=2(q-\delta)^2\cdot\rho$ so that $|C_L(v)|=2^n e^2\rho$ (for $L \cap \langle s, t\rangle=\{1\}$). Now $\langle d_1^{2^n z}, d_1 d_2, w\rangle$ has a normal 2-complement, namely, $\langle d_1^{2^n z}, (d_1 d_2)^{2^{n-1}}\rangle$ and

$$\begin{aligned}\langle d_1^{2^n z}, (d_1 d_2)^{2^{n-1}}\rangle \cap A^z &= \langle d_1^{2^n z}(d_1 d_2)^{2^{n-1} z}\rangle \cap A^z \\ &= (\langle d_1^{2^n}, (d_1 d_2)^{2^{n-1}}\rangle \cap A)^z = \{1\}.\end{aligned}$$

Thus $|\langle d_1^{2^n z}, d_1 d_2, w\rangle\cdot A^z|=2^n e^2\rho=|C_L(v)|$ and so $C_L(v)=\langle d_1^{2^n z}, d_1 d_2, w\rangle A^z$.

Clearly $L=O^2(C_G(u))$.

Now $M/O(L)=(L/O(L))'$ implies that $M=L'O(L) \lhd C_G(u)$. Moreover $L \cap \langle t\rangle =\{1\}$ and $L\langle t\rangle$ is a subgroup of $C_G(u)$ such that $O(L\langle t\rangle)=O(L)$ and $L\langle t\rangle/O(L)$ contains a normal subgroup $M/O(L)$ of index twice an odd integer. Now Lemma 3.6 applies, and we conclude that $L\langle t\rangle/M$ is Abelian. Hence $[d_1^{2^n z}, u^z]=[d_1^{2^n z}, tv] =d_1^{-2^n z}d_2^{2^n z}=(d_1^{-1}d_2)^{2^n z} \in M$; but $(d_1^{-1}d_2)^{2^{n-1}}$ has order e, therefore $(d_1^{-1}d_2)^{2^{n-1} z} \in M$. Suppose that $w=w^z(d_1 d_2)^{mz}tu \in \mathfrak{H}^z A^z$. Then $w \in \mathfrak{H}^z$, $tu \in \mathfrak{H}^z$ and $tu \in \mathfrak{H}$ which is impossible. Thus $C_L(v) \cap \mathfrak{H}^z A^z=\langle d_1^{2^n z}, d_1 d_2\rangle A^z$ and $C_L(v) \cap \mathfrak{H}^z=\langle d_1^{2^n z} d_1 d_2\rangle =\langle (d_1 d_2^{-1})^{2^{n-1} z}, d_1 d_2\rangle$ is normalized by A^z. Also

$$C_M(v) = \langle (d_1 d_2^{-1})^{2^{n-1} z}, d_1 d_2, w\rangle(A^z \cap M)$$

is normalized by A^z and $C_M(v, w) \geqq \langle (d_1 d_2^{-1})^{2^{n-1} z}\rangle \times \langle v, w\rangle$. However,

$$C_{M/O(L)}(\langle v, w\rangle O(L)) = \langle v, w\rangle O(L)/O(L)$$

since $M/O(L)=\mathrm{PSL}(2, r)$. Thus $(d_1 d_2^{-1})^{2^{n-1} z} \in O(L)$ and A^z normalizes $C_L(v) \cap \mathfrak{H}^z \cap O(L)=\langle (d_1 d_2^{-1})^{2^{n-1} z}\rangle$. Thus A normalizes $\langle (d_1 d_2^{-1})^{2^{n-1}}\rangle=\langle (d_1 d_2^{-1})^{2^n}\rangle$ which is a group of order $e>1$. In the proof of Lemma 2.4, we saw that A normalizes both $\langle d_1\rangle$ and $\langle d_2\rangle$. Hence A normalizes $\langle d_1^{2^n}\rangle$ and $\langle d_2^{2^n}\rangle$ which are groups of order $e>1$. Let $a \in [A, u]$ and let $((d_1 d_2^{-1})^{2^n})^a=(d_1 d_2^{-1})^{2^n\gamma}=d_1^{2^n\gamma}d_2^{-2^n\gamma}$ where γ is an integer relatively prime to e. Let $d_1^{2^n a}=d_1^{2^n\alpha}$ where α is an integer relatively prime to

