TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 163, January 1972

SOME INVARIANT σ-ALGEBRAS FOR MEASURE-PRESERVING TRANSFORMATIONS

BY
PETER WALTERS

Abstract. For an invertible measure-preserving transformation T of a Lebesgue measure space $(X, \mathscr{B}, m)$ and a sequence N of integers, a T-invariant partition $\alpha_N(T)$ of $(X, \mathscr{B}, m)$ is defined. The relationship of these partitions to spectral properties of T and entropy theory is discussed and the behaviour of the partitions $\alpha_N(T)$ under group extensions is investigated. Several examples are discussed.

0. **Introduction.** For an invertible measure-preserving transformation T of a Lebesgue space $(X, \mathscr{B}, m)$ and a sequence $N=\{n_i\}_{i=1}^{\infty}$ of integers we define a σ-algebra by $\mathscr{A}_N(T)=\{A \in \mathscr{B} \mid m(T^{n_i}A \Delta A) \to 0\}$. Our aim is to study these σ-algebras. In §1 we evaluate those elements of $L^2(X, \mathscr{B}, m)$ which are measurable with respect to $\mathscr{A}_N(T)$. The connections the algebras $\mathscr{A}_N(T)$ have with discrete spectrum and entropy theory are discussed in §2. Every ergodic T with discrete spectrum has $\mathscr{A}_N(T)=\mathscr{B}$ for some sequence N and every T with $\mathscr{A}_N(T)=\mathscr{B}$ for some sequence N has zero entropy. It turns out that the algebras $\mathscr{A}_N(T)$ have properties in common with the σ-algebra generated by the eigenfunctions of T and also properties in common with the σ-algebra generated by all the finite algebras having zero entropy relative to T. Some of these properties are noted in §3 which also contains remarks on the relationship of the σ-algebras $\mathscr{A}_N(T)$ to mixing properties of T. The behaviour of the σ-algebras $\mathscr{A}_N(T)$ under group extensions is discussed in §4 and in §5 we use Gaussian processes to give examples of weak mixing transformations with $\mathscr{A}_N=\mathscr{B}$ for some sequence N. §6 is devoted to a discussion of further properties of the algebras $\mathscr{A}_N(T)$.

Throughout T will denote an invertible measure-preserving transformation of a Lebesgue space $(X, \mathscr{B}, m)$ [14]. $\mathscr{N}$ will denote the trivial σ-algebra consisting of those members of $\mathscr{B}$ with measure 0 or 1. ν will denote the trivial partition and ε will denote the partition into points of any Lebesgue space. Greek letters ξ, η, ζ etc. will be used to denote measurable partitions. We shall use partitions and their associated σ-algebras interchangeably. The factor space of X by ζ will be denoted by X/ζ and if $T\zeta=\zeta$ the factor transformation induced by T on X/ζ will be denoted by T_ζ. If $\mathscr{A}$ denotes the σ-algebra generated by the members of ζ then $L^2(\zeta)$ and $L^2(\mathscr{A})$ will both denote the collection of all elements of $L^2(X, \mathscr{B}, m)$ measurable

Received by the editors November 6, 1970.

AMS 1970 *subject classifications.* Primary 28A65, 47A35.

Key words and phrases. Measure-preserving transformation, ergodic, partition, σ-algebra, entropy, mixing, spectral measure, group extension, Gaussian process.

with respect to $\mathscr{A}$. In particular $L^2(\varepsilon)$ and $L^2(\mathscr{B})$ will stand for $L^2(X, \mathscr{B}, m)$. U_T will denote the unitary operator of $L^2(\mathscr{B})$ defined by $f \to f \circ T$ and $\|\cdot\|_2$ will denote the norm on an L^2-space. We shall repeatedly use the spectral theorem which implies that for each $f \in L^2(\mathscr{B})$ there is a Borel measure σ_f on the unit circle K with $(U^n f, f) = \int_K \lambda^n \, d\sigma_f(\lambda)$ $\forall n \in Z$. K will always denote the unit circle. σ_f is called the spectral measure of f.

I would like to thank W. Parry for valuable discussions.

1. **The σ-algebras $\mathscr{A}_N(T)$.** For a sequence $N=\{n_i\}$ of integers let $\mathscr{A}_N(T) = \{A \in \mathscr{B} \mid m(T^{n_i}A \Delta A) \to 0\}$. We show below that $\mathscr{A}_N(T)$ is a σ-algebra. The corresponding partition will be denoted by $\alpha_N(T)$. Let $\mathscr{A}(T) = \bigvee_N \mathscr{A}_N(T)$ (the refinement is taken over all sequences N of integers) and let $\alpha(T)$ denote the corresponding partition. We have $T\mathscr{A}_N(T) = \mathscr{A}_N(T)$, $T\alpha_N(T) = \alpha_N(T)$, $T\mathscr{A}(T) = \mathscr{A}(T)$ and $T\alpha(T) = \alpha(T)$. When T is understood we shall write $\mathscr{A}_N$, α_N, $\mathscr{A}$ and α.

THEOREM 1. *$\mathscr{A}_N(T)$ is a σ-algebra.*

Proof. Clearly $\varnothing \in \mathscr{A}_N$. Since $T^{n_i}(X \backslash A) \Delta (X \backslash A) = T^{n_i}A \Delta A$ $\mathscr{A}_N$ is closed under complementation. $\mathscr{A}_N$ is finitely additive since if $A_1, A_2 \in \mathscr{A}_N$ then

$$T^{n_i}(A_1 \cup A_2) \Delta (A_1 \cup A_2) \subset (T^{n_i}A_1 \Delta A_1) \cup (T^{n_i}A_2 \Delta A_2)$$

implies $A_1 \cup A_2 \in \mathscr{A}_N$. It remains to show that if $A_j \in \mathscr{A}_N$ ($j \geqq 1$) and $A_1 \subset A_2 \subset A_3 \subset \cdots$ then $A = \bigcup_{j=1}^{\infty} A_j \in \mathscr{A}_N$. Let $\varepsilon > 0$ be given. Choose j_0 so that $m(A \backslash A_{j_0}) < \varepsilon$. Choose I so that $i > I \Rightarrow m(T^{n_i}A_{j_0} \Delta A_{j_0}) < \varepsilon$. Then

$$i > I \Rightarrow m(T^{n_i}A \Delta A) \leqq m(T^{n_i}A \Delta T^{n_i}A_{j_0}) + m(T^{n_i}A_{j_0} \Delta A_{j_0}) + m(A_{j_0} \Delta A) < 3\varepsilon.$$

Therefore $A \in \mathscr{A}_N$ and $\mathscr{A}_N$ is countably additive. | |

Our next aim is to show $L^2(\mathscr{A}_N(T)) = \{f \in L^2(\mathscr{B}) \mid \|U_T^{n_i}f - f\|_2 \to 0\}$.

