BRANCHED STRUCTURES ON RIEMANN SURFACES

BY

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Abstract. Following results of Gunning on geometric realizations of projective structures on Riemann surfaces, we investigate more fully certain generalizations of such structures.

We define the notion of a branched analytic cover on a Riemann surface M (of genus g) and specialize this to the case of branched projective and affine structures. Establishing a correspondence between branched projective and affine structures on M and the classical projective and affine connections on M we show that if a certain linear homogeneous differential equation involving the connection has only meromorphic solutions on M then the connection corresponds to a branched structure on M. Utilizing this fact we then determine classes of positive divisors on M such that for each divisor $\mathfrak{D}$ in the appropriate class the branched structures having $\mathfrak{D}$ as their branch locus divisor form a nonempty affine variety. Finally we apply some of these results to study the structures on a fixed Riemann surface of genus 2.

1. **Introduction.** The classical [1], [3], [5], [6], [14] representation of a Riemann surface as the quotient space of a subspace of the complex projective line P by a Fuchsian group provides a special coordinate covering of the Riemann surface, with the property that the local analytic coordinates are related to one another by projective transformations. Conversely, [1], [3], [7], [8] in general, a special coordinate covering can be realized geometrically as a fixed point free Fuchsian group acting on a subspace of P. The question arises as to what type of coordinate cover would admit a realization corresponding to Fuchsian groups with elliptic elements. The natural answer would seem to be coordinate covers which admit branched coordinate functions. The aim of this paper is to investigate the nature of such branched covers. In addition we also investigate how branching affects the existence of affine covers, which in the unbranched case exist only for open Riemann surfaces and compact surfaces of genus one.

Briefly the contents of the paper are the following. §2 develops the notation and terminology to be used and defines the structures to be studied. §3 establishes a correspondence between branched projective and affine structures on the Riemann surface M (of genus g) and the classical projective and affine connections on M.

In §§4 and 5 we establish the existence of linear homogeneous differential equations on M involving the projective and affine connections, respectively, whose solution spaces determine whether a specified branched structure on M exists or not. Utilizing this fact we then determine classes of positive divisors on M such that for each divisor $\mathfrak{D}$ in the appropriate class, the branched projective or affine structures having $\mathfrak{D}$ as their branch locus divisor form an affine variety containing a submanifold of dimension $3g - 3$ in the projective case and g in the affine case. Lastly, in §6 we apply some of these results to study the structures on a fixed Riemann surface of genus 2.

2. **Notation and terminology.** A *coordinate covering* $\{U_\alpha, z_\beta\}$ of the two-dimensional manifold M consists of an open covering $\{U_\alpha\}$ of M together with homeomorphisms $z_\alpha \colon U_\alpha \to V_\alpha$ from the sets U_α to open subsets V_α of the complex line C. On each nonempty intersection $U_\alpha \cap U_\beta$ we can define *coordinate transition* functions

$$f_{\alpha\beta} = z_\alpha \circ z_{\alpha\beta}^{-1} \colon z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta),$$

and we sometimes write our coverings $\{U_\alpha, z_\alpha, f_{\alpha\beta}\}$ if we wish to emphasize the transition functions.

As in [1, §1] we can define G-covers on our manifold M, where G is any pseudo-group of mappings of domains in C, by insisting that all the coordinate transition functions belong to G. We then define a G-structure on M as an equivalence class of G-covers, where two G-covers $\{U_\alpha, z_\alpha\}, \{W_\alpha, w_\alpha\}$ are equivalent if their union is a G-cover. If G is the pseudo-group of differentiable or complex analytic maps we then simply have the standard *differentiable 2-manifolds* or *complex 2-manifolds* (i.e. *Riemann surfaces*).

We wish to generalize the concept of coordinate cover to include the possibility that $z_\alpha \colon U_\alpha \to V_\alpha$ is not exactly a homeomorphism but is instead a locally branched covering map. In order to achieve maximum generality we will also assume that the sets V_α lie in the projective line P rather than C. Coordinate covers for which all the V_α are in C will be called *regular coordinate covers*. Before continuing we recall the following facts about locally branched covering maps.

Let M and N be two-dimensional manifolds and let $f \colon M \to N$ be a continuous map. Then f is a *locally branched covering* if and only if for each $p \in M$ there is an open neighborhood $U \subseteq M$ containing p, such that

$$f \,|\, U - \{p\} \to f(U) - f(p)$$

exhibits $U - p$ as an m-sheeted covering space for some integer m.

We let $O_f(p) = m - 1$ and call this the *branching order* of the mapping f at p. The point p is *regular* if and only if $O_f(p) = 0$, and a branch point of f if $O_f(p) > 0$. We note immediately that (i) the branch points are a discrete subset of M, finite if M is compact, and (ii) f is an open mapping and is a local homeomorphism in a neighborhood of any regular point.

We call $\sum_{p \in M} O_f(p)$ the *total branching order* of f. If f is constant we let $O_f(p) = -\infty$.

Suppose $f: M \to N$ is any nontrivial complex analytic mapping between Riemann surfaces. Then it is known that f is a locally branched covering and the branching order at a point p is the order of the zero of the derivative of f in any local *regular* coordinate system in M and N, that is $O_f(p) = \nu_p(f')$.

DEFINITION. Suppose $\{U_\alpha, z_\alpha\}$ is a coordinate covering of M. Suppose for each α, $w_\alpha: U_\alpha \to Y_\alpha$ is a locally branched covering from the set U_α to the open subset Y_α of the projective line P. Suppose further that for each nonempty intersection $U_\alpha \cap U_\beta$ there exists a homeomorphism

$$\phi_{\alpha\beta}: w_\beta(U_\alpha \cap U_\beta) \to w_\alpha(U_\alpha \cap U_\beta)$$

such that $w_\alpha(p) = \phi_{\alpha\beta} \circ w_\beta(p)$ for $p \in U_\alpha \cap U_\beta$. Then we say $\{U_\alpha, w_\alpha\}$ or $\{U_\alpha, w_\alpha, \phi_{\alpha\beta}\}$ is a branched cover on M.

We note that the existence of a branched cover on M presupposes the existence of a coordinate cover on M. As in the case of coordinate covers we say two branched covers $\{U_\alpha, w_\alpha\}$, $\{U_\alpha', w_\alpha'\}$ are equivalent if their union is a branched cover and we define a branched structure as an equivalence class of branched covers. Also by restricting the homeomorphisms $\phi_{\alpha\beta}$ to suitable pseudo-groups G we can also speak of branched G-structures on M. It is easily verifiable that if $\{U_\alpha, w_\alpha\}$, $\{U_\alpha', w_\alpha'\}$ are branched covers of M then if $p \in U_\alpha \cap U_\beta$ then $O_{w_\alpha}(p) = O_{w_\beta}(p)$ and if $g_\alpha: w_\alpha(U_\alpha) \to w_\alpha'(U_\alpha)$ is a homeomorphism such that $w_\alpha' = g_\alpha \circ w_\alpha$ then $O_{w_\alpha}(p) = O_{w_\alpha'}(p)$. Therefore branch points are invariant under equivalence and we can speak of the branch points of a structure rather than of only a cover.

