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A METHOD OF SYMMETRIZATION AND APPLICATIONS

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Abstract. In this paper we define a method of symmetrization for plane domains that includes as special cases methods of symmetrization considered by Szegö and by Marcus. We prove that under this method of symmetrization the mapping radius of a fixed point is not decreased. This fact is used to obtain some results concerning covering properties of Bieberbach-Eilenberg functions.

1. **Introduction.** In this paper we define a method of symmetrization for plane domains that generalizes methods of symmetrization considered by Szegö [6] and Marcus [4]. Our main result is that under this method of symmetrization, the mapping radius at a fixed point is not decreased. We will also show that this method of symmetrization preserves certain classes of domains. We conclude with some applications concerning covering properties for the class of Bieberbach-Eilenberg functions.

2. **Definitions and remarks.** The notation used in defining our method of symmetrization will, as far as possible, follow that used by Marcus [4].

Let D be a domain in the plane. In the sequel there will be no loss of generality to assume that $0 \in D$. If the disk $|z| < \rho$ is contained in D then, following Marcus, we define

$$L_\rho(\phi) = \int_E \frac{dr}{r} \tag{2.1}$$

where $E = E_\rho(\phi)$ is the intersection of D with the set $\{z \mid |z| \geqq \rho, \arg z = \phi\}$,

$$R(\phi) = \rho \exp L_\rho(\phi). \tag{2.2}$$

$R(\phi)$ is of course independent of ρ.

Let $A = \{\alpha_k\}_{k=1}^n$ and $B = \{\beta_k\}_{k=1}^n$ be two sequences of real numbers with $|\alpha_k| = 1$. We define

$$R^{(n)}(\phi) = R^{(n)}(\phi, A, B) = \left[\prod_{k=1}^n R(\alpha_k \phi + \beta_k)\right]^{1/n}. \tag{2.3}$$

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DEFINITION 2.1. *Let D be a domain in the plane which contains the origin. The $P_n[A, B]=P_n$ symmetrized set of D, denoted P_nD, is defined by*

$$P_nD = \{z \mid z = re^{i\phi}, 0 \leqq r < R^{(n)}(\phi), 0 \leqq \phi < 2\pi\}.$$

REMARKS. (1) It follows immediately from the definition that P_nD is a simply connected domain in the plane which is starlike with respect to the origin.

(2) If in Definition 2.1 we take $\alpha_k=1$ $(1\leqq k\leqq n)$ and $\beta_k=2\pi/n$, then P_n symmetrization reduces to the method of symmetrization introduced by Marcus [4].

(3) If D is a starlike domain (here and in the sequel starlike means starlike with respect to 0) then in (2.2), $R(\phi)$ is just the distance from 0 to the boundary of D along the ray $\arg z=\phi$. Thus if the boundary of D has the polar representation $r=r(\phi)$ $(0\leqq\phi<2\pi)$, then $r(\phi)=R(\phi)$. Thus for starlike domains, P_n symmetrization generalizes the method of symmetrization considered by Szegö [6].

(4) If we apply Marcus' method of symmetrization with $n=1$ (i.e. radial symmetrization) to D, then the symmetrized domain D^* is defined by

$$D^* = \{z \mid z = re^{i\phi}, 0 \leqq r < R(\phi), 0 \leqq \phi < 2\pi\},$$

where $R(\phi)$ is defined by (2.2). D^* is a starlike domain. If we now apply P_n symmetrization to D^* we obtain precisely the same domain as if we had originally applied P_n symmetrization to D. Thus P_n symmetrization defined above is a composition of radial symmetrization as defined by Marcus and P_n symmetrization for starlike domains. This observation will be important for the proof of our main theorem. Using the special case of the results of Marcus for radial symmetrization, we can reduce the proof of the main theorem to a consideration of P_n symmetrization for starlike domains.

(5) In Definition 2.1 the case $n=2$, $\alpha_1=1$, $\alpha_2=-1$, $\beta_1=0$ and $\beta_2=0$ will be particularly important for our applications. The symmetrized domain P_2D in this case is not only starlike with respect to 0, but has the additional property that the distances from 0 to the boundary of P_2D along the rays $\arg z=\pm\phi$ $(0<\phi<\pi)$ are equal.