LEMMA 1. *Let $f \in L^2(\mathscr{B})$ be real valued and nonconstant. Let $N=\{n_i\}$ be a sequence of integers. If $\|U^{n_i}f - f\|_2 \to 0$ then $f^{-1}(\mathscr{C}) \subset \mathscr{A}_N(T)$, where $\mathscr{C}$ denotes the σ-algebra of Borel subsets of R.*

Proof. Let $b \in R$. Put $B = \{x \mid f(x) \leqq b\}$ and $B_\varepsilon = \{x \mid f(x) \leqq b + \varepsilon\}$. Let $\delta > 0$ be given. On $T^{-n_i}B \backslash B_\varepsilon$ we have $|f(T^{n_i}x) - f(x)| \geqq \varepsilon$ and therefore $m(T^{n_i}B \backslash B_\varepsilon) \to 0$ as $i \to \infty$. Since $B_\varepsilon \backslash B$ decreases with ε and $\bigcap_{\varepsilon > 0} (B_\varepsilon \backslash B) = \varnothing$, choose ε_0 so that $m(B_{\varepsilon_0} \backslash B) < \delta$. Choose i_0 so that $i > i_0$ implies $m(T^{-n_i}B \backslash B_{\varepsilon_0}) < \delta$. Then $m(T^{-n_i}B \backslash B) \leqq m(T^{-n_i}B \backslash B_{\varepsilon_0}) + m(B_{\varepsilon_0} \backslash B) < 2\delta$ if $i > i_0$. Therefore $m(T^{n_i}B \Delta B) \to 0$. We have shown $f^{-1}(-\infty, b] \in \mathscr{A}_N(T)$ and by Theorem 1 $f^{-1}(\mathscr{C}) \subset \mathscr{A}_N(T)$. | |

THEOREM 2. $L^2(\mathscr{A}_N(T)) = \{f \in L^2(\mathscr{B}) \mid \|U_T^{n_i}f - f\|_2 \to 0\}$.

Proof. Let $\mathscr{H}$ denote the right-hand side. Certainly $L^2(\mathscr{A}_N(T)) \subset \mathscr{H}$. Suppose $f \in \mathscr{H} \backslash L^2(\mathscr{A}_N(T))$. We can assume f is real valued since either the real or imaginary part of f does not belong to $L^2(\mathscr{A}_N(T))$ but belongs to $\mathscr{H}$. By Lemma 1 $f^{-1}(\mathscr{C}) \subset \mathscr{A}_N(T)$ (where $\mathscr{C}$ = Borel subsets of R) and hence $f \in L^2(\mathscr{A}_N(T))$, a contradiction. | |

Our aim is to study the algebras $\mathscr{A}_N(T)$. Of particular interest are those transformations with $\mathscr{A}_N(T)=\mathscr{B}$ ($\alpha_N(T)=\varepsilon$) for some sequence N, those with $\mathscr{A}(T)=\mathscr{B}$ ($\alpha(T)=\varepsilon$) and those with $\mathscr{A}(T)=\mathscr{N}$ ($\alpha(T)=\nu$). The condition $\mathscr{A}_N(T)=\mathscr{B}$ means T^{n_i} converges to the identity in the space of invertible measure-preserving transformations of $(X, \mathscr{B}, m)$ with the weak topology [6] or equivalently $U_T^{n_i}$ converges to I in the space of unitary operators of $L^2(\mathscr{B})$ with the weak (or strong) topology. The following result relates the property $\mathscr{A}_N(T)=\mathscr{B}$ to the maximal spectral type of T. For the theory of spectral measures and types see [13].

THEOREM 3. $\mathscr{A}_N(T)=\mathscr{B} \Leftrightarrow \int_K |\lambda^{n_i}-1|^2\, d\sigma(\lambda)\to 0$ *where σ denotes a finite measure on $K=\{z \mid |z|=1\}$ whose type is the maximal spectral type of T.*

Proof. Suppose the right-hand side holds and $h\in L^1(\sigma)$. We shall show $\int_K |\lambda^{n_i}-1|^2 h(\lambda)\, d\sigma(\lambda)\to 0$. Let $\delta>0$ be given and choose h_1, h_2 so that $h=h_1+h_2$, h_1 is bounded ($|h_1(\lambda)|\leqq c_\delta$ say) and $\int |h_2(\lambda)|\, d\sigma(\lambda)<\delta$.

$$\left|\int |\lambda^{n_i}-1|^2 h(\lambda)\, d\sigma(\lambda)\right| \leqq \int |\lambda^{n_i}-1|^2 |h_1(\lambda)|\, d\sigma(\lambda)+\int |\lambda^{n_i}-1|^2 |h_2(\lambda)|\, d\sigma(\lambda)$$
$$< c_\delta \int |\lambda^{n_i}-1|^2\, d\sigma(\lambda)+4\delta \;<\; 5\delta$$

if $i>i_0$ and i_0 is chosen so that $i>i_0$ implies $\int |\lambda^{n_i}-1|^2\, d\sigma(\lambda)<\delta/c_\delta$. Hence $\int |\lambda^{n_i}-1|^2 h(\lambda)\, d\sigma(\lambda)\to 0$. If $f\in L^2(\mathscr{B})$ the spectral measure σ_f of f is absolutely continuous with respect to σ and by the above $\|U^{n_i}f-f\|_2^2=\int |\lambda^{n_i}-1|^2\, d\sigma_f(\lambda)\to 0$. Hence $f\in L^2(\mathscr{A}_N(T))$ and $\mathscr{A}_N(T)=\mathscr{B}$.