e. Then $d_2^{2^n\alpha}=d_1^{2^nu\alpha}=d_1^{2^n\alpha^{-1}u}=d_1^{2^n\beta u}=d_2^{2^n\beta}$ where β is an integer relatively prime to e such that $\alpha\beta\equiv 1 \pmod e$ (since u inverts a). Hence $(d_1d_2^{-1})^{2^n\alpha}=d_1^{2^n\gamma}d_2^{-2^n\gamma}=d_1^{2^n\alpha}d_2^{-2^n\beta}$. Thus $\alpha\equiv\beta \pmod e$ and $\alpha^2\equiv 1 \pmod e$. Since a is of odd order, a centralizes $d_1^{2^n}$ and $d_2^{2^n}$. Consequently a centralizes $\langle a_1\rangle\times\langle d_1^{2^n}\rangle=\langle d_1\rangle$ and similarly for $\langle d_2\rangle$. But a subgroup of order $q-\delta$ contained in SL $(2, q)$ is self-centralizing in Aut (SL $(2, q)$); thus there exist elements $d_1'\in\langle d_1\rangle$ and $d_2'\in\langle d_2\rangle$ such that $d_1'd_2'a$ centralizes $L_1L_2=\mathfrak{H}$. Hence $d_1'd_2'a\in C_G(\mathfrak{H})=\langle t\rangle$ and $a\in\mathfrak{H}\cap A=\{1\}$. We have proved that $[A, u]=1$ and thus u centralizes A and z normalizes $A=C_A(u)$. Hence $C_L(v, w)=(\langle(d_1d_2^{-1})^{2^{n-1}z}\rangle\times\langle v, w\rangle)A$ has a normal 2-complement $\langle(d_1d_2^{-1})^{2^{n-1}z}\rangle A$ and $C_M(v, w)=(\langle(d_1d_2^{-1})^{2^{n-1}z}\rangle\times\langle v, w\rangle)(M\cap A)$. Again $C_{M/O(L)}(\langle v, w\rangle O(L))=\langle v, w\rangle O(L)$ and $M\cap A\leqq O(L)$ so that $M\cap A=\{1\}$. Consequently $C_M(v)=\langle(d_1d_2^{-1})^{2^{n-1}z}, d_1d_2, w\rangle$. Since $C_{M/O(L)}(vO(L))=C_M(v)O(L)/O(L)$, taking orders, we conclude that $r\pm 1=q-\delta$ (because $O(L)\cap\langle(d_1d_2^{-1})^{2^{n-1}z}, d_1d_2, w\rangle=\langle(d_1d_2^{-1})^{2^{n-1}z}\rangle$). But r is a power of q; thus $q=r$ and $M=BO(L)$. Since $|C_M(v)|=e(q-\delta)=d^22^n$, $|C_M(v)\cap O(L)|=e$ and w, vw are conjugate to v in B, we also have $|C_{O(L)}(w)|=|C_{O(L)}(w)|=e$. But $|O(L)\cap C_M(v, w)|=e$, and we may apply [3, Lemma 3] to conclude that $|O(L)|=e$. Thus $O(L)=\langle(d_1d_2^{-1})^{2^{n-1}z}\rangle$. Since $C_M(O(L))\lhd M$, $C_M(O(L))$ contains $\langle O(L), v\rangle$ and $M/O(L)$ is simple, we conclude that $C_M(O(L))=M$; thus $M=B\times O(L)$. We have $B=M'$ so that (b) follows. Also all of the involutions of B are conjugate in B and $|L/B|$ is odd. Hence $|C_L(v):C_M(v)|=|L:M|=|A|$ and $L=(B\times O(L))A$ where $(B\times O(L))\cap A=M\cap A=\{1\}$. This finishes the proof of the lemma.

COROLLARY 3.7.1. $O^{2'}(C_G(t))=\mathfrak{H}\langle u\rangle$.

Lemma 3.5(b) yields

LEMMA 3.8. $C_{AB}(B)=\{1\}$.

Consequently AB acts faithfully by conjugation on B, and we have an injection $\theta\colon AB=BA\to \text{Aut}(B)$ such that θ maps B onto Inn (B). Thus

$$|AB| = \rho|B| \mid |\text{Aut}(B)| = 2|B|f$$

so that ρ divides f. Moreover, the image of θ, Im (θ), is the unique subgroup of Aut (B) which contains Inn (B) as a subgroup of index ρ. (This follows from the structure of Aut (B) and the fact that ρ is odd.) Also $A\cong AB/B$ must be cyclic.

Let F_q denote the field of $q=p^f$ elements and let SL $(2, F_q)$ denote the multiplicative group of 2×2 matrices with coefficients in F_q of determinant 1. Fix an isomorphism

$$\phi_1\colon \text{SL}(2, F_q)\to L_1. \tag{12}$$

Let

$$\phi_2 = I_u\circ\phi_1\colon \text{SL}(2, F_q)\to L_2 \tag{13}$$

where I_u denotes conjugation by u. Clearly

$$\phi_1\left(\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right) = t = \phi_2\left(\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right).$$

Thus we have the obvious isomorphism

$$\phi\colon \mathrm{PSL}(2, F_q) = \mathrm{SL}(2, F_q)\Big/\left\langle\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right\rangle \to B \tag{14}$$

defined by:

$$\text{if } x \in \mathrm{SL}(2, F_q), \text{ then } \phi\left(x\left\langle\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right\rangle\right) = \phi_1(x)\phi_2(x).$$

Since $\rho | f$, we may let $f = \rho f_1$ where f_1 is a positive integer. Let $\bar\sigma$ denote the automorphism of F_q of order ρ defined by

$$\bar\sigma(x) = x^{p^{f_1}} \quad \text{for } x \in F_q.$$

Then $\bar\sigma$ induces an automorphism of order ρ of $\mathrm{SL}(2, F_q)$ and of $\mathrm{PSL}(2, F_q) = \mathrm{SL}(2, F_q)/\langle\left[\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right]\rangle$. Since the natural semidirect product $\langle\bar\sigma\rangle\,\mathrm{Inn}(\mathrm{PSL}(2, F_q))$ is the unique subgroup of $\mathrm{Aut}(\mathrm{PSL}(2, F_q))$ which contains $\mathrm{Inn}(\mathrm{PSL}(2, F_q))$ of index ρ, it follows from the above discussion of the subgroup AB that there exists an element $\sigma \in AB$ of order ρ such that $AB = \langle\sigma\rangle B$, $B \cap \langle\sigma\rangle = \{1\}$ and

$$\phi\left(x\left\langle\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right\rangle\right)^{\sigma} = \phi\left(x^{\bar\sigma}\left\langle\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right\rangle\right) \quad \text{for all } x \in \mathrm{SL}(2, F_q). \tag{15}$$