Now suppose $\{U_\alpha, w_\alpha, \phi_{\alpha\beta}\}$ is a branched cover of the Riemann surface M.

We say $\{U_\alpha, w_\alpha, \phi_{\alpha\beta}\}$ is an *analytic branched cover* of M iff w_α is meromorphic on $U_\alpha \subset M$. This immediately implies that the $\phi_{\alpha\beta}$ are meromorphic. (If the w_α are holomorphic $\{U_\alpha, w_\alpha, \phi_{\alpha\beta}\}$ is a regular analytic branched cover.) If in addition all the $\phi_{\alpha\beta}$ are projective (resp. affine) transformations then we say $\{U_\alpha, w_\alpha, \phi_{\alpha\beta}\}$ is a *projective (resp. affine) branched cover*. In the obvious fashion we can also speak of analytic (resp. projective; resp. affine) branched structures on the Riemann surface M.

Lastly suppose M is a Riemann surface with a branched analytic structure. Then we can always choose a representative branched cover $\{U_\alpha, w_\alpha\}$ such that

(i) each U_α has at most one branch point,

(ii) if p is a branch point and $p \in U_\alpha \cap U_\beta$ then $\alpha = \beta$ and $w_\alpha(p) = 0$,

(iii) if U_α has no branch points then w_α is a homeomorphism of U_α into a unit cell in P.

We shall call such a cover a *restricted cover* on M and will assume that all our covers are restricted.

We shall henceforth fix M as a compact Riemann surface of genus $g \geqq 2$.

3. **Structures and connections.** Having introduced the notion of affine and projective structures on a Riemann surface it remains to be seen under what circumstances a Riemann surface admits such structures and what is the nature of their branching. To investigate this question we introduce (Gunning [1], [3], [4]) the notion of an $\mathscr{F}_\nu$ *connection* ($\nu = 1, 2$) on a Riemann surface.

We recall the definition of the differential operators $\theta_i \colon \mathscr{M}_U \to \mathscr{M}_U$ ($i = 1, 2$), $U \subset P$; namely:

(1)
$$\theta_1 f(z) = f''(z)/f'(z) = (d/dz)(\log f'(z)) \quad \text{for } z \in U,$$
$$\theta_2 f(z) = (\theta_1 f(z))' - \tfrac{1}{2}(\theta_1 f(z))^2 = f'''(z)/f'(z) - \tfrac{3}{2}(f''(z)/f'(z))^2$$

(θ_2 is, of course, simply the Schwarzian derivative). We note that by computation (Gunning [2, Lemma 24]) we have that if $h = g \circ f$ and $w = f(z)$ then

(2) $$\theta_\nu h(z) = \theta_\nu g(w) \cdot f'(z)^\nu + \theta_\nu f(z) \qquad (\nu = 1, 2).$$

Letting $\mathscr{F}_\nu = \{f \mid \text{for some open subdomain } U \subset P; f \in M_U \text{ and } \theta_\nu f = 0\}$ we immediately find that $\mathscr{F}_1$ is the complex affine group of dimension 1, that is GA $(1, C)$ while $\mathscr{F}_2 = $ complex projective group of dimension $1 = $ PL $(1, C)$.

Now suppose $\mathscr{U} = \{U_\alpha, z_\alpha, f_{\alpha\beta}\}$ is a representative unbranched complex analytic coordinate cover for the Riemann surface M. We can associate to $\mathscr{U}$ the cocycles $(\sigma_{\nu\alpha\beta}) \in Z^1(\mathscr{U}, \mathscr{M}(\kappa^\nu))$, ($\nu = 1, 2$) (where κ is the canonical line bundle on M, $\kappa_{\alpha\beta}(p) = (dz_\beta/dz_\alpha)_p$) defined as follows:

(3)
$$\sigma_{\nu\alpha\beta}(z_\beta(p)) = \theta_\nu f_{\alpha\beta}(z_\beta(p)) \quad \text{where } p \in U_\alpha \cap U_\beta,$$
$$\sigma_{\nu\alpha\beta}(z_\gamma(p)) = \kappa^\nu_{\gamma\beta}(p) \cdot \sigma_{\nu\alpha\beta}(z_\beta(p)) \quad \text{where } p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

By computation [1, p. 168] we see that $(\sigma_{\nu\alpha\beta})$ is in fact a cocycle for $\nu = 1, 2$.

We then have

DEFINITION. An $\mathscr{F}_\nu$ connection for the covering $\mathscr{U}$ is a zero cochain $h = (h_\alpha) \in C^0(\mathscr{U}, \mathscr{M}(\kappa^\nu))$ such that $\delta h = \sigma_\nu$.

Explicitly an $\mathscr{F}_\nu$ connection for $\mathscr{U}$ consists of sections $(h_\alpha) \in \Gamma(U_\alpha, \mathscr{M}(\kappa^\nu))$ such that $\sigma_{\nu\alpha\beta}(p) = h_\beta(p) - h_\alpha(p)$ for $p \in U_\alpha \cap U_\beta$, or in terms of the local mapping function z_α on U_α,

$$\sigma_{\nu\alpha\beta}(z_\beta(p)) = h_\beta(z_\beta(p)) - \kappa_{\alpha\beta}^{-\nu} h_\alpha(z_\alpha(p)).$$

Using the usual notion of equivalence we call two connections h, h' for coverings U, U' equivalent if h and h' can be extended to form a connection for $U \cup U'$. An equivalence class of connections will be called an $\mathscr{F}_\nu$ connection for M. An $\mathscr{F}_1$ connecting being simply the classical affine connection while an $\mathscr{F}_2$ connection is just a projective connection [4].

DEFINITION. Suppose $\tau = \{\tau_\alpha\}$ is an $\mathscr{F}_\nu$ connection for $\mathscr{U}$. Suppose further that there exist meromorphic sections $\{h_\alpha\} \in C^0(\mathscr{U}, \mathscr{M})$ such that if τ and h are realized as functions $h_\alpha(z_\alpha)$, $\tau_\alpha(z_\alpha)$ on $z_\alpha(U_\alpha)$, then

$$\theta_\nu h_\alpha(z_\alpha(p)) = \tau_\alpha(z_\alpha(p)).$$

We will then call τ an integrable $\mathscr{F}_\nu$ connection.

THEOREM 1. *There is a canonical bijection between the integrable $\mathscr{F}_v$-connections and the $\mathscr{F}_v$-structures on M.*

Proof. We first need the following lemma.

LEMMA 1. *Suppose h, k are $\mathscr{F}_v$-connections for M. Then $h = k + \tau$ for some $\tau = (\tau_\alpha) \in \Gamma(M, \mathcal{M}(\kappa^v))$.*

Proof of lemma. Representing h by $\{h_\alpha\}$ and k by $\{k_\alpha\}$, we let $\tau_\alpha = h_\alpha - k_\alpha$. We then note that since $\{\tau_\alpha\} \in C^0(\mathscr{U}, \mathcal{M}(\kappa^v))$ and by computation $\delta(\tau_\alpha) = 0$; $\{\tau\}$ is in fact in $\Gamma(M, \mathcal{M}(\kappa^v))$. (In words: The difference between two projective (resp. affine) connections is a meromorphic quadratic (resp. abelian) differential.)