Conversely if $\mathscr{A}_N(T)=\mathscr{B}$ then choosing $f\in L^2(\mathscr{B})$ with spectral measure σ_f of maximal type we have $\int |\lambda^{n_i}-1|^2\, d\sigma_f(\lambda)=\|U^{n_i}f-f\|_2^2\to 0$. By the above, if σ is any measure whose type is the maximal spectral type of T then $\int |\lambda^{n_i}-1|^2\, d\sigma(\lambda)\to 0$.

2. **Some properties of $\mathscr{A}_N(T)$.** The simplest examples of transformations with $\mathscr{A}_N=\mathscr{B}$ are given by

THEOREM 4. *If T is ergodic with discrete spectrum there exists a sequence $N=\{n_i\}$ with $\mathscr{A}_N(T)=\mathscr{B}$.*

Proof. We can suppose T is an ergodic rotation $Tx=ax$ on a compact abelian group G [6, p. 48]. Choose $N=\{n_i\}$ so that $a^{n_i}\to e$ the identity element of G. If γ is a character of G $\|U_T^{n_i}\gamma-\gamma\|_2^2=|\gamma(a^{n_i})-1|^2\to 0$. Since the characters generate $L^2(G)$ we have $\mathscr{A}_N(T)=\mathscr{B}$. | |

Later we shall give more examples of transformations with $\mathscr{A}_N=\mathscr{B}$.

The algebras $\mathscr{A}_N(T)$ are related to the work of Katok and Stepin [7] and Chacon and Schwartzbauer [1] on approximation by periodic transformations. It is easily checked that if T admits an approximation of the second kind by periodic transformations (a.p.t.II) with speed $o(1/n)$ in the sense of Katok and Stepin [7, p. 78] then $\mathscr{A}_{\{p_n\}}(T)=\mathscr{B}$. Also if T admits an approximation by periodic automorphisms in Chacon and Schwartzbauer's sense [1] then $\mathscr{A}_{\{q_n\}}(T)=\mathscr{B}$.

Now we discuss the relationship of the partitions $\alpha_N(T)$ to entropy theory. The notations for entropy are from [15]. Let $\mathscr{Z}$ denote the set of partitions with finite entropy. Pinsker [12] has defined the maximum partition with zero entropy for T as $\pi(T)=\bigvee\{\xi\in\mathscr{Z}\mid h(T,\xi)=0\}$. We have $T\pi(T)=\pi(T)$ and if $\eta\in\mathscr{Z}$ then $\eta\leqq\pi(T)$ if and only if $h(T,\eta)=0$. Using the concept of sequence entropy introduced by Kushnirenko [8], one can define the maximum partition with zero N-entropy for T (for a sequence of integers N) by $\pi_N(T)=\bigvee\{\xi\in\mathscr{Z}\mid h_N(T,\xi)=0\}$. It is straightforward to check that $T\pi_N(T) = \pi_N(T)$ and if $\eta\in\mathscr{Z}$ then $\eta\leqq\pi_N(T)$ if and only if $h_N(T,\eta)=0$. The main result of Kushnirenko's paper [8] implies $\pi_\infty(T)=\bigwedge_N \pi_N(T)$ is the maximum partition for T such that the associated factor transformation has discrete spectrum. In other words $\pi_\infty(T)$ is the partition generated by the eigenfunctions of T.

THEOREM 5. *For every sequence N of integers, $\alpha_N(T)\leqq\pi(T)$ and $\alpha_N(T)\leqq\pi_N(T)$. Hence $\alpha(T)\leqq\pi(T)$.*

Proof. We first show $\alpha_N(T)\leqq\pi_N(T)$. Suppose ξ is a finite partition and $\xi\leqq\alpha_N(T)$. We have

$$H(T^{n_1}\xi\vee T^{n_2}\xi\vee\cdots\vee T^{n_k}\xi)$$
$$\leqq H(T^{n_1}\xi)+H(\xi/T^{n_1-n_2}\xi)+H(\xi/T^{n_2-n_3}\xi)+\cdots+H(\xi/T^{n_{k-1}-n_k}\xi)$$

so if $H(\xi/T^{n_{k-1}-n_k}\xi)\to 0$ then $h_N(T,\xi)=0$ and $\xi\leqq\pi_N(T)$. But $\mathscr{A}_{\{n_i\}}(T)=\mathscr{B}$ implies $\mathscr{A}_{\{n_{i-1}-n_i\}}(T)=\mathscr{B}$ and this readily implies $H(\xi/T^{n_{i-1}-n_i}\xi)\to 0$. Hence $\alpha_N(T)\leqq\pi_N(T)$.

Similarly $\alpha_N(T)\leqq\pi(T)$ since if $\xi\leqq\alpha_N(T)$ is finite $h(T,\xi)=H(\xi/\bigvee_{n=1}^\infty T^n\xi)$ $\leqq\lim_{i\to\infty} H(\xi/T^{n_i}\xi)=0$.

COROLLARY 5.1. $h(T_{\alpha_N(T)})=0$, $h_N(T_{\alpha_N(T)})=0$ *and* $h(T_{\alpha(T)})=0$.

By Theorem 4 we know there exists a sequence N such that $\pi_\infty(T)\leqq\alpha_N(T)$. This and Theorem 5 indicate that the partitions $\alpha_N(T)$ may inherit some of the properties of $\pi_\infty(T)$ and some of the properties of $\pi(T)$. This is shown to be so in the later sections.