But $AB \leqq N_G(L_1) \cap N_G(L_2)$ so that $\phi_1(x)^\sigma\phi_2(x)^\sigma = \phi_1(x^{\bar\sigma})\phi_2(x^{\bar\sigma})$. Thus $\phi_1(x)^{-\sigma}\phi_1(x^{\bar\sigma}) \in L_1 \cap L_2 = \langle t\rangle$ for all $x \in \mathrm{SL}(2, F_q)$. It follows as in the proof of Lemma 3.5(a) that

$$\phi_i(x)^\sigma = \phi_i(x^{\bar\sigma}) \quad \text{for } i = 1, 2 \text{ and all } x \in \mathrm{SL}(2, F_q). \tag{16}$$

Moreover, $\mathfrak{H} \cap (AB) = B = \mathfrak{H} \cap (\langle\sigma\rangle B)$ implies that $\mathfrak{H} \cap \langle\sigma\rangle = 1$ and

$$(B \times O(L)) \cap (AB) = B = (B \times O(L)) \cap (\langle\sigma\rangle B)$$

implies that $(B \times O(L)) \cap \langle\sigma\rangle = \{1\}$. It follows that we may replace A by $\langle\sigma\rangle$, consequently from now on, we will assume

$$A = \langle\sigma\rangle. \tag{17}$$

If $\delta = 1$, let ε denote a generator of the multiplicative group $F_q^\times$ of F_q and set

$$d = \begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{18}$$

If $\delta = -1$, note that f is odd and $p \equiv -1 \pmod 4$ also. Let ε denote a generator of the multiplicative group $F_p^\times$ of F_p. Since $F_q^\times$ has no element of order 4, $\sqrt{\varepsilon} \notin F_q$ so that, using the norm from $F_{p^2}^\times \to F_p^\times$, there exist elements $\lambda, \mu \in F_p$ such that

$\lambda^2-\varepsilon\mu^2=-1$ and, using the norm from $F_{q^2}\to F_q$, there exist elements $\alpha, \beta\in F_q$ such that $\alpha+\beta\sqrt{\varepsilon}$ is a generator of the subgroup of elements of $F_{q^2}^{\times}$ of norm 1. Set

(19) $$d = \begin{bmatrix}\alpha & \beta\\ \varepsilon\beta & \alpha\end{bmatrix} \quad\text{and}\quad b = \begin{bmatrix}\lambda & \mu\\ -\varepsilon\mu & -\lambda\end{bmatrix}.$$

Thus $b, d\in \mathrm{SL}(2, F_q)$ and it is easy to see that $\bar\sigma$ centralizes b in both cases and $d^{\bar\sigma}=d^{p^{f_1}}$ if $\delta=1$ and $d^{\bar\sigma}=d^{-p^{f_1}}$ if $\delta=-1$. Moreover, $b^2=-I$ and b inverts d. By (16), we have $\phi_i(b)^\sigma=\phi_i(b)$ for $i=1, 2$ and $\phi_i(d)^\sigma=\phi_i(d)^{\delta p^{f_1}}$ for $i=1, 2$. Hence $\phi_i(d^e)^\sigma=\phi_i(d)^{\delta e p^{f_1}}$ and since $p^{f_1}\equiv\delta \pmod{2^n}$ we have $\phi_i(d^e)^\sigma=(\phi_i(d)^e)^\sigma=\phi_i(d)^e$. Now note that we may assume

(20) $$b_i = \phi_i(b),\quad a_i = \phi_i(d)^e,\quad d_i = \phi_i(d) \qquad \text{for } i = 1, 2.$$

LEMMA 3.9. *For a suitable integer k and $s=a_1^z w(d_1d_2)^k$, we have*

(21) $$C_G(u) = L\langle s, t\rangle = ((B\times O(L))\langle\sigma\rangle)\langle s, t\rangle$$

where $B\cong \mathrm{PSL}(2, q)$, $O(L)$ is cyclic of order e, and $\langle t, s\rangle$ is dihedral of order 2^{n+1} where $t^2=s^{2^n}=1$, $s^t=s^{-1}$ and $s^{2^{n-1}}=u$. Also B, $O(L)$, $L=(B\times O(L))\langle\sigma\rangle$ and $(B\times O(L))\langle s, t\rangle$ are normal in $C_G(u)$ and $O^{2'}(C_G(u))=(B\times O(L))\langle s, t\rangle$. The involution t centralizes B and inverts $O(L)$; the element s centralizes $O(L)$ and for any

$$x\in \mathrm{SL}(2, F_q)$$

we have

(22) $$\phi\left(x\left\langle\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right\rangle\right)^s = \phi\left(x^{\bar s}\left\langle\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right\rangle\right)$$

where $\bar s=\left[\begin{smallmatrix}0 & \varepsilon\\ -1 & 0\end{smallmatrix}\right]$ $(\delta=1)$ or $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ $(\delta=-1)$. (Here ε denotes a generator of the multiplicative group of F_q.) Also, if $\delta=-1$, then s and σ commute, and if $\delta=1$, then $(sw)^e$ and σ commute and $[\sigma, s]\in\langle(d_1d_2)^{2^n}\rangle$.