Proof of theorem. Let $\{U_\alpha, z_\alpha, f_{\alpha\beta}\}$ be a representative complex analytic coordinate cover for M.

Suppose $\{U_\alpha, w_\alpha, T_{\alpha\beta}\}$ represents an $\mathscr{F}_v$ structure on M. Thus in U_α, $w_\alpha = h_\alpha(z_\alpha)$ for some $h_\alpha \in \mathcal{M}_{z_\alpha(U_\alpha)}$. Now let $\tau_\alpha(z_\alpha(p)) = \theta_v h_\alpha(z_\alpha(p))$ for $p \in U_\alpha$. We claim $\tau = \{\tau_\alpha\}$ will be an integrable $\mathscr{F}_v$-connection. We note that $h_\alpha \circ f_{\alpha\beta}(z_\beta(p)) = T_{\alpha\beta} \circ h_\beta(z_\beta(p))$ for $p \in U_\beta$. Thus by (3)

$$\theta_v h_\alpha(z_\alpha) \cdot f'^v_{\alpha\beta}(z_\beta) + \theta_v f_{\alpha\beta}(z_\beta) = \theta_v T_{\alpha\beta}(w_\beta) \cdot h'^v_\beta(z_\beta) + \theta_v h_\beta(z_\beta).$$

However $\{U_\alpha, w_\alpha, T_{\alpha\beta}\}$ is an $\mathscr{F}_v$-structure, so that $\theta_v T_{\alpha\beta}(w_\beta) = 0$. So we have

$$\tau_\alpha(z_\alpha) \cdot \kappa_{\alpha\beta}^{-v}(p) + \sigma_{v\alpha\beta}(z_\beta) = \tau_\beta(z_\beta),$$

where $\kappa_{\alpha\beta}(p) = [f'_{\alpha\beta}(z_\beta(p))]^{-1}$, and therefore $\{\tau_\alpha\}$ is an $\mathscr{F}_v$ connection which is obviously integrable. Furthermore any equivalent $\mathscr{F}_v$-cover $\{U_\alpha, \tilde{w}_\alpha, \tilde{T}_{\alpha\beta}\}$ will give rise to an equivalent connection, since if $\{U_\alpha, \tilde{w}_\alpha, \tilde{T}_{\alpha\beta}\}$ is equivalent to $\{U_\alpha, w_\alpha, T_{\alpha\beta}\}$, then $\tilde{w}_\alpha = T_\alpha \circ w_\alpha$ for some $T_\alpha \in \mathscr{F}_v$, and therefore $\theta_v \tilde{w}_\alpha = \theta_v w_\alpha$.

Now suppose $\tau \in C^0(\mathscr{U}, \mathcal{M}(\kappa^v))$ is an integrable $\mathscr{F}_v$-connection; and let $\{h_\alpha\} \in C^0(U, M)$ be such that $\theta_v h_\alpha(z_\alpha) = \tau_\alpha(z_\alpha)$. Let $w_\alpha = h_\alpha(z_\alpha)$. We claim $\{U_\alpha, w_\alpha\}$ will define an $\mathscr{F}_v$-structure which will in turn correspond to the connection τ under the construction in the first part of the proof.

We can assume $\{U_\alpha\}$ is a restricted cover and therefore define transition functions $T_{\alpha\beta}$ for $\{U_\alpha, w_\alpha\}$ by means of inverses $T_{\alpha\beta} = w_\alpha \circ w_\beta^{-1}$ in $U_\alpha \cap U_\beta$.

$$T_{\alpha\beta} \circ h_\beta = (h_\alpha \circ z_\alpha) \circ (h_\beta \circ z_\beta)^{-1} \circ h_\beta = h_\alpha \circ f_{\alpha\beta} \circ h_\beta^{-1} \circ h_\beta = h_\alpha \circ f_{\alpha\beta}.$$

Then applying (3) we have

$$\theta_v T_{\alpha\beta}(w_\beta) \cdot h'_\beta(z_\beta)^v = [\tau_\beta(z_\beta) - \kappa_{\alpha\beta}^{-v}(p)\tau_\alpha(z_\alpha)] - \sigma_{v\alpha\beta}(z_\beta).$$

But τ is an $\mathscr{F}_v$ connection, so

$$[\theta_v T_{\alpha\beta}(w_\beta)][h'_\beta(z_\beta)]^v \equiv 0 \quad \text{in } U_\beta.$$

However $h'_\beta(z_\beta) = 0$ at most at one point in U_β and $\theta_v T_{\alpha\beta}$ is meromorphic so $\theta_v T_{\alpha\beta} \equiv 0$ in U_β, and therefore $\{U_\alpha, z_\alpha, T_{\alpha\beta}\}$ is an $\mathscr{F}_v$-cover. So each $\mathscr{F}_v$ cover corresponds uniquely to an $\mathscr{F}_v$-connection and we can immediately verify that this bijective correspondence preserves equivalence, thus proving the theorem.

4. Branched projective structures. Suppose $\{U_\alpha, w_\alpha\}$ is a branched analytic cover on M. We can then canonically associate to it the positive divisor $\mathfrak{D}_{\{U_\alpha, w_\alpha\}} = \sum_{p \in M} O_{w_\alpha}(p) \cdot p$ for some α such that $p \in U_\alpha$. Since M is compact $\sum_{p \in M} O_{w_\alpha}(p) \cdot p$ is finite and therefore recalling that equivalence of covers preserves branch points we have a map $\mathfrak{D}$: {structures on M} $\to$ {positive divisors on M}. Given a positive divisor $\mathfrak{D}$ we shall then say the branched analytic cover $\{U_\alpha, w_\alpha\}$ is of type $\mathfrak{D}$ if and only if $\mathfrak{D}_{\{U_\alpha, w_\alpha\}} = \mathfrak{D}$. We let $B_{\{U_\alpha, w_\alpha\}} = B(\{U_\alpha, w_\alpha\}) = B(\mathfrak{D}_{\{U_\alpha, w_\alpha\}}) = B(\mathfrak{D}) = \sum_{p \in M} O_{w_\alpha}(p)$ and call this the total order or total branching order of the structure. We shall also let $|\mathfrak{D}_{\{U_\alpha, w_\alpha\}}| = |\{U_\alpha, w_\alpha\}| = \text{card } \{p_j \in M \mid O_{w_\alpha}(p_j) \neq 0\}$. This will be called the multiplicity of the structure.

Lastly if $\mathfrak{D}$ is a positive divisor such that $\mathfrak{D} = \sum_{j=1}^{j=n} (r_j - 1) \cdot p_j$, $r_j > 1$; we shall say that a projective connection τ on M is of type $\mathfrak{D}$ if and only if for each p_j we have (assuming $z_\alpha(p_j) = 0$)

$$\tau_\alpha(z_\alpha) = \frac{1 - r_j^2}{2z_\alpha^2} + t_{-1} z_\alpha^{-1} + \sum_{i=0}^{\infty} t_i z_\alpha^i,$$

with $\sum t_i z_\alpha^i$ convergent, in a neighborhood U_α of p_j, while τ_α is holomorphic in a neighborhood of any nonbranch point.