THEOREM 6. (a) *$\bigcap_N \mathscr{A}_N(T)$ is the σ-algebra of T-invariant members of $\mathscr{B}$.*

(b) *The class of invertible measure-preserving transformation with $\mathscr{A}_N=\mathscr{B}$ is closed under* (i) *factors,* (ii) *countable direct products, and* (iii) *inverse limits. In fact* (iii) *can be strengthened to the property*:

if $\xi_n\nearrow\xi$ and $T\xi_n=\xi_n$, $T\xi=\xi$, then $\alpha_N(T_{\xi_n})\nearrow\alpha_N(T_\xi)$.

Proof. (a) Let $A\in\bigcap_N\mathscr{A}_N(T)$. Then $m(TA\Delta A)\leqq m(T^{n+1}A\Delta A)+m(A\Delta T^nA)\to 0$.

(b) (i) is trivial.

(ii) Let T_i act on $(X_i,\mathscr{B}_i,m_i)$ and let $T_\infty=\prod_{i=1}^\infty T_i$ acting on $(X,\mathscr{B},m)$ $=\prod_{i=1}^\infty (X_i,\mathscr{B}_i,m_i)$. Assume $\mathscr{A}_N(T_i)=\mathscr{B}_i$ for each i. It is easy to show that each measurable rectangle is in $\mathscr{A}_N(T_\infty)$ and hence $\mathscr{A}_N(T_\infty)=\mathscr{B}$.

(iii) It suffices to take $\xi=\varepsilon$. Let $f\in L^2(\alpha_N(T))$ and put $f_n=E(f/\xi_n)$, where $E(f/\xi_n)$ is the conditional expectation of f relative to the σ-algebra generated by ξ_n. Then $\|f-f_n\|_2\to 0$ and $\|U^{n_i}_{T_{\xi_n}}f_n-f_n\|_2\leqq\|U^{n_i}_Tf-f\|_2\to 0$. Hence $f_n\in L^2(\alpha_N(T_{\xi_n}))$ and $\alpha_N(T_{\xi_n})\to\alpha_N(T_\xi)$.

Let $\mathscr{W}$ denote the class of invertible measure-preserving transformations of $(X,\mathscr{B},m)$ with the weak topology [6]. $\mathscr{W}$ is a complete metric space and hence has the Baire property that a countable intersection of open dense sets is dense. From Theorem 1.1 of [7] it follows that the collection of all transformations with $\mathscr{A}_N=\mathscr{B}$ for some N contains a dense G_δ in the space $\mathscr{W}$. Since the weak-mixing transformations form a dense G_δ in $\mathscr{W}$ ([6]) it follows that the class of all weak-mixing transformations with $\mathscr{A}_N=\mathscr{B}$ for some N contains a dense G_δ in $\mathscr{W}$. It follows from this and the next theorem that the class of weak-mixing transformations which are not strong-mixing contains a dense G_δ in $\mathscr{W}$.

3. **Mixing properties.** An example of a property of $\alpha_N(T)$ inherited from $\pi_\infty(T)$ is

THEOREM 7. *There are no nonconstant mixing functions in* $L^2(\mathscr{A}_N(T))$ (*i.e.* $(U^n_Tf,f)\to(f,1)(1,f)$ *for* $f\in L^2(\mathscr{A}_N(T))$ *implies* $f=$ *constant*).

Proof. If $(U^n_Tf,f)\to(f,1)(1,f)$ and $f\in L^2(\mathscr{A}_N(T))$ then $(f,f)=(f,1)(1,f)$ and f is constant. | |

COROLLARY 7.1. $T_{\alpha_N(T)}$ *has singular spectrum.*

Proof. If the spectrum is not singular there exists $f\in L^2(\mathscr{A}_N(T))$ with absolutely continuous spectral measure σ_f. Then $(U^n_Tf,f)=\int\lambda^n\,d\sigma_f(\lambda)\to 0$ by the Riemann-Lebesgue lemma and since $(U^{n_i}_Tf,f)\to\|f\|_2^2$ we have $f=0$. | |

COROLLARY 7.2. *If T is totally ergodic with quasi-discrete spectrum then* $\alpha_N(T)\leqq\pi_\infty(T)$ *for all sequences N and there exists a sequence N with* $\alpha_N(T)=\pi_\infty(T)$. *Hence* $\alpha(T)=\pi_\infty(T)$.

Proof. $L^2(\varepsilon)=L^2(\pi_\infty(T))\oplus\mathscr{H}$ where $U_T\mathscr{H}=\mathscr{H}$ and $U_T|\mathscr{H}$ has Lebesgue spectrum. Let $f\in L^2(\alpha_N(T))$ and $f=f_1+f_2$, $f_1\in L^2(\pi_\infty(T))$, $f_2\in\mathscr{H}$. Then $\|U^{n_i}_Tf-f\|_2^2=\|U^{n_i}_Tf_1-f_1\|_2^2+\|U^{n_i}_Tf_2-f_2\|_2^2$ implies $f_1\in L^2(\alpha_N(T))$ and $f_2\in L^2(\alpha_N(T))$. By Theorem 7 $f_2=0$ and hence $\alpha_N(T)\leqq\pi_\infty(T)$. The rest of the corollary follows from Theorem 4. | |

Later we shall give examples of weak-mixing transformations with $\mathscr{A}_N(T)=\mathscr{B}$ for some N. Consideration of $\mathscr{A}(T)$ gives an interesting connection with mixing. We first give a definition.

DEFINITION 1. T is intermixing if whenever $m(A)>0$ and $m(B)>0$, $A,B\in\mathscr{B}$, we have $\liminf_{n\to\infty} m(T^nA\cap B)>0$.

Friedman and Ornstein [2] give examples of intermixing transformations which are not strong-mixing.