Proof. We have already seen that $L=O^2(C_G(u))=(B\times O(L))\langle\sigma\rangle$, that $B\cong \mathrm{PSL}(2, q)$ is characteristic in L, and that $\langle t, s\rangle$ has the properties of the lemma for every integer k. Also, t centralizes B, s centralizes $O(L)$, $s^2=[t, s]$ so that s^2 centralizes B, s inverts $\langle(d_1d_2)^z\rangle=\langle d_1d_2\rangle$ and $w^s=w(d_1d_2)^{2k}(a_1a_2)^z$ for every integer k. The argument of [8, Lemma 3.7] now applies to yield (22). Since $\delta p^{f_1}\equiv 1 \pmod{2^n}$, let $\delta p^{f_1}=1+m2^n$. Then $\sigma^{-1}(d_1d_2)\sigma=(d_1d_2)^{\delta p^{f_1}}$ so that $[d_1d_2, \sigma]=(d_1d_2)^{m2^n}$. Since z normalizes $A=\langle\sigma\rangle$ and a_1, w centralize σ, we have $\sigma^s=\sigma(d_1d_2)^k=\sigma(d_1d_2)^{-mk2^n}$ and $\sigma^{s^2}=\sigma$ since s inverts (d_1d_2). Thus $(B\times O(L))\langle s, t\rangle\lhd C_G(u)$. But t inverts $O(L)$ and B is simple, hence, $(B\times O(L))\langle s, t\rangle\leqq O^{2'}(C_G(u))$; therefore,

$$O^{2'}(C_G(u)) = (B\times O(L))\langle s, t\rangle.$$

Moreover, when $\delta=-1$, $[s, \sigma]\in C_G(B)\cap B=\{1\}$ so that $[s, \sigma]=1$. When $\delta=1$, then $\sigma^{sw}=\sigma(d_1d_2)^{mk2^n}$ so that $\sigma^{(sw)^e}=\sigma(d_1d_2)^{mke2^n}=\sigma$, finishing the proof.

A well-known lemma of John Thompson is

LEMMA 3.10. *Let G be a finite group with more than one class of involutions. Let α, β be two nonconjugate involutions and let R consist of a set of representatives for the conjugacy classes of involutions of G. For $t \in R$, let $a(\alpha, \beta, t)$ denote the number of ordered pairs of involutions (σ, λ) where $\sigma, \lambda \in C_G(t)$ such that σ is conjugate in G to α, λ is conjugate in G to β, and such that $t \in \langle(\sigma\lambda)\rangle$. Then*

$$|G| = |C_G(\alpha)|\,|C_G(\beta)| \sum_{t\in R} \frac{a(\alpha, \beta, t)}{|C_G(t)|}.$$

We can now prove

LEMMA 3.11. $|G| = |\mathrm{PSp}\,(4, q)|\rho = \frac{1}{2}q^4(q^2+1)(q^2-1)\rho$.

Proof. All involutions of $C_G(t)$ and $C_G(u)$ are contained in $O^{2'}(C_G(t)) = (L_1L_2)\langle u\rangle$ and $O^{2'}(C_G(u)) = (B \times O(L))\langle s, t\rangle$ respectively. Moreover, the structures of $O^{2'}(C_G(t))$ and $O^{2'}(C_G(u))$ are completely determined.

It is easy to see that the G-fusion of involutions in both $C_G(u)$ and $C_G(t)$ is determined. Since the semidirect product $H = \langle\bar{\sigma}\rangle\,\mathrm{PSp}\,(4, q)$ satisfies the hypotheses of the theorem, it follows from the above discussion and Lemmas 2.6 and 3.10 that $|G| = |H| = \rho|\mathrm{PSp}\,(4, q)| = \frac{1}{2}q^4(q^2+1)(q^2-1)\rho$.

4. **The p-structure of G.** For $\alpha \in F_q$, set

$$\theta(\alpha) = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}. \tag{23}$$

Then $\{\theta(\alpha) \mid \alpha \in F_q\}$ is a Sylow p-subgroup of $\mathrm{SL}\,(2, F_q)$. Set

$$\theta_i = \phi_i \circ \theta \colon F_q \to L_i \tag{24}$$

and

$$P_i = \mathrm{Image}\,(\theta_i) \quad \text{for } i = 1, 2. \tag{25}$$

Then $\theta_i \colon F_q \to P_i$ is an isomorphism of the additive group of F_q onto P_i which is a Sylow p-subgroup of L_i for $i=1, 2$. Thus P_i is elementary Abelian of order q and is normalized by $A = \langle\sigma\rangle$. The subgroup

$$R = P_1P_2 = P_1 \times P_2 \tag{26}$$

is a Sylow p-subgroup of $\mathfrak{H}$. The subgroup

$$D_1 = \{\theta_1(\alpha)\theta_2(\alpha) \mid \alpha \in F_q\} \tag{27}$$

is a Sylow subgroup of B and is elementary Abelian of order q. Set

$$D_2 = \{\theta_1(\alpha)\theta_2(-\alpha) \mid \alpha \in F_q\}. \tag{28}$$

Then

$$R = P_1 \times D_1 = D_1 \times D_2. \tag{29}$$

Put

$$h = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} \quad \text{and} \quad h_i = \phi_i(h) \quad \text{for } i = 1, 2, \tag{30}$$

where ε is a generator of the multiplicative group of F_q. If $\delta=1$, use the same generator ε as that used in the element $\bar{s}$ of Lemma 3.9. Note that $h=d$ when $\delta=1$, but not when $\delta=-1$. Then

$$H = \langle h_1, h_2 \rangle \tag{31}$$

is an Abelian subgroup of $\mathfrak{H}$ of order $\frac{1}{2}(q-1)^2$ and

$$h_1^{q-1/2} = h_2^{q-1/2} = t. \tag{32}$$

Clearly

$$N_G(R) \cap C_G(t) = (RH\langle u \rangle)\langle \sigma \rangle. \tag{33}$$

Put

$$\begin{aligned} y &= (sw)^{2^n-2e} \quad \text{if } \delta = 1, \quad \text{and} \\ y &= s^{2^n-2e} \qquad \text{if } \delta = -1, \end{aligned} \tag{34}$$

where s is the element of $C_G(u)$ referred to in Lemma 3.9. Then

$$y \in N_G(D_1) \quad \text{and} \quad [y, \sigma] = 1. \tag{35}$$

$$\begin{aligned} (h_1h_2)^y = h_1h_2 \quad \text{and} \quad w^y = vw \quad \text{if } \delta = 1 \quad \text{and} \\ w^y = w \quad \text{if } \delta = -1. \end{aligned} \tag{36}$$

Also

$$y^2 = uv \quad \text{if } \delta = 1 \quad \text{and} \quad y^2 = u \quad \text{if } \delta = -1 \tag{37}$$

and

$$t^y = tu \quad \text{and} \quad (tu)^y = t. \tag{38}$$

Thus R^y is a Sylow p-subgroup of $O^{2'}(C_G(tu))$ by (38).