This last definition is justified by

PROPOSITION 2. *For each positive divisor $\mathfrak{D}$ on M there exists a canonical bijection between the projective structures of type $\mathfrak{D}$ on M and the integrable projective connections of type $\mathfrak{D}$ on M.*

Proof. It suffices to show that if $f: U \to V$ is a meromorphic function with $O \in U$ and $O_f(0) = r - 1$, r an integer ≥ 1, then $\theta_2 f(z) = (1 - r^2)/2z^2 + t_{-1}/z + A(z)$ where $t_{-1} = 0$ if $r = 1$ and A is analytic in U. However if $O_f(0) = r - 1$ then we can write locally $f(z) = (g(z))^{\pm r}$ where $g(0) = 0$; $g'(0) \neq 0$. But then by direct calculation we see that $\theta_2 f(z) = ((1 - r^2)/2)(g'(z)/g(z))^2 + \theta_2 g(z)$. Thus $g'(z)/g(z) = 1/z + (\text{holomorphic function})$ if $r > 1$ while $\theta_2 f(z) = \theta_2 g(z) = (\text{holomorphic function})$ if $r = 1$, proving the theorem.

We will call a projective connection τ holomorphic if it has a representative $\{\tau_\alpha\}$ where all the τ_α are holomorphic. We then have as an obvious consequence of the proposition,

COROLLARY 2.1 (SEE [1], [3]). *There is a canonical bijection between the unbranched projective structures on M and the integrable holomorphic projective connections on M.*

We thus have shown that the study of projective structures of a given type is equivalent to the study of integrable projective connections of that type. Let us then fix $\mathfrak{D}$ as a positive divisor with the expansion $\mathfrak{D} = \sum_{j=1}^{j=n} (r_j - 1) \cdot p_j$; $r_j > 1$ on M. We then fix n as the multiplicity of $\mathfrak{D}$ and B as the total branching order.

THEOREM 3. (i) *The integrable projective connections of type $\mathfrak{D}$ have the structure of a complex affine subvariety of the $3g-3+n$-dimensional complex linear manifold of all projective connections of type $\mathfrak{D}$.*

(ii) *Furthermore if $n=1$ or $B \leqq 2g-2$ then this subvariety is nonempty and its regular points form a $3g-3$-dimensional submanifold of the space of all projective connections of type $\mathfrak{D}$.*

Proof. We will first show that the space of all projective connections of type $\mathfrak{D}$ forms a $3g-3+n$-dimensional complex linear manifold. We recall that by Lemma 1, given a projective connection, we can produce all other projective connections by simply adding appropriate quadratic differentials. Thus suppose τ is a projective connection of type $\mathfrak{D}$. By adding to τ any quadratic differential which is holomorphic except for at most simple poles at p_j we produce all the projective connections of type $\mathfrak{D}$. That is

{projective connections of type $\mathfrak{D}$}

$\quad = \tau + \{$quadratic differentials with at most simple poles at p_j, $(j = 1, \ldots, n)\}$.

However by the Riemann-Roch theorem the space of quadratic differentials with at most simple poles at p_j, $(j=1, \ldots, n)$ is isomorphic to a complex number space of dimension $3g-3+n$. Thus if there exists one projective connection of type $\mathfrak{D}$, then there exists a $3g-3+n$-complex linear manifold of such connections. However by [3, Corollary 2 to Theorem 4] we see that there exists a projective connection σ of type 0 on M. Now utilizing the notation of [1], [2] and [3] $\Gamma(M, \mathcal{O}_m(\kappa^2, -2 \cdot p_j))$ is the space of quadratic differentials with at most a double pole at p_j while $\Gamma(M, \mathcal{O}_m(\kappa^2, -1 \cdot p_j))$ is the space of quadratic differentials with at most a single pole at p_j. However by the Riemann-Roch theorem

$$\dim \Gamma(M, \mathcal{O}_m(\kappa^2, -2 \cdot p_j)) = 3g-1 \quad \text{and} \quad \dim \Gamma(M, \mathcal{O}_m(\kappa^2, -1 \cdot p_j)) = 3g-2.$$

Therefore for each p_j there exists a quadratic differential θ_j which has exactly a pole of order 2 at p_j and no other singularities. Letting a_j be the coefficient of $z_{\alpha_j}^{-2}$ in the Laurent expansion of θ_j around $z_\alpha(p_j)=0$ we let $\theta = \sum_{j=1}^{j=n} ((1-r_j^2)/2a_j)\theta_j$. Then it is immediately apparent that θ is a quadratic differential such that $\tau = \sigma + \theta$ is a projective connection of type $\mathfrak{D}$. We are therefore left with the task of showing that integrable projective connections of type $\mathfrak{D}$ form a complex affine subvariety.

Let $F_2(M, \mathfrak{D})$ be the space of projective connections of type $\mathfrak{D}$. Let $\tau \in F_2(M, \mathfrak{D})$. Let us now choose a basis for the vector space of quadratic differentials with at most a simple pole at p_j, $j=1, \ldots, n$. By the constructions we made previously we can choose this basis as follows:

Let $_jq$, $j=1, \ldots, 3g-3$, be a basis for the space of holomorphic quadratic differentials. Let $_j\theta$ be a quadratic differential which is holomorphic everywhere except at p_j where it has the expansion $_j\theta(z_\alpha)=(1/z_\alpha)+\sum_{i=0}^{\infty} {}_j\theta(\alpha)_i z_\alpha^i$. We let $\boldsymbol{q}=(_1q, \ldots, _{3g-3}q)$, $\boldsymbol{\theta}=(_1\theta, \ldots, _n\theta)$, and for each $(\boldsymbol{t}, \boldsymbol{\sigma})=(t_1, \ldots, t_n, \sigma_1, \ldots, \sigma_{3g-3})$

$\in C^n \times C^{3g-3}$, we let $_{(t,\sigma)}\mu = \tau + \theta \cdot t + q \cdot \sigma$ be the element of $F_2(M, \mathfrak{D})$ determined by (t, σ). Therefore we see that

$$IF_2(M, \mathfrak{D}) = \{\text{integrable projective connections of type } \mathfrak{D}\}$$

is equivalent to

$\{(t, \sigma) \in C^n \times C^{3g-3} \mid \theta_2 w_\alpha(z_\alpha) = {}_{(t,\sigma)}\mu_\alpha(z_\alpha)$ has a meromorphic solution for all $\alpha\}$.