THEOREM 8.

$$T \text{ strong-mixing} \underset{\not\Leftarrow}{\Rightarrow} T \text{ intermixing} \Rightarrow \mathscr{A}(T) = \mathscr{N} \underset{\not\Leftarrow}{\Rightarrow} T \text{ weak-mixing}.$$

Proof. T strong-mixing $\Rightarrow$ T intermixing is clear. The example of Friedman and Ornstein mentioned above shows the converse is false. If T is intermixing and $0<m(A)<1$, $A \in \mathscr{B}$, then $\lim\inf_{n\to\infty} m(T^nA \cap A^c)>0$ and so $A \notin \mathscr{A}_N(T)$ for any sequence N. Therefore $\mathscr{A}(T)=\mathscr{N}$. Theorem 4 shows $\mathscr{A}(T)=\mathscr{N} \Rightarrow T$ weak-mixing, and the converse is false by the examples of §5. | |

We do not know if $\mathscr{A}(T)=\mathscr{N}$ implies T is intermixing. Pinsker [12] has shown that if $T\xi=\xi$ and $\pi(T_\xi)=\nu$ then $\pi(T)$ and ξ are independent partitions. We shall show the corresponding result for the partitions $\alpha_N(T)$. Two Borel measures (or types of measures) on the unit circle K will be called singular modulo $\{1\}$ if their restrictions to $K-\{1\}$ are singular.

THEOREM 9. *If $\mathscr{H}$ is a U_T-invariant subspace of $L^2(\mathscr{B})$ with $L^2(\mathscr{A}_N(T)) \cap \mathscr{H}=\{0\}$ or the constants, then the maximal spectral types of $U_T|L^2(\mathscr{A}_N(T))$ and $U_T|\mathscr{H}$ are singular modulo $\{1\}$.*

Proof. Let σ be a measure with type equal to the maximal spectral type of $T_{\alpha_N(T)}$ and μ a measure with type the maximal spectral type of $U_T|\mathscr{H}$. If σ and μ are not singular modulo $\{1\}$ there exists a measure τ not concentrated on $\{1\}$ with $\tau \leqq \sigma$ and $\tau \leqq \mu$. As in the proof of Theorem 3, $\int |\lambda^{n_i}-1|^2\, d\tau \to 0$. Let $g \in \mathscr{H}$ have spectral measure τ. g is not constant and $\|U_T^{n_i}g-g\|_2^2=\int |\lambda^{n_i}-1|^2\, d\tau \to 0$ so $g \in L^2(\mathscr{A}_N(T))$, a contradiction. | |

The next corollary is the analogue of the result of Pinsker mentioned above.

COROLLARY 9.1. *Suppose $T\xi=\xi$ and $\alpha_N(T_\xi)=\nu$. Then ξ and $\alpha_N(T)$ are independent partitions.*

Proof. By Theorem 9, T_ξ and $T_{\alpha_N(T)}$ have singular types mod $\{1\}$. Let $f \in L^2(\alpha_N(T))$ and $g \in L^2(\xi)$ both have integral zero. Then f and g have singular spectral types and hence are orthogonal [13, p. 124].

COROLLARY 9.2. *If $\alpha_N(T)=\varepsilon$ then T is disjoint from all strong-mixing transformations.* (*For the definition of disjointness see* [3].)

Proof. By Theorem 8 and Corollary 9.1.

This corollary is a strengthening of Theorem 7. The converse to Corollary 9.2 is false since the transformation of the 2-torus $T(z, w)=(e^{2\pi i a}z, zw)$, where a is irrational, is disjoint from all strong-mixing transformations [3] and yet $\alpha(T)\neq\varepsilon$ by Corollary 7.2 since T is totally ergodic with quasi-discrete spectrum.

4. **Group extensions.** We now investigate how the partitions $\alpha_N(T)$ behave under group extensions.

THEOREM 10. *Let G be a compact abelian metric group acting as a group of measure-preserving transformations of $(X, \mathscr{B}, m)$ such that $gT=Tg$. Let $\xi(G)$ denote the partition of X into orbits of G. If $\alpha_N(T_{\xi(G)})=\nu$ then $T_{\alpha_N(T)}$ is conjugate to a rotation on a factor group of G. (The triviality of this factor group means $\alpha_N(T)=\nu$ and this will occur if T is weak-mixing.)*

Proof. $gT=Tg$ implies $g\alpha_N(T)=\alpha_N(T)$ and so G acts on $X/\alpha_N(T)$. We first show that G acts ergodically on $X/\alpha_N(T)$. Let $\xi(G, N)$ denote the partition of X determined by the partition of the space $X/\alpha_N(T)$ into orbits of G. Then $\xi(G, N)\leqq\xi(G)$ and so $\alpha_N(T_{\xi(G,N)})=\nu$. But $\xi(G, N)\leqq\alpha_N(T)$ and therefore $\xi(G, N)=\nu$. That $T_{\alpha_N(T)}$ is conjugate to a rotation on a factor group of G follows from Lemma 3 of [11]. | |

Results of this nature have been proved about $\pi(T)$ by Parry [11] and Thomas [17].

COROLLARY 10.1. *Suppose T is totally ergodic and G is a finite group acting as measure-preserving transformations of $(X, \mathscr{B}, m)$ so that $gT=Tg$ for each $g\in G$. If $\alpha_N(T_{\xi(G)})=\nu$ then $\alpha_N(T)=\nu$.*

Proof. By Theorem 10, $X/\alpha_N(T)$ is a finite space and the total ergodicity of T implies it is one point. Hence $\alpha_N(T)=\nu$. | |

Let G be a compact connected abelian metric group which acts freely as a group of homeomorphisms on a compact metric space X. Let $T: X\to X$ be a homeomorphism with $gT=Tg$ for every $g\in G$. Suppose T and G preserve a measure m defined on the completion of the Borel subsets of X. T induces a homeomorphism $T_G: X/G\to X/G$ of the orbit space and every lift of T_G to X is of the form $x\to\phi(x)T(x)$ where $\phi\in C_0(X, G)=\{\phi: X\to G \mid \phi$ is continuous and $\phi(gx)=\phi(x)$ $\forall g\in G, x\in X\}$ [4]. $C_0(X, G)$ becomes a complete metric space when endowed with the metric $D(\phi, \psi)=\sup_{x\in X} d(\phi(x), \psi(x))$ where d is an invariant metric for G. T_G preserves the measure on X/G determined by m and the maps $x\to\phi(x)T(x)$ preserve the measure m. An (unpublished) result of Jones and Parry announced in [4] states that if T_G is weak-mixing the set of ϕ making $x\to\phi(x)Tx$ weak-mixing contains a dense G_δ in $C_0(X, G)$. From this, Theorem 8 and Theorem 10 we conclude