Now $D_1 \leqq R \cap R^y \leqq C_G(t) \cap C_G(tu) = C_G(t, u) = (\langle t, u \rangle \times B)\langle \sigma \rangle$. But

$$R \leqq O^{2'}(C_G(t)) = \mathfrak{H}\langle u \rangle$$

so that $R \cap R^y \leqq O^{2'}(C_G(t)) \cap C_G(t, u) = (\mathfrak{H}\langle u \rangle) \cap C_G(t, u) = \langle t, u \rangle \times B$. Hence

$$D_1 = R \cap R^y. \tag{39}$$

The proof of [8, Lemma 4.1] yields

LEMMA 4.1. *If* $\{1\} < F \leqq O(L)$, *then* $N_G(F) \cap R = N_G(F) \cap R^y = D_1$.

LEMMA 4.2. *The group* $C_G(D_1)$ *has a normal* 2*-complement* M *which is a semi-direct product* $M = O(L)Q$, $Q \lhd M$, $O(L) \cap Q = \{1\}$ *and where* $Q = R^yR$ *and* $|Q| = q^3$.

Proof. It is easy to see that

$$C_G(D_1) \cap C_G(u) = (D_1 \times O(L))\langle t, s^2\rangle \tag{40}$$

and

$$C_G(D_1) \cap C_G(t) = R\langle t, u\rangle. \tag{41}$$

Thus

$$C_G(D_1) \cap C_G(tu) = R^y\langle t, u\rangle. \tag{42}$$

Now the proof of [8, Lemma 4.2] applies to finish the proof.

Note that Q is characteristic in M, which is characteristic in $C_G(D_1)$, which is normal in $N_G(D_1)$ so that

$$N_G(D_1) \leqq N_G(Q). \tag{43}$$

LEMMA 4.3. *The group Q is elementary Abelian of order q^3 and is the normal 2-complement of $C_G(R)$. Also*

$$C_G(Q) = Q. \tag{44}$$

Proof. From $N_G(R) \cap C_G(t) = (RH\langle u\rangle)\langle\sigma\rangle$, we conclude that $C_G(R) \cap C_G(t) = R\langle t\rangle$. Hence $\langle t\rangle$ is a Sylow 2-subgroup of $C_G(R)$ so that $C_G(R)$ has a normal 2-complement K. The four group $\langle t, u\rangle$ normalizes R and hence acts on K. Also $C_K(t) = R$ and $C_G(u) \cap C_G(R) \leqq C_G(u) \cap C_G(D_1) = (D_1 \times O(L))\langle t, s^2\rangle$. But

$$C_G(R) \cap O(L) = \{1\}$$

by Lemma 4.1 so that $C_K(u) = D_1$; consequently, $C_K(tu) \geqq C_K(t, u) = D_1$. Assume that $C_K(tu) \cap R^y = D_1$. Then

$$\begin{aligned} C_G(tu) \cap K = K \cap C_G(tu) \cap C_G(D_1) &= C_K(tu) \cap (R^y\langle t, u\rangle) \\ &= C_K(tu) \cap R^y = D_1. \end{aligned}$$

Then [8, Lemma 3.6] implies that $K = R$. The Frattini argument implies that

$$\begin{aligned} N_G(R) &= C_G(R)(N_G(R) \cap C_G(t)) = (R\langle t\rangle)(N_G(R) \cap C_G(t)) \\ &= N_G(R) \cap C_G(t) = (RH\langle u\rangle)\langle\sigma\rangle. \end{aligned}$$

Hence $R\langle\sigma\rangle$ contains a Sylow p-subgroup of $N_G(R)$. However, $Q \cap \langle\sigma\rangle \leqq C_G(D_1) \cap \langle\sigma\rangle = \{1\}$ and $\langle\sigma\rangle \leqq N_G(Q)$ so that $N_Q(R)\langle\sigma\rangle \leqq N_G(R)$. Since $N_Q(R) > R$, $R\langle\sigma\rangle$ cannot contain a Sylow p-subgroup of $N_G(R)$ which is a contradiction. Thus $C_K(tu) \cap R^y > D_1$, and the argument of [8, Lemma 4.3] yields $K = Q$, $R^y \leqq C_G(R)$ and $C_G(Q) = Q$, completing the proof.