Following [1], [13] we note that the problem of solving

$$(4) \qquad\qquad\qquad \theta_2 w(z) = S(z)$$

for meromorphic $S(z)$ is equivalent to the simpler problem of finding two linear independent solutions w_1 and w_2 of the linear homogeneous second order equation

$$(5) \qquad\qquad\qquad w''(z) + \tfrac{1}{2} S(z) w(z) = 0.$$

That is, if w_1 and w_2 solve (5) then w_1/w_2 solves (4), the general solution being $(aw_1 + bw_2)/(cw_1 + dw_2)$ with $ad - bc = 1$. Furthermore it is immediately verifiable that if $S(z)$ is holomorphic then so are w_1 and w_2 and a solution w_h to (4) can be found such that w_h is holomorphic and $w_h'(z) \neq 0$. Thus to determine $IF_2(M, \mathfrak{D})$ we must find out what

$\{(t, \sigma) \in C^n \times C^{3g-3} \mid w_\alpha''(z_\alpha) + \tfrac{1}{2} {}_{(t,\sigma)}\mu_\alpha(z_\alpha) = 0$ has two linearly

independent meromorphic solutions in each U_α containing a $p_j\}$

is. Now we recall that in a neighborhood U_α of p_j,

$$_{(t,\sigma)}\mu_\alpha(z_\alpha) = \frac{1-r_j^2}{2z_\alpha^2} + {}_{(t,\sigma)}\mu_{-1}(\alpha)z_\alpha^{-1} + \sum_{i=0}^{\infty} {}_{(t,\sigma)}\mu_i(\alpha)z_\alpha^i.$$

Thus by [12], [13] there exists a polynomial $A(r_j)$ in $C[x_1, \ldots, x_r]$ (where

$$A(r)(x_1, \ldots, x_r) = \sum_{(i_1, \ldots, i_r)} \alpha_{(i_1, \ldots, i_r)} x_1^{i_1} \cdots x_r^{i_r}$$

and $(i_1, \ldots, i_r)$ are such that $\sum_{k=1}^{k=r} k i_k = r$); with leading term $\alpha_1 x_1^r$ (with $\alpha_1 \neq 0$) such that $w_\alpha''(z_\alpha) + \tfrac{1}{2} {}_{(t,\sigma)}\mu_\alpha(z_\alpha) = 0$ has two linearly independent meromorphic solutions in U_α if and only if

$$A(r_j)({}_{(t,\sigma)}\mu_{-1}(\alpha), \ldots, {}_{(t,\sigma)}\mu_{r-2}(\alpha)) = 0.$$

However $_{(t,\sigma)}\mu_{-1}(\alpha) = \tau_{-1}(\alpha) + t_1$ and, for $k \geq 0$, $_{(t,\sigma)}\mu_k(\alpha) = \tau_k(\alpha) + \theta_k(\alpha) \cdot t + q_k(\alpha) \cdot \sigma$ where $\tau_k(\alpha)$, $\theta_k(\alpha)$, $q_k(\alpha)$ are just the coefficients of z_α^k in the Laurent expansion around $z_\alpha(p_j) = 0$ of $\tau(\alpha)$, $\theta(\alpha)$, $q(\alpha)$ respectively.

Thus there exists a polynomial $K_j \in C[x_1, \ldots, x_{3g-3+n}]$ with leading term $\alpha_0^{(j)} x_j^{r_j}$, $\alpha_0^{(j)} \neq 0$; and total degree of any other term $\leq [r_j/2]$; such that

$$A(r_j)({}_{(t,\sigma)}\mu_{-1}(\alpha), \ldots, {}_{(t,\sigma)}\mu_{r-2}(\alpha)) = 0 \quad \text{iff} \quad K_j(t, \sigma) = 0.$$

Then the K_j $(j = 1, \ldots, n)$ form a system of n-polynomials in $n + 3g - 3$ variables. If we now let

$$_pV_{\mathfrak{D}} = \{(t, \sigma) \in C^n \times C^{3g-3} \mid K_j(t, \sigma) = 0, j = 1, \ldots, n\},$$

then $IF_2(M, \mathfrak{D}) \approx {_pV_{\mathfrak{D}}}$ which is the required subvariety.

Furthermore if we look at the mapping $K: C^{3g-3+n} \to C^n$ given by $K(t, \sigma) = (K_1(t, \sigma), \ldots, K_n(t, \sigma))$ we can show by a direct calculation that "in general" the rank of $J(K)$, the Jacobian matrix of the mapping K, will be n. Therefore if $_pV_{\mathfrak{D}}^0 \subset {_pV_{\mathfrak{D}}}$ consists of the regular points of $_pV_{\mathfrak{D}}$ then if $_pV_{\mathfrak{D}} \neq 0$, $_pV_{\mathfrak{D}}^0$ is in turn a submanifold of dimension $3g - 3$. We see immediately that if $n = 0$, $_pV_{\mathfrak{D}} = C^{3g-3}$, and if $n = 1$, $_pV_{\mathfrak{D}}$ is just a union of hypersurfaces in C^{3g-2}.

Now suppose $B \leq 2g - 2$. We will show $_pV_{\mathfrak{D}} \neq 0$, thus concluding our theorem.

Suppose for each j, we could pick a quadratic differential $_j\theta$ such that $_j\theta$ is holomorphic everywhere except that

   (i) around p_j, $_j\theta(z_\alpha) = 1/z_\alpha + \sum_{i=0}^{\infty} {_j\theta(\alpha)_i z_\alpha^i}$ and

   (ii) around $p_i \neq p_j$, $_j\theta(z_\beta) = z_\beta^{r_i-1} \sum_{k=0}^{\infty} {_j\theta(\beta)_k z_\beta^k}$.

The system $_jq$, $j = 1, \ldots, 3g - 3$, $_j\theta$, $j = 1, \ldots, n$, will again form a basis for the quadratic differentials with at most a simple point at p_j. However if we now calculate $K_j(t, \sigma)$ we find that, since the $_i\theta$ for $i \neq j$ have no terms of degree less than $r_j - 1$ around p_j, $K_j(t, \sigma) = K_j((0, \ldots, t_j, \ldots, 0), \sigma)$. Thus for each j, K_j will be a polynomial in t_j and σ with leading term $\alpha_0^{(j)} t_j^{r_j}$, $a_0^{(j)} \neq 0$. Thus for any fixed σ, if we let t_j^* be a root of $K_j(t_j, \sigma) = 0$ for $j = 1, \ldots, n$ we immediately see that $(t^*, \sigma) \in {_pV}$ where $t^* = (t_1^*, \ldots, t_n^*)$. Therefore $_pV_{\mathfrak{D}} \neq 0$.

So we must prove that we can pick the $_j\theta$ as described above.

Now let ϕ_j be the divisor $(r_1 - 1) \cdot p_. + \cdots + (-1) \cdot p_j + \cdots + (r_n - 1) \cdot p_n$. Let $Q(\mathfrak{D})$, for any divisor $\mathfrak{D}$, be the space of quadratic differentials whose divisors are multiples of $\mathfrak{D}$.

Then if for each j, $Q(\phi_j) - Q(\phi_j + p_j) \neq 0$, we have our basis, since if we let $_j\psi \in Q(\phi_j) - Q(\phi_j + p_j)$ for each j, then we simply let $_j\theta = {_j\psi}/\mathrm{Res}_{p_j}({_j\psi})$ and the $_j\theta$, $j = 1, \ldots, n$, are as required.