COROLLARY 10.2. (i) *If $\alpha_N(T_G)=\nu$ and T_G is weak-mixing the set of $\phi\in C_0(X, G)$ having the property that $x\to\phi(x)Tx$ has $\alpha_N=\nu$ contains a dense G_δ in $C_0(X, G)$.*

(ii) *If $\alpha(T_G)=\nu$ the set of $\phi\in C_0(X, G)$ having the property that $x\to\phi(x)T(x)$ has $\alpha=\nu$ contains a dense G_δ in $C_0(X, G)$.*

We now consider the problem of extending a transformation with $\alpha_N=\varepsilon$ to obtain one with the same property. We shall consider only extensions by $Z^2=\{1, -1\}$. The measure on Z^2 is always taken to be the measure giving weight $\frac{1}{2}$ to each point.

THEOREM 11. *Let $(Y, \mathscr{C}, \mu)$ be a Lebesgue space and let $X=Y\times Z^2$. Define $T: X\to X$ by $T(y, \varepsilon)=(Sy, \phi(y)\varepsilon)$ where $S: Y\to Y$ is measure-preserving and*

$\phi\colon Y \to Z^2$ is measurable. If $\mathscr{A}_N(S)=\mathscr{C}$ then

$$\mathscr{A}_N(T) = \mathscr{B} \Leftrightarrow \mu(\{y \mid \phi(S^{n_i-1}y)\phi(S^{n_i-2}y)\cdots\phi(y) = -1\}) \to 0.$$

Proof. Suppose $\mathscr{A}_N(T)=\mathscr{B}$ and take $f(y, \varepsilon)=\varepsilon$. Then

$$\int |\phi(S^{n_i-1}y)\cdots\phi(y)-1|^2\, d\mu(y) = \|U_T^{n_i}f-f\|_2^2 \to 0 \quad \text{as } i\to\infty$$

and hence $\mu(\{y \mid \phi(S^{n_i-1}y)\cdots\phi(y)=-1\})\to 0$.

Conversely if this condition holds, the above function f belongs to $L^2(\mathscr{A}_N(T))$ and hence $\mathscr{A}_N(T)=\mathscr{B}$. | |

We use this in the following theorem the proof of which comes from ideas in [7].

THEOREM 12. *Let $T\colon K\times Z^2 \to K\times Z^2$ be defined by $T(z, \varepsilon)=(e^{2\pi i a}z, \phi(z)\varepsilon)$ where*

$$\begin{aligned}\phi(z) &= -1 \quad \text{if } \arg z \leqq \gamma 2\pi \\ &= 1 \quad\ \ \text{if } \arg z > \gamma 2\pi\end{aligned} \qquad (0<\gamma<1).$$

Then $\mathscr{A}_N(T)=\mathscr{B}$ if there exist integers p_i and even integers r_i with $(p_i, n_i)=1$, $|a-p_i/n_i| =o(1/n_i^2)$ and $|\gamma-r_i/n_i|=o(1/n_i)$.

Proof. For the proof we shall consider the circle group K as the additive group $[0, 1)$ with addition modulo 1. Set

$$\phi_i(x) = \phi(x)\phi(a+x)\cdots\phi((n_i-1)a+x)$$

and

$$\phi_i^*(x) = \phi(x)\phi(p_i/n_i+x)\cdots\phi(((n_i-1)/n_i)p_i+x);$$

$$\{x \mid \phi_i(x) = -1\} \subset \{x \mid \phi_i^*(x) = -1\} \cup \{x \mid \phi_i(x) \neq \phi_i^*(x)\}.$$

Let μ denote Lebesgue measure on [0, 1).

$$\begin{aligned}\mu(\{x \mid \phi_i(x) \neq \phi_i^*(x)\}) &\leqq \sum_{j=0}^{n_i-1} \mu\{x \mid \phi(ja+x) \neq \phi(jp_i/n_i+x)\} \\ &\leqq \sum_{j=0}^{n_i-1} |ja-jp_i/n_i| \leqq (n_i(n_i+1)/2)o(1/n_i^2) \to 0.\end{aligned}$$

If $n_i\gamma-r_i \geqq 0$ then

$$\begin{aligned}\phi_i^*(x) &= (-1)^{r_i+1} \quad \text{if } \{n_i x\} \leqq n_i\gamma-r_i, \\ &= (-1)^{r_i} \qquad \text{if } \{n_i x\} > n_i\gamma-r_i.\end{aligned}$$

If $n_i\gamma-r_i<0$ then

$$\begin{aligned}\phi_i^*(x) &= (-1)^{r_i} \qquad \text{if } \{n_i x\} \leqq 1+n_i\gamma-r_i, \\ &= (-1)^{r_i-1} \quad \text{if } \{n_i x\} > 1+n_i\gamma-r_i.\end{aligned}$$

Therefore

$$\begin{aligned}\mu(\{x \mid \phi_i^*(x) = -1\}) &= \mu(\{x \mid \{n_i x\} \leqq n_i\gamma-r_i\} \cup \{x \mid \{n_i x\} > 1+n_i\gamma-r_i\}) \\ &\leqq 2|n_i\gamma-r_i| \to 0 \quad \text{as } i\to\infty.\end{aligned}$$

Hence $\mu(\{x \mid \phi_i(x)=-1\}) \to 0$ as $i\to\infty$ and $\mathscr{A}_N(T)=\mathscr{B}$ by Theorem 11. | |

5. **Further examples.** In this section we shall consider some weak-mixing transformations with $\mathscr{A}_N=\mathscr{B}$ for some N. T will be the shift generated by a stationary Gaussian process. Let $X=\prod_{-\infty}^{\infty} R$, $\mathscr{C}$ the product σ-algebra generated by the Borel subsets of R and let p_j denote the jth coordinate function. Hence if $x=\{x_n\}$ then $p_j(x)=x_j$. One assigns a probability measure to $(X, \mathscr{C})$ by requiring that $\{p_j\}$ be a stationary Gaussian process with covariance sequence $R(n)$ where $R(n) =\int_K \lambda^n \, d\mu(\lambda)$ and μ is a finite measure on the unit circle K symmetric with respect to the real axis. μ is called the covariance measure of the process. Let $\mathscr{B}$ denote the completion of $\mathscr{C}$ and let the measure on $\mathscr{B}$ be m. T is then defined by $p_{i-1}(x) =p_i(Tx)$, and is an invertible measure-preserving transformation of $(X, \mathscr{B}, m)$. Hence every symmetric finite Borel measure on K is the covariance measure of a stationary Gaussian process.