Note that $C_G(R) = Q\langle t\rangle$ and $N_G(R) = (Q\langle t\rangle)(N_G(R) \cap C_G(t)) = (QH\langle u\rangle)\langle\sigma\rangle$. Also Q is characteristic in $C_G(R)$ which is normal in $N_G(R)$. Hence

$$N_G(R) \leqq N_G(Q). \tag{45}$$

Set

$$(46)\qquad P_3 = D_2^y \quad\text{and}\quad \theta_3(\alpha) = (\theta_1(\alpha)\theta_2(-\alpha))^y \quad\text{for } \alpha\in F_q.$$

Then, by (29) and (35), we have $R^y=D_1\times P_3$ so that

$$(47)\qquad Q = P_1\times P_2\times P_3.$$

Moreover, $H\leqq N_G(R)\leqq N_G(Q)$, and the proof of [8, Lemma 4.4] (provided we observe that $[v,\sigma]=1$ so that replacing s by sv, if necessary, is permitted) yields

LEMMA 4.4. *The action of H on Q is given by*

$$h_1\colon \theta_1(\alpha)\to\theta_1(\varepsilon^2\alpha),\ \theta_2(\alpha)\to\theta_2(\alpha),\ \theta_3(\alpha)\to\theta_3(\varepsilon\alpha),$$
$$h_2\colon \theta_1(\alpha)\to\theta_1(\alpha),\ \theta_2(\alpha)\to(\varepsilon^2\alpha),\ \theta_3(\alpha)\to\theta_3(\varepsilon\alpha).$$

LEMMA 4.5. *Let $V=O(C_G(P_1))$. Then $C_G(P_1)=L_2V$, $L_2\cap V=\{1\}$ and $N_G(P_1)=(L_2V\langle h_1\rangle)\langle\sigma\rangle$. The group V/P_1 is Abelian, V is nilpotent of class at most 2 and $Q\cap V=P_1\times P_3$ and $(L_2V\langle h_1\rangle)\cap\langle\sigma\rangle=\{1\}$.*

Proof. Clearly $N_G(P_1)\cap C_G(t)=(P_1\langle h_1\rangle L_2)\langle\sigma\rangle$ so that $C_G(P_1)\cap C_G(t)=P_1L_2$. The proof of [8, Lemma 4.5] applies to give all but the last equation of the lemma.

Finally, by considering the action of an element of $(C_G(P_1)\langle h_1\rangle)\cap\langle\sigma\rangle$ on P_1, we have $(C_G(P_1)\langle h_1\rangle)\cap\langle\sigma\rangle\leqq C_G(P_1)\cap\langle\sigma\rangle=1$. This completes the proof.

LEMMA 4.6. *$C_G(P_3)$ has a normal 2-complement, and $O(C_G(P_3))\leqq QH$.*

Proof. If $\delta=1$, then $a_2^{2^n-2}\colon D_1\to D_2$, so that $C_G(D_2)$ has a normal 2-complement by Lemma 4.2. If $\delta=-1$, then $N_G(D_2)\cap C_G(t)=(R\langle h_1h_2\rangle\langle t,u\rangle)\langle\sigma\rangle$ implies that $C_G(D_2)\cap C_G(t)=R\langle t\rangle$. Thus $\langle t\rangle$ is a Sylow 2-subgroup of $C_G(D_2)$ so that $C_G(D_2)$ has a normal 2-complement. Since $P_3=D_2^y$, $C_G(P_3)$ always has a normal 2-complement. In either case, $\langle t,u\rangle\leqq N_G(D_2)$ so that $\langle t,u\rangle$ acts on $O(C_G(D_2))$. From the above, we have $O(C_G(D_2))\cap C_G(t)=R$. Since $D_1\leqq B\lhd C_G(u)$ and every pair of distinct Sylow p-subgroups of B generates B, every odd ordered subgroup of $C_G(u)$ containing D_1 is contained in $N_G(D_1)\cap C_G(u)$. By Lemma 3.9, this group has a normal 2-complement $\langle\sigma\rangle FO(L)$ where $F=D_1\langle(h_1h_2)^{2^m}\rangle$ is the normal 2-complement of $N_B(D_1)$. If $C_G(D_2)$ contains an element of form $\sigma^i xz$ where $x\in F$ and $z\in O(L)$, then $[t,\sigma^i xz]=z^2\in C_G(D_2)$. By Lemma 4.1, $z^2=1$ so that $z=1$. Hence $O(C_G(D_2))\cap C_G(u)\leqq\langle\sigma\rangle F$. However, no nontrivial element of form $(h_1h_2)^j\sigma^i$ centralizes D_2. Hence $O(C_G(D_2))\cap C_G(u)=D_1$. Since D_1 is normalized by $\langle\sigma\rangle$, there exists a maximal odd order subgroup X of $C_G(t)$ containing D_1 which is normalized by $\langle\sigma\rangle$. Thus $X\leqq\mathfrak{H}\langle\sigma\rangle$ and $X=(X\cap\mathfrak{H})\langle\sigma\rangle$. But $D_1\leqq X\cap\mathfrak{H}$ and RH contains a unique largest odd order subgroup of $\mathfrak{H}$ containing D_1 and hence $X\leqq RH\langle\sigma\rangle$. Since $\langle\sigma\rangle\leqq N_G(D_2^{y^{-1}})$ and $D_1\leqq O(C_G(D_2^{y^{-1}}))\cap C_G(t)$, we conclude that $O(C_G(D_2^{y^{-1}}))\cap C_G(t)\leqq RH\langle\sigma\rangle$, so that $O(C_G(D_2))\cap C_G(tu)\leqq R^yH^y\langle\sigma\rangle$. Hence $O(C_G(D_2))\leqq D_1RR^yH^y\langle\sigma\rangle=QH^y\langle\sigma\rangle$. Thus $O(C_G(P_3))\leqq QH\langle\sigma\rangle$. Since no element of $H\langle\sigma\rangle-H$ centralizes P_3, $O(C_G(P_3))\leqq QH$.