However by the Riemann-Roch theorem provided $\deg(\phi_j) \leq 2g - 4$ we have

$$\dim Q(\phi_j) = 3g - 3 - \deg(\phi_j) \geq g + 1$$

while

$$\dim Q(\phi_j + p_j) = 3g - 3 - \deg(\phi_j + p_j) = 3g - 3 - \deg(\phi_j) - 1 \geq g.$$

Thus $\dim Q(\phi_j) - \dim Q(\phi_j + p_j) \geq 1$ and $Q(\phi_j) - Q(\phi_j + p_j) \neq 0$, provided $\deg(\phi_j) \leq 2g - 4$. However

$$\deg(\phi_j) < \deg(\phi_j + p_j) < \deg(\phi_j + p_j^{r_1}) = B \leq 2g - 2.$$

Thus $\deg(\phi_j) \leq 2g - 4$ and our theorem is proven.

We note that our result can be extended to $B \leq 3g - 2$ provided the points $p_1 \cdots p_n$ avoid a one-codimensional subset $A \subset M^n$ (see [1], [10] and [11]).

5. Branched affine structures. Having discussed $\mathscr{F}_\nu$ structures and connections for $\nu=2$ we now turn to the case of $\nu=1$, the affine structures. So suppose $h=(h_\alpha) \in C^0(\mathscr{U}, \mathscr{M}(\kappa))$ is an affine connection for U. We recall that this means

$$\sigma_{1\alpha\beta}(z_\beta) = f''_{\alpha\beta}(z_\beta)/f'_{\alpha\beta}(z_\beta) = (d/dz_\beta) \log f'_{\alpha\beta} = h_\beta(z_\beta) - \kappa_{\alpha\beta}^{-1} h_\alpha(z_\alpha)$$

where $\mathscr{U} = \{U_\alpha, z_\alpha, f_{\alpha\beta}\}$ is a coordinate cover of M.

Paralleling the development in [9] we now let $p \in U_\alpha \cap U_\beta$ and let $\Gamma \subset U_\alpha \cap U_\beta$ be a simple closed curve bounding a cell $D \ni p$ in $U_\alpha \cap U_\beta$. We then define

$$\text{Res } (h, p) = \frac{1}{2\pi i} \int_\Gamma h_\alpha \, dz_\alpha$$

as the residue of h at p, and

$$\text{Res } (h) = \sum_{p \in M} \text{Res } (h, p).$$

We immediately note that Res (h, p) is independent of the choice of representatives for h since

$$d \log f'_{\alpha\beta}(z_\beta) + h_\alpha(z_\alpha) \, dz_\alpha = h_\beta(z_\beta) \, dz_\beta,$$

and $f'_{\alpha\beta} \neq 0$ in $U_\alpha \cap U_\beta$. Furthermore since any two affine connections h, k differ at most by an abelian differential τ, which has total residue 0 on the compact surface M, Res $(h) = \text{Res } (k)$. Thus the total residue of all affine connections are the same fixed constant. We then have

LEMMA 2. *Let* $h=(h_\alpha)$ *be an affine connection for* U. *Then* Res $(h)=2g-2$.

Proof. Let $g=\{g_\alpha\} \in \Gamma(M, \mathscr{M}(\kappa))$ represent an abelian differential on M. Let

$$k_\alpha(z_\alpha) = (d/dz_\alpha) \log g_\alpha(z_\alpha) = g'_\alpha(z_\alpha)/g_\alpha(z_\alpha).$$

Now $g_\alpha(z_\alpha) = \kappa_{\alpha\beta}(p) g_\beta(z_\beta)$. Thus since $d/dz_\beta = \kappa_{\alpha\beta}^{-1}(d/dz_\alpha)$, we have

$$\kappa_{\alpha\beta}^{-1}(p) \cdot k_\alpha(z_\alpha) = (d/dz_\beta) \log \kappa_{\alpha\beta}(p) + k_\beta(z_\beta).$$

But

$$\sigma_{1\alpha\beta}(z_\beta) = f''_{\alpha\beta}(z_\beta)/f'_{\alpha\beta}(z_\beta) = -(d/dz_\beta) \log \kappa_{\alpha\beta}(p).$$

Thus

$$\sigma_{1\alpha\beta}(z_\beta) = \kappa_\beta(z_\beta) - \kappa_{\alpha\beta}^{-1}(p) \cdot k_\alpha(z_\alpha)$$

and therefore $k=\{k_\alpha\}$ is an affine connection for U. However

$$\text{Res } (k) = \frac{1}{2\pi i} \sum_{p \in M} \int \frac{d}{dz_\alpha} \log g_\alpha(z_\alpha) = \sum_{p \in M} \nu_p(g_\alpha).$$

But since $g_\alpha \in \Gamma(M, \mathscr{M}(\kappa))$, $\sum_{p \in M} \nu_p(g_\alpha) = c(\kappa) = 2g-2$. So Res $(k)=2g-2$ and, since all affine connections have the same residue, Res $(h)=2g-2$ also.

Now suppose $\{U_\alpha, w_\alpha\}$ is a branched affine structure with a branch point of order $r-1$ at p_j. Thus locally in $U_\alpha \ni p_j$ we have

$$w_\alpha = [g_\alpha(z_\alpha)]^{\pm r} \quad \text{where } g_\alpha \colon z_\alpha(U_\alpha) \to C$$

with

$$g_\alpha(0) = 0, \quad g'_\alpha(0) \neq 0, \quad \text{and} \quad z_\alpha(p_j) = 0.$$

But affine transformations preserve poles so that, in contradistinction to the projective case, $[g_\alpha(z_\alpha)]^{+r}$ is not equivalent to $[g_\alpha(z_\alpha)]^{-r}$. Thus in addition to specifying the order of a branch point we must also specify whether it arises from a pole or not. To this end we introduce the notion of the "residue class" divisor χ as follows.

Suppose $\{U_\alpha, w_\alpha\}$ represents an affine structure on M with poles of order ν_j at $S_j, j=1, \ldots, \sigma$, and nonpolar branch points of order r_j-1 at $p_j, j=1, \ldots, n$. (We note that $O_{w_j}(s_j) = |\nu_j| - 1$.)