THEOREM 13. *Let T be the shift on a stationary Gaussian process with covariance measure μ. Then $\mathscr{A}_N(T)=\mathscr{B} \Leftrightarrow \int |\lambda^{n_i}-1|^2 \, d\mu(\lambda) \to 0$.*

Proof. Suppose $\mathscr{A}_N(T)=\mathscr{B}$. $\int |\lambda^{n_i}-1|^2 \, d\mu(\lambda)=\|U_T^{n_i}p_1-p_1\|_2^2 \to 0$. Conversely $\int |\lambda^{n_i}-1|^2 \, d\mu(\lambda) \to 0$ implies $p_1 \in L^2(\mathscr{A}_N(T))$ and hence $p_k \in L^2(\mathscr{A}_N(T))$ for each k and hence $L^2(\mathscr{A}_N(T))=L^2(\mathscr{B})$. | |

THEOREM 14. *Let μ be a continuous symmetric finite measure concentrated on $D \cup D^{-1}$ where D is a Kronecker subset of K. Let T be the shift on the Gaussian process determined by μ. Then T is weak-mixing and $\mathscr{A}_N(T)=\mathscr{B}$ for some sequence N (and hence is not strong-mixing or intermixing).*

Proof. The conclusion about mixing is in [10]. We have

$$\int_{D\cup D^{-1}} |\lambda^n-1|^2 \, d\mu(\lambda) \leqq \int_D |\lambda^n-1|^2 \, d\mu(\lambda)+\int_{D^{-1}} |\lambda^n-1|^2 \, d\mu(\lambda)$$
$$= 2\int_D |\lambda^n-1|^2 \, d\mu(\lambda).$$

Let $\varepsilon_i \to 0$ and for each i choose $n_i \in Z$ with $\sup_{z\in D} |1-z^{n_i}| < \varepsilon_i$. This is possible since D is a Kronecker set. Then $\int_D |\lambda^{n_i}-1|^2 \, d\mu(\lambda) < \varepsilon_i^2\mu(D) \to 0$ as $i \to \infty$. Therefore $\mathscr{A}_N(T)=\mathscr{B}$ by Theorem 13 if $N=\{n_i\}$. | |

We also note the following

THEOREM 15. *If S is an invertible measure-preserving transformation with $\alpha_N(S) =\varepsilon$ and S does not have discrete spectrum there exists a weak-mixing shift T of a stationary Gaussian process with $\alpha_N(T)=\varepsilon$.*

Proof. Let μ_S denote a measure having type equal to the maximal spectral type of S. μ_S can be chosen symmetric with respect to the real axis. Let μ be its continuous part which is nontrivial (by the assumption that S does not have discrete spectrum) and is symmetric. By the proof of Theorem 3, since $\mu \ll \mu_S$ we have $\int_K |\lambda^{n_i}-1| \, d\mu(\lambda) \to 0$. So letting T be the shift defined on the Gaussian process with covariance measure μ we obtain a transformation with $\alpha_N(T)=\varepsilon$ by Theorem 13, and T is weak-mixing since μ is a continuous measure [9]. | |

Other examples of weak-mixing transformations with $\mathscr{A}_N=\mathscr{B}$ for some N are constructed in [7] by taking transformations induced from rotations of the unit circle.

6. **Problems.** We now discuss whether properties of ergodic transformations with discrete spectrum carry over to ergodic transformations with $\alpha_N(T)=\varepsilon$ for some N. Ergodic transformations with discrete spectrum have simple spectrum and this may account for the fact that some properties do not carry over. We first note that if T is ergodic and $\alpha_N(T)=\varepsilon$ then T need not have simple spectrum, for we could choose T weak-mixing and then $T\times T$ does not have simple spectrum but is ergodic and $\alpha_N(T\times T)=\varepsilon$ (Theorem 6).

If T is ergodic with discrete spectrum then T is coalescent, i.e. if S is measure-preserving and $ST=TS$ then S is invertible. All transformations with simple spectrum have this property but it does not hold for all ergodic T with $\alpha_N(T)=\varepsilon$. Let T acting as $(X, \mathscr{B}, m)$ be weak-mixing and $\alpha_N(T)=\varepsilon$ then $T_\infty=\prod_{i=1}^{\infty} T$ acting on $Y=\prod_{i=1}^{\infty} X$ is ergodic with $\alpha_N(T_\infty)=\varepsilon$ but commutes with the 1-sided shift with state space X.

The main result of [1] is that if T admits an approximation by periodic automorphisms (in the sense of Chacon and Schwarzbauer) and if S is an invertible measure-preserving transformation commuting with T there exists a sequence $\{j_n\}$ of integers such that $m(T^{j_n}A\Delta SA)\to 0$ for every $A\in\mathscr{B}$. This property is, of course, true for an ergodic T with discrete spectrum since every measure-preserving transformation commuting with an ergodic rotation of a compact abelian group is itself a rotation. It is not true in general for ergodic transformations with $\alpha_N=\varepsilon$ as the following example shows. Let T acting on $(X, \mathscr{B}, m)$ be weak-mixing with $\alpha_N(T)=\varepsilon$. Put $T_\infty=\prod_{i=1}^{\infty} T$, $X_\infty=\prod_{i=1}^{\infty} X$, $\mathscr{B}_\infty=\prod_{i=1}^{\infty}\mathscr{B}$, $m_\infty=\prod_{i=1}^{\infty} m$, $S=\prod_{i=1}^{\infty} T^i$. T_∞ and S both act on X_∞, $\alpha_N(T_\infty)=\varepsilon$ and $T_\infty S=ST_\infty$. However there is no sequence $\{j_n\}$ with $m_\infty(T_\infty^{j_n}A\Delta SA)\to 0$ for all $A\in\mathscr{B}_\infty$. However if T admits an approximation by periodic automorphisms in Chacon and Schwartzbauer's sense then T has simple spectrum and so we could pose the following problem that we have been unable to solve. If T has simple spectrum and $\alpha_N(T)=\varepsilon$ for some N and if S is an invertible measure-preserving transformation with $ST=TS$ then does there exist a sequence $\{j_n\}$ of integers with $m(T^{j_n}A\Delta SA)\to 0$ for every $A\in\mathscr{B}$?