3. **Main theorem.** Let $f(z)$ be analytic in the unit disc, $\mathscr{U}$, and let $D=f(\mathscr{U})$. Without loss of generality we may assume $f(0)=0$. If $f(z)$ is univalent then the mapping radius of D at 0, $r(D, 0)=r(D)$, is defined by

$$r(D) = |f'(0)|.$$

If $f(z)$ is not univalent, then $r(D)$ is defined by a limiting process (see [3, p. 79]). Hayman has shown [3, p. 80] that

$$|f'(0)| \leqq r(D) \tag{3.1}$$

with equality if and only if $f(z)$ is univalent. A very powerful tool in the theory of conformal mapping is the fact that if D^* is obtained from D by Pólya, Steiner or Marcus symmetrization then

$$r(D) \leqq r(D^*) \tag{3.2}$$

(see [3, p. 81] and [4, Theorem 3]). We will now prove the analogous result for the method of symmetrization given in Definition 2.1. The proof uses (3.2) for the special case of Marcus' radial symmetrization and a method due to Szegö.

THEOREM 3.1. *Let $f(z)$ be analytic in $\mathcal{U}$ with $f(0)=0$ and let $D=f(\mathcal{U})$. If P_nD is the P_n symmetrized domain of D and P_nD is not the whole plane, then $r(D)\leqq r(P_nD)$.*

Proof. We recall Remark 4 of §2. P_nD is obtained from D by a composition of Marcus symmetrization with $n=1$, S_1, (i.e. radial symmetrization) and P_n symmetrization for starlike domains. If P_nD is not the plane then S_1D is not the plane and hence $r(D)\leqq r(S_1D)$ [4, Theorem 3]. Thus it suffices to show that P_n symmetrization for starlike domains does not decrease the mapping radius. To this end, we may suppose that the boundary of D is an analytic Jordan curve. Indeed if not, and $f(z)$ maps $\mathcal{U}$ 1-1 onto D with $f(0)=0$, then consider the domain D_s $(0<s<1)$ which is the image of $\mathcal{U}$ under $f(sz)$. D_s is starlike with respect to 0, is bounded by an analytic Jordan curve and

$$\lim_{s\to 1} r(D_s) = r(D). \tag{3.3}$$

Consequently if we have proved the result for domains with an analytic boundary, then $r(P_nD)\geqq r(P_nD_s)$ (by the principle of subordination) which together with (3.3) implies $r(P_nD)\geqq r(D)$.

In order to prove our results for starlike domains with analytic boundaries, we will need to make use of some terminology and results concerning condensers in the plane. The reader is referred to [3, Chapter 4] as a convenient reference.

Let E_0 be the complement of D and E_1 a small closed disc with center 0 contained in D. Using a standard argument [3, p. 83] we will deduce our result if we can show that P_n symmetrization decreases the capacity of the condenser (D, E_0, E_1). The capacity of a condenser is defined as

$$I_D(\omega) = \iint_D \left[\left(\frac{\partial\omega}{\partial x}\right)^2+\left(\frac{\partial\omega}{\partial y}\right)^2\right] dx\, dy, \tag{3.4}$$

where $\omega(z)$ is harmonic in $D-E_1$, continuous on $(D-E_1)^-$ with values 0 on E_0 and 1 on E_1. A fundamental fact that our proof requires is that if $u(z)$ is continuous in the extended plane, 0 on E_0, 1 on E_1 and is Lipschitz on compact subsets of D, then

$$I_D(u) \geqq I_D(\omega) \tag{3.5}$$

(see for example [3, p. 65]).

Consider the condenser (P_nD, P_nE_0, P_nE_1) where P_nE_0 is the complement of P_nD and $P_nE_1=E_1$. We will compare the capacity c^* of this condenser with the capacity c of (D, E_0, E_1). Changing to polar coordinates, we have from (3.4) that

$$I_D(\omega) = \iint_D \left[\left(\frac{\partial\omega}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\omega}{\partial\phi}\right)^2\right] r\, dr\, d\phi. \tag{3.6}$$

Let L_λ denote the λ level curve of $\omega(z)$, i.e., $L_\lambda=\{z \mid \omega(z)=\lambda,\ 0\leqq\lambda\leqq 1\}$. Since D is a starlike domain, L_λ is starlike with respect to 0 [6, p. 115]. Let $r=r(\lambda, \phi)$ be the polar equation of L_λ. On L_λ we have

$$\omega(r(\lambda, \phi); \phi) \equiv \lambda \tag{3.7}$$

and hence $\omega_r r_\lambda=1$ and $\omega_r r_\phi+\omega_\phi=0$. Substituting in (3.6), we obtain

$$\begin{aligned} I_D(\omega) &= \int_0^1\int_{-\pi}^{\pi} [r_\lambda^{-2}+r^{-2}(r_\phi/r_\lambda)^2]rr_\lambda\, d\phi\, d\lambda \\ &= \int_0^1\int_{-\pi}^{\pi} \frac{1+(r_\phi/r)^2}{r_\lambda/r}\, d\phi\, d\lambda. \end{aligned} \tag{3.8}$$