We now define the divisor (not necessarily positive)

$$\chi_{(U_\alpha, w_\alpha)} = \sum_{j=1}^{n} (r_j-1) \cdot p_j + \sum_{j=1}^{\sigma} (\nu_j - 1) \cdot s_j.$$

We thus have a map

$$\chi \colon \{\text{affine structures on } M\} \to \{\text{divisors on } M\}.$$

We let $\rho(\chi) = \sum_{j=1}^{j=n} (r_j-1) + \sum_{j=1}^{j=\sigma} (\nu_j - 1)$ and $|\chi| = \eta(\chi) + \sigma(\chi)$ where if $\chi = \sum x_j \cdot p_j$ then

$$\eta(\chi) = \operatorname{card} \{p_j \mid x_j > 0\} \quad \text{and} \quad \sigma(\chi) = \operatorname{card} \{p_j \mid x_j < 0\}.$$

Now given a divisor χ with $\rho(\chi) = 2g-2$ we shall say that an affine connection $h = (h_\alpha)$ is of class χ iff h has only simple poles and

$$\chi = \sum_{j=1}^{n} x_j \cdot p_j \Rightarrow \operatorname{Res}(h, p_j) = x_j.$$

If χ has no terms of the form $-1 \cdot p_j$ we shall say it is of the second kind. We now have

PROPOSITION 4. *For each divisor of the second kind χ with $\rho(\chi) = 2g-2$ there is a canonical bijection between the affine structures of class χ on M and the integrable affine connections of class χ on M.*

Proof. It suffices to show that if $f \colon U \to V$ is a meromorphic function with $O \in U$ and $\nu_0(f') = r-1$ with $|r| \geq 1$, then $\theta_1 f(z) = (r-1)/z + A(z)$ where A is analytic in a neighborhood of 0. However if $\nu_0(f') = r-1$ we can write locally $f(z) = (g(z))^{\pm r}$ with $g(0) = 0, g'(0) \neq 0$. But then by direct calculation

$$\theta_1 f(z) = (r-1)(g'(z)/g(z)) + \theta_1 g(z).$$

However $\theta_1 g(z)$ is holomorphic and $g'(z)/g(z) = 1/z + $ (holomorphic function). Thus $\theta_1 f(z) = (r-1)/z + A(z)$ where $A(z)$ is holomorphic and $\theta_1 f$ is holomorphic iff $r = 1$.

We note that the proof implies that if $\{U_\alpha, w_\alpha\}$ represents an affine structure on M and if h is an affine connection corresponding to $\{U_\alpha, w_\alpha\}$ then necessarily $\chi_{\{U_\alpha, w_\alpha\}}$ is of the second kind with $\rho(\chi) = \text{Res}(h) = 2g - 2$.

We recall that a coordinate cover $\{U_\alpha, z_\alpha\}$ is regular iff z_α is holomorphic for each α. Since holomorphicity is preserved by affine transformations we shall say $\{U_\alpha, w_\alpha\}$ represents a regular affine structure iff it is a regular affine cover. We immediately see that $\{U_\alpha, w_\alpha\}$ is regular iff $\chi_{\{U_\alpha, w_\alpha\}}$ is positive or equivalently iff $\chi_{\{U_\alpha, w_\alpha\}} = \mathfrak{D}_{\{U_\alpha, w_\alpha\}}$.

Similarly we shall say that h is a regular affine connection iff $\text{Res}(h, p) \geqq 0$ for all $p \in M$.

THEOREM 5. *Let $\mathfrak{D}$ be a positive divisor with $B(\mathfrak{D}) = 2g - 2$. Then the integrable affine connections of class $\mathfrak{D}$ form a g-dimensional complex linear manifold.*

Proof. Suppose h is an affine connection of class $\mathfrak{D}$. Then so is $h + \tau$ for any holomorphic abelian differential τ. But the space of holomorphic abelian differentials is isomorphic to C^g. Thus it suffices to show that

(i) there exists an affine connection of class $\mathfrak{D}$,

(ii) any affine connection of class $\mathfrak{D}$ is integrable.

We prove (ii) first. So it suffices to show h is integrable. Now locally we have $h(z) = (r-1)/z + A(z)$ with A analytic and $r \geqq 1$. For $h(z)$ to be integrable we must find a meromorphic solution to the linear differential equation

$$(6) \qquad\qquad w''/w' = (r-1)/z + A(z).$$

So let $v = w'$ and rewrite this as $v'/v = (r-1)/z + A(z)$. Then $v(z) = Kz^{r-1}e^{A(z)}$; but since $r \geqq 1$ and A is analytic then v is holomorphic and therefore so is w, since $w' = v$.

We now are left to show that an affine connection of type $\mathfrak{D}$ exists.

So let $\tau = \{\tau_\alpha\}$ be a holomorphic differential on M. Let $\chi_\tau = \sum_{p \in M} \nu_p(\sigma_j) \cdot p$ and let $k_\alpha(z_\alpha) = (d/dz_\alpha) \log \tau_\alpha(z_\alpha)$ in U_α. Then by the proof of Lemma 2 we see that $k = \{k_\alpha\}$ is an affine connection of class χ_τ, and $\rho(\chi_\tau) = 2g - 2$.

Now consider the divisor $\mathfrak{D} - \chi_\tau$. Since

$$\rho(\mathfrak{D} - \chi_\tau) = \rho(\mathfrak{D}) - \rho(\chi_\tau) = 0,$$

by [14] and [5] there exists an abelian differential of the third kind θ with only simple poles such that

$$\sum_{p \in M} \text{Res}(\theta, p) \cdot p = \mathfrak{D} - \chi_\tau.$$

Therefore $k + \theta$ will in fact be an affine connection of class $\mathfrak{D}$.

We note that the first and third parts of our proof will be true not only for a positive divisor $\mathfrak{D}$ but for any divisor of the second kind χ with $\rho(\chi) = 2g - 2$. We then have

COROLLARY 5.1. *Let χ be a divisor of the second kind with $\rho(\chi)=2g-2$. Then the affine connection of class χ form a g-dimensional complex linear manifold and the integrable connections of class χ comprise a complex affine subvariety of this manifold.*

Proof. The existence of the g-dimensional complex linear manifold of affine connections of class χ follows directly from the proof of the theorem. It thus suffices to examine the nature of the solutions to the differential equation $w''/w' = (r-1)/z + A(z)$ in the case where $r \leq -1$. Again letting $v = w'$ we have

$$v(z) = Kz^{r-1}e^{A(z)} = Ke^{a_0} \sum_{n=0}^{\infty} C(n)z^{n+r-1}$$

where $A(z) = \sum_{i=0}^{\infty} a_i z^i$, $C_0 = 1$, and for $n \geq 1$

$$C(n) = \sum_{(i_1,\ldots,i_n)} \alpha_{(i_1,\ldots,i_n)} a_1^{i_1} \cdots a_n^{i_n}$$

and $(i_1,\ldots,i_n)$ are such that $\sum_{k=1}^{k=n} ki_k = n$ and $\alpha_{(1,0,0,\ldots,0)} = 1/n!$. Thus we have

$$w(z) = Ke^{a_0} \sum_{n=0}^{\infty} C(n) \int z^{n+r-1} \, dz.$$

This is meromorphic if and only if $C_{-r} = C_{|r|} = 0$. Therefore, in general, for each affine connection h, there exists a polynomial

$$H_{|r|} \in C[x_1,\ldots,x_r]$$

such that h is integrable iff $H_{|r|}(a_1,\ldots,a_r) = 0$ where $a_1,\ldots,a_r$ are defined as above.