Another property enjoyed by an ergodic transformation T with discrete spectrum is that if S is measure-preserving and $ST=TS$ then $\mathscr{A}_M(S)=\mathscr{B}$ for some sequence M. It is possible that if T has simple spectrum and $\mathscr{A}_N(T)=\mathscr{B}$ for some sequence N then each measure-preserving transformation S commuting with T has $\mathscr{A}_M(S)=\mathscr{B}$ for some sequence M. This is false if the condition of simplicity of the spectrum of T is replaced by ergodicity since we could take T to be the 2-sided direct product of a weak-mixing transformation with $\mathscr{A}_N=\mathscr{B}$ and then T commutes with the 2-sided shift S which is invertible and $\mathscr{A}_M(S)=\mathscr{N}$ for every sequence M (Theorem 8).

Another property of ergodic transformations with discrete spectrum is that $\{B \in \mathscr{B} \mid (B, X\backslash B)$ is a generator$\}$ is dense in the metric space $\mathscr{B}$ (mod 0) with the symmetric difference metric [16]. This is also true for totally ergodic transformations with quasi-discrete spectrum [5]. We have been unable to decide whether it is true for ergodic T with $\mathscr{A}_N(T)=\mathscr{B}$ for some N.

7. Noninvertible transformations.

Suppose now that T is a noninvertible measure-preserving transformation of a Lebesgue space $(X, \mathscr{B}, m)$. If we define $\mathscr{A}_N(T)=\{A \in \mathscr{B} \mid m(T^{-n_i}A\Delta A)\to 0\}$ for a sequence $N=\{n_i\}_{i=1}^{\infty}$ of nonnegative integers and let $\alpha_N(T)$ denote the corresponding partition, then $T^{-1}\alpha_N(T)\leqq\alpha_N(T)$ and one can show, as in the proof of Theorem 5, that $\alpha_N(T)\leqq\pi(T)$ for each sequence N. Since $T_{\pi(T)}$ is an invertible measure-preserving transformation with zero entropy we have $T^{-1}\alpha_N(T)=\alpha_N(T)$ mod 0 for each sequence N. Hence to study the algebras $\alpha_N(T)$ for a noninvertible T it suffices to study $\alpha_N(T_{\pi(T)})$ for the invertible transformation $T_{\pi(T)}$.

References

1. R. V. Chacon and T. Schwartzbauer, *Commuting point transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1969), 277–287. MR **39** #2939.

2. N. A. Friedman and D. S. Ornstein, *On mixing and partial mixing* (to appear).

3. H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49. MR **35** #4369.

4. L. W. Green and W. Parry, *Ergodic theory of G-spaces*, Bull. London Math. Soc. (to appear).

5. F. J. Hahn and W. Parry, *Some characteristic properties of dynamical systems with quasi-discrete spectra*, Math. Systems Theory **2** (1968), 179–190. MR **37** #6435.

6. P. R. Halmos, *Lectures on ergodic theory*, Publ. Math. Soc. Japan, no. 3, Math. Soc. Japan, Tokyo, 1956; reprint, Chelsea, New York, 1960. MR **20** #3958.

7. A. B. Katok and A. M. Stepin, *Approximations in ergodic theory*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 81–106 = Russian Math. Surveys **22** (1967), no. 5, 77–102. MR **36** #2776.

8. A. G. Kušnirenko, *Metric invariants of entropy type*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 57–65 = Russian Math. Surveys **22** (1967), no. 5, 53–67. MR **36** #348.

9. G. Maruyama, *The harmonic analysis of stationary stochastic processes*, Mem. Fac. Sci. Kyūsyū Univ. A. **4** (1949), 45–106. MR **11**, 257.

10. D. Newton, *On Gaussian processes with simple spectrum*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **5** (1966), 207–209. MR **34** #868.

11. W. Parry, *Ergodic properties of affine transformations and flows on nilmanifolds*, Amer. J. Math. **91** (1969), 757–771. MR **41** #5595.

12. M. S. Pinsker, *Dynamical systems with completely positive or zero entropy*, Dokl. Akad. Nauk SSSR **133** (1960), 1025–1026 = Soviet Math. Dokl. **1** (1961), 937–938. MR **27** #2603.

13. A. I. Plesner and V. A. Rohlin, *Spectral theory of linear operators*. II, Uspehi Mat. Nauk **1** (1946), no. 1 (11), 71–191; English transl., Amer. Math. Soc. Transl. (2) **62** (1967), 29–175. MR **9**, 43.

14. V. A. Rohlin, *On the fundamental ideas of measure theory*, Mat. Sb. **25 (67)** (1949), 107–150; English transl., Amer. Math. Soc. Transl. (1) **10** (1962), 1–54. MR **11**, 18.

15. V. A. Rohlin, *Lectures on the entropy theory of transformations with invariant measure*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 3–56 = Russian Math. Surveys **22** (1967), no. 5, 1–52. MR **36** #349.

16. ———, *On endomorphisms of compact commutative groups*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 329–340. (Russian) MR **11**, 40.

17. R. K. Thomas, *Metric properties of transformations of G-spaces*, J. London Math. Soc. (to appear).

Mathematics Institute, University of Warwick, Coventry, England