LEMMA 4.7. *$V=O(C_G(P_1))$ is a p-group and $V=P_1P_3P_4$ where $P_4=P_3^{c_2}$ with $c=\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$ and $c_2=\phi_2(c)$.*

Proof. Now

$$\begin{aligned} C_V(P_3) &\leqq V\cap O(C_G(P_3)) \\ &= V\cap O(C_G(P_3))\cap QH\cap C_G(P_1) = V\cap O(C_G(P_3))\cap (Q\langle h_2\rangle) \leqq V\cap Q \\ &= P_1\times P_3 \leqq C_V(P_3) \end{aligned}$$

so that $C_V(P_3)=P_1\times P_3$ which is a p-group. Since V is nilpotent, V is a p-group. Also $c_2\in C_G(t)\cap C_G(P_1)$ so that $P_4=P_3^{c_2}\leqq V$. Since $h_1h_2^{-1}\in C_G(P_3)$, we have $(h_1h_2^{-1})^{c_2}=h_1h_2\in C_G(P_4)$. But h_1h_2 acts fixed point free on $P_1\times P_3$ by Lemma 4.4; hence $P_1P_3\cap P_4=\{1\}$. We have seen that $P_1\geqq V'$ so $P_1P_3P_4$ is a subgroup of V of order q^3. But $|P_2V\langle\sigma\rangle|=q\rho|V|$ so that $|V|\leqq q^3$ by Lemma 3.11. Thus $V=P_1P_3P_4$ which completes the proof of the lemma.

Note that $[\sigma, c_2]=1$ so that if

(48) $$\theta_4(\alpha) = \theta_3(\alpha)^{c_2} \quad \text{for } \alpha\in F_q,$$

then

(49) $$\theta_4(\alpha)^\sigma = (\theta_3(\alpha)^\sigma)^{c_2} = \theta_3(\alpha^{\bar\sigma})^{c_2} = \theta_4(\alpha^{\bar\sigma}) \quad \text{for } \alpha\in F_q.$$

Also c_2 centralizes h_1 and inverts h_2 so that

(50) $$\theta_4(\alpha)^{h_1} = \theta_4(\varepsilon\alpha) \quad\text{and}\quad \theta_4(\alpha)^{h_2} = \theta_4(\varepsilon^{-1}\alpha) \quad \text{for } \alpha\in F_q$$

and

(51) $$H\langle\sigma\rangle \leqq N_G(P_4).$$

LEMMA 4.8. *If $U=P_2V=P_1P_2P_3P_4$, then $N_G(U)\cap N_G(P_1)=UH\langle\sigma\rangle\leqq N_G(Q)$. Also $(UH)\cap\langle\sigma\rangle=\{1\}$, $P_1P_3\lhd U$ and U/P_1P_3 is Abelian.*

Proof. Since $U=QP_4$, we have $N_{P_4}(Q)>\{1\}$. But if $\alpha\in F_q$, then $\theta_4(\alpha)^{h_1}=\theta_3(\alpha)^{c_2h_1}=\theta_3(\alpha)^{h_1c_2}=\theta_4(\varepsilon\alpha)$ by Lemma 4.4 and $h_1\in N_G(Q)$. Thus $\langle h_1\rangle$ normalizes $N_{P_4}(Q)$ and acts irreducibly on P_4. Consequently, $N_{P_4}(Q)=P_4$ so $Q\lhd U$. Then

$$\begin{aligned} N_G(U)\cap N_G(P_1) &= N_G(U)\cap ((L_2V\langle h_1\rangle)\langle\sigma\rangle) = ((N_G(U)\cap L_2)V\langle h_1\rangle)\langle\sigma\rangle \\ &= ((N_G(P_2)\cap L_2)V\langle h_1\rangle)\langle\sigma\rangle = UH\langle\sigma\rangle \leqq N_G(Q). \end{aligned}$$

Clearly $P_1\leqq Z(U)$ and $UH\langle\sigma\rangle\leqq N_G(P_1)$. By considering the action of $UH\langle\sigma\rangle$ on P_1, we conclude that $UH\cap\langle\sigma\rangle=\{1\}$. Also $P_1P_3=Q\cap V\lhd U$ and $U=QV$. However, $[Q, V]\leqq Q\cap V=P_1\times P_3$ and Q/P_1P_3 and V/P_1P_3 are Abelian, consequently U/P_1P_3 is Abelian.

5. **The (BN)-pair.** The proof of [8, Lemma 5.1] yields

LEMMA 5.1.

$$u\colon \theta_1(\alpha)\to\theta_2(\alpha),\ \theta_2(\alpha)\to\theta_1(\alpha),\ \theta_3(\alpha)\to\theta_3(-\alpha)$$

and

$$c_2\colon \theta_1(\alpha)\to\theta_1(\alpha),\ \theta_3(\alpha)\to\theta_4(\alpha),\ \theta_4(\alpha)\to\theta_3(-\alpha)$$

for all $\alpha\in F_q$.

Since $\langle t\rangle$ is characteristic in H, $N_G(H)=N_G(H)\cap C_G(t)=\langle H, u, c_2\rangle\langle\sigma\rangle$. Set

$$N = (\mathfrak{H}\langle u\rangle)\cap N_G(H) = \langle H, u, c_2\rangle. \tag{52}$$

Clearly $\langle\sigma\rangle$ normalizes N.

LEMMA 5.2. *The structure of* $N=\langle H, u, c_2\rangle$ *is determined by the relations*: $h_1^{q-1/2}=h_2^{q-1/2}=c_2^2=t$, $t^2=u^2=1$, $[h_1, h_2]=[h_1, c_2]=1$, $h_1^u=h_2$, $h_2^{c_2}=h_2^{-1}$, $(uc_2)^4=1$. *The group* $W=N/H$ *is dihedral of order* 8.