So let $h = \{h_\alpha\}$ be an affine connection of class

$$\chi = \sum_{j=1}^{n} (r_j-1)p_j + \sum_{j=1}^{\sigma} (\nu_j-1)\cdot s_j$$

and let $_1\phi,\ldots,_g\phi$ be a basis for the holomorphic differentials on M. Now for each $t \in C^g$, let $_tk = h + \boldsymbol{\phi} \cdot t$, where $\boldsymbol{\phi} = (_1\phi,\ldots,_g\phi)$. We then see immediately that $_tk$ is an affine connection of class χ and that therefore $IF_1(M,\chi) = \{$integrable connections of class $\chi\}$ is equivalent to $\{t \in C^g \mid _tk$ is integrable around s_j for $j=1,\ldots,\sigma\}$. Now for any $t \in C^g$ we note that around s_j with $z_\alpha(s_j) = 0$ we have

$$_tk(z_\alpha) = \frac{\nu_j-1}{z_\alpha} + A(z_\alpha) + \boldsymbol{\phi} \cdot t = \frac{\nu_j-1}{z_\alpha} + \sum {}_tk_m(\alpha)z_\alpha^m,$$

and, in general,

$$_tk_m(\alpha) = A_m(\alpha) + \boldsymbol{\phi}_m(\alpha) \cdot t$$

where $A_m(\alpha)$, $\boldsymbol{\phi}_m(\alpha) = (_1\phi_m(\alpha),\ldots,_g\phi_m(\alpha))$ are simply the coefficients of z_α^m in the Taylor expansion around $z_\alpha(s_j) = 0$ of $A(\alpha)$ and $\boldsymbol{\phi}(\alpha)$ respectively. Thus there exists a polynomial $L_j \in C[x_1,\ldots,x_g]$ such that $H_{|\nu_j|}(_tk_1,\ldots,_tk_{|\nu_j|}) = 0$ iff $L_j(t) = 0$. So $_tk$ is integrable around s_j iff $L_j(t) = 0$; and therefore if we now let

$$_aV^\chi = \{t \in C^g \mid L_j(t) = 0 \text{ for } j = 1,\ldots,\sigma\},$$

then $IF_1(M,\chi) \approx {}_aV^\chi$, which is the required subvariety.

COROLLARY 5.2. *Suppose* $\chi = \sum_{j=1}^{n} (r_j - 1) \cdot p_j + (\nu - 1) \cdot s$. *Then,*

(i) *if s is not a hyperelliptic Weierstrass point or ν is even, $_aV^\chi \neq \varnothing$ and is in fact a union of hypersurfaces in C^g; and*

(ii) *if ν is odd and s is a hyperelliptic Weierstrass point, then if $_aV^\chi \neq \varnothing$ it is either C^g or a union of hypersurfaces in C^g.*

Proof. From the proof of Corollary 5.1 we see that $_aV^\chi = \{t \in C^g \mid L(t) = 0\}$ where in our case $L(t) = H_{|\nu|}({}_tk_1, \ldots, {}_tk_{|\nu|})$. Now if s is not a hyperelliptic Weierstrass point, the basis $_1\phi, \ldots, {}_g\phi$ of abelian differentials used in constructing $_tk$ could have been chosen so that $_2\phi$ was the only term with nonzero coefficient of z_α'. Thus $_tk_1 = a_1 + {}_2\phi_1 \cdot t_2$ and we can assume $_2\phi_1 = 1$. Thus our polynomial $L(t)$ has leading term $(1/|\nu|!)t_2^1 +$ other lower order terms and this clearly shows that $_aV^\chi$ is a union of hypersurfaces in C^g.

Furthermore even if s is a hyperelliptic point then it is still impossible for any basis of the abelian differentials to have all coefficients of both z_α^1 and z_α^2 zero. If some abelian differential has a nonzero z_α^1 coefficient the above construction shows that $_aV^\chi$ is a union of hypersurfaces. However even if all the z_α^1 coefficients are zero at least one z_α^2 coefficient is nonzero. We can thus always rechoose the basis so that $_2\phi$ is the only differential in the basis with $_2\phi_2 \neq 0$, and we can assume $_2\phi_2 = 1$. Then if $\nu_1 = -2m$ for some $m \geq 1$, $L(t)$ would have leading term $(1/m!)t_2^m$ and again $_aV^\chi$ would be a union of hypersurfaces in C^g. However if ν_1 is odd it is possible that $L(t)$ reduces to a constant so that if $L(t) = k$ then if $k = 0$, $_aV^\chi = C^g$, if $k \neq 0$ then $_aV^\chi = \varnothing$ and otherwise we again have a union of hypersurfaces.

What this corollary shows us is that on any Riemann surface if we avoid a "suitably small" set of points we will always have $_aV^\chi \neq \varnothing$.

Using techniques of [10] and [11] this can be generalized to

COROLLARY 2.6.4. *Suppose*

$$\chi = \sum_{j=1}^{j=n} (r_j - 1) \cdot p_j + \sum_{j=1}^{j=\sigma} (\nu_j - 1) \cdot s_j$$

and

$$0 < \left| \sum_{j=1}^{j=\sigma} (\nu_j - 1) \right| \leq g - 1.$$

Then there exists an analytic subset $A \subset M^{\sigma(\chi)}$ of codimension ≥ 1 such that if $(s_1, \ldots, s_\sigma) \in M^\sigma - A$ then $_aV^\chi \neq \varnothing$ and $_a^0V^\chi$ has dimension $g - \sigma(\chi)$, where $_a^0V^\chi$ is the submanifold of regular points of $_aV^\chi$.

However we will pursue the subject no further.

6. **Examples.** To indicate the wealth of new structures now apparent on Riemann surfaces we briefly examine the case of surfaces of genus 2. Let M be such a surface and suppose $\mathfrak{D} = 1 \cdot p_1 + 1 \cdot p_2$ is a divisor on M with p_1 possibly equal to p_2. Then $\rho(\mathfrak{D}) = 2 = 2g - 2$ and the space of regular affine structures with branching order $\mathfrak{D}$ is just C^2. Furthermore the space of distinct positive divisors on M of total branching order 2 is just SM^2, the 2-fold symmetric product of M (that is,

$M^2/\sim$ where $(a_1, a_2) \sim (b_1, b_2)$ iff $\{a_1, a_2\} = \{b_1, b_2\}$ as sets). Thus any Riemann surface of genus 2 has associated to it the space $SM^2 \times C^2$ of distinct regular branched affine structures.

If we now examine the projective structures associated to M we find they form a subvariety $_pV_{\mathfrak{D}}$ of C^5 containing a 3-dimensional submanifold $_pV_{\mathfrak{D}}^0$. Thus the space of distinct branched projective structures of total branching order 2 form the fiber space $_pV_2 = \{(\mathfrak{D}, V_{\mathfrak{D}}) \mid \mathfrak{D} \in SM^2\}$ and this space will itself contain a submanifold of dimension 5, namely $_pV_2^0 = \{(\mathfrak{D}, _pV_{\mathfrak{D}}^0) \mid \mathfrak{D} \in SM^2\}$. Thus it is immediately apparent that whereas in the case of $g = 1$ all projective structures of order $2g - 2 = 0$ reduce to affine structures [1, §9.3], this is decidedly *not the case* when $g = 2$. In a forthcoming article we hope to apply some of this additional structure to study the holomorphic vector bundles of rank 2 on a Riemann surface [2].

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