The proof of this lemma is the same as that of [8, Lemma 5.2].

Set $r_1=uH$ and $r_2=c_2H$; then r_1, r_2 are involutions which generate $W=N/H$. The elements of W written in shortest possible form in terms of r_1 and r_2 are as given in [8, p. 30]. For $\alpha\in W$, let $\lambda(\alpha)$ be the numbers of factors r_i when α is expressed in its shortest form. Also define ω: $W\to N$ as in [8, p. 30] so that $\alpha=\omega(\alpha)H$ for all $\alpha\in W$. If K is any subgroup of G containing H, write αK and $K\alpha$ for the cosets $\omega(\alpha)K$ and $K\omega(\alpha)$ respectively. If K is any subgroup normalized by H, write K^α for $K^{\omega(\alpha)}$. Set

$$\mathfrak{B} = UH. \tag{53}$$

Clearly $\mathfrak{B}\cap N=H$.

The proofs of the following two lemmas are the same as those of [8, Lemmas 5.3 and 5.4].

LEMMA 5.3. *Let* $G_i=\mathfrak{B}\cup\mathfrak{B}r_i\mathfrak{B}$ *for* $i=1, 2$. *Then* G_1 *and* G_2 *are subgroups of* G.

LEMMA 5.4. *If* $\alpha\in W$ *and* $\lambda(r_i\alpha)\geqq\lambda(\alpha)$ *for* $i=1$ *or* 2, *then* $r_i\mathfrak{B}\alpha\subseteq\mathfrak{B}r_i\alpha\mathfrak{B}$.

LEMMA 5.5. *Set* $G_0=\mathfrak{B}N\mathfrak{B}$. *Then* G_0 *is a subgroup of* G, *and* G_0 *is the disjoint union of the eight double cosets* $\mathfrak{B}\alpha\mathfrak{B}$ *for* $\alpha\in W$. *Also* $\langle\sigma\rangle\leqq N_G(\mathfrak{B})\cap N_G(N)\leqq N_G(G_0)$ *and* $G_0\cap\langle\sigma\rangle=\{1\}$.

Proof. The first statement follows from Lemmas 5.3 and 5.4 by a theorem of Tits [7]. Since the second statement is already known, it remains to prove that $G_0\cap\langle\sigma\rangle=\{1\}$. But $G_0=UNU$ and $N\cap(U\langle\sigma\rangle)=N\cap C_G(t)\cap(U\langle\sigma\rangle)=N\cap(R\langle\sigma\rangle)$ $=\{1\}$ by the structure of $C_G(t)$ and the lemma follows.

Using our knowledge of the action of H on P_4 as given in (50), the same proof of [8, Lemma 5.6] yields

LEMMA 5.6. $U\cap U^{r_1r_2r_1r_2}=\{1\}$.

Similarly the arguments of [8, Lemmas 5.7 and 5.8] yield

LEMMA 5.7. $|G_0|=|\mathrm{PSp}(4, q)|$.

LEMMA 5.8. $G_0\cong\mathrm{PSp}(4, q)$, $G_0\lhd G_0\langle\sigma\rangle$, $\langle\sigma\rangle\cap G_0=\{1\}$ *and* $O(G)=\{1\}=C_G(G_0)$. *Finally* G *is isomorphic to the semidirect product* $\langle\bar{\sigma}\rangle\,\mathrm{PSp}(4, q)$ *described in the introduction.*

Proof. By Lemmas 5.5, 5.7 and 3.11, we have that $G=G_0\langle\sigma\rangle\triangleright G_0$. We now prove that $G_0\cong \mathrm{PSp}(4,q)$.

Clearly G_0 contains $\langle P_2, P_2^{c_2}\rangle=L_2$ and hence G_0 contains $L_1=L_2^u$. Thus $C_G(t)\cap G_0=(L_1L_2)\langle u\rangle$ by Lemma 5.5. Since $|G/G_0|$ is odd, t and uv are involutions in G_0 which are fused in G_0 by Lemma 2.6. Now [8] implies that $G_0\cong\mathrm{PSp}(4,q)$. The rest follows easily.

References

1. R. W. Carter, *Simple groups and simple Lie algebras*, J. London Math. Soc. **40** (1965), 193–240. MR **30** #4855.

2. P. Fong and W. J. Wong, *A characterization of the finite simple groups* PSp (4, q), $G_2(q)$, $D_4^2(q)$. I, Nagoya Math. J. **36** (1969), 143–184. MR **41** #326.

3. D. Gorenstein and J. H. Walter, *On finite groups with dihedral Sylow* 2-*subgroups*, Illinois J. Math. **6** (1962), 553–593. MR **26** #188.

4. ———, *The characterization of finite groups with dihedral Sylow* 2-*subgroups*. I, J. Algebra **2** (1965), 85–151. MR **31** #1297.

5. B. Huppert, *Endliche Gruppen*. I. Die Grundlehren der math. Wissenschaften, Band 134, Springer-Verlag, Berlin and New York, 1967. MR **37** #302.

6. D. S. Passman, *Permutation groups*, Benjamin, New York, 1968. MR **38** #5908.

7. J. Tits, *Théorème de Bruhat et sous-groupes paraboliques*, C. R. Acad. Sci. Paris **254** (1962), 2910–2912. MR **25** #2149.

8. W. J. Wong, *A characterization of the finite projective symplectic groups* PSp_4 (q), Trans. Amer. Math. Soc. **139** (1969), 1–35. MR **40** #5724.

Department of Mathematics, University of Illinois at Chicago Circle, Chicago, Illinois 60680