For future reference we set

$$I_\lambda = \int_{-\pi}^{\pi} \frac{1+(r_\phi/r)^2}{r_\lambda/r}\, d\phi.$$

The curves L_λ correspond to the curves L_λ^* in the symmetrized condenser (P_nD, P_nE_0, P_nE_1) where L_λ^* is the curve with polar equation

$$\rho = \rho(\lambda, \phi) = \left[\prod_{k=1}^{n} r(\lambda, \alpha_k\phi+\beta_k)\right]^{1/n}. \tag{3.9}$$

Since the L_λ are analytic and starlike, it is clear that the L_λ^* are analytic and mutually disjoint. We define $\omega^*(z)$ in the extended plane by taking $\omega^*(z)=0$ on P_nE_0, $\omega^*(z)=1$ on P_nE_1 and

$$\omega^*(\rho(\lambda, \phi); \phi) = \lambda \tag{3.10}$$

for $z=\rho(\lambda, \phi)e^{i\phi}$ in $P_nD-P_nE_1$. It follows that $\omega^*(z)$ is continuous in the extended plane, 0 on P_nE_0, 1 on P_nE_1 and Lipschitz on compact subsets of $P_nD-P_nE_1$ (in fact (3.7), (3.9) and (3.10) show that ω^* is analytic in $P_nD-P_nE_1$). Consequently from (3.5) we have that $I_{P_nD}(\omega^*)$ is larger than the capacity of (P_nD, P_nE_0, P_nE_1), and so it remains to show that

$$I_{P_nD}(\omega^*) \leqq I_D(\omega). \tag{3.11}$$

To show this inequality we use a modification of an argument due to Szegö [6]. Repeating the argument used to obtain (3.8), we have

$$I_{P_nD}(\omega^*) = \int_0^1\int_{-\pi}^{\pi} \frac{1+(\rho_\phi/\rho)^2}{\rho_\lambda/\rho}\, d\lambda\, d\phi \tag{3.12}$$

where ρ is given by (3.9).

Now, let

$$\begin{aligned} I_\lambda^* &= \int_{-\pi}^{\pi} \frac{1+(\rho_\phi/\rho)^2}{\rho_\lambda/\rho}\, d\phi \\ &= \int_{-\pi}^{\pi} \frac{1+[(1/n)\sum_{k=1}^{n} \alpha_k r_\phi(\lambda, \alpha_k\phi+\beta_k)/r(\lambda, \alpha_k\phi+\beta_k)]^2}{(1/n)\sum_{k=1}^{n} r_\lambda(\lambda, \alpha_k\phi+\beta_k)/r(\lambda, \alpha_k\phi+\beta_k)}\, d\phi. \end{aligned} \tag{3.13}$$

In [6] Szegö proved the inequality

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1+x_k^2}{|y_k|} \geqq \frac{1+((1/n)\sum_{k=1}^{n}|x_k|)^2}{(1/n)\sum_{k=1}^{n}|y_k|}. \tag{3.14}$$

We set

$$x_k = \frac{\alpha_k r_\phi(\lambda, \alpha_k\phi+\beta_k)}{r(\lambda, \alpha_k\phi+\beta_k)}, \qquad y_k = \frac{r_\lambda(\lambda, \alpha_k\phi+\beta_k)}{r(\lambda, \alpha_k\phi+\beta_k)}.$$

Since L_λ is starlike, $r_\lambda > 0$ and hence it follows from (3.13) and (3.14) (recall that $|\alpha_k| = 1$) that

$$I_\lambda^* \leqq \frac{1}{n}\sum_{k=1}^{n}\int_{-\pi}^{\pi}\frac{1+[r_\phi(\lambda, \alpha_k\phi+\beta_k)/r(\lambda, \alpha_k\phi+\beta_k)]^2}{[r_\lambda(\lambda, \alpha_k\phi+\beta_k)/r(\lambda, \alpha_k\phi+\beta_k)]}\, d\phi.$$

Since $r(\lambda, \phi)$ is periodic with period 2π, each integral on the right side of the above inequality equals

$$I_\lambda = \int_{-\pi}^{\pi}\frac{1+[r_\phi(\lambda, \phi)/r(\lambda, \phi)]^2}{r_\lambda(\lambda, \phi)/r(\lambda, \phi)}\, d\phi.$$

Thus $I_\lambda^* \leqq I_\lambda$. Integrating this inequality from 0 to 1 and recalling (3.8) and (3.12), we see that (3.11) is established and the proof is complete.

Remark. With minor modifications in the above argument, we can prove that, more generally, if (D, E_0, E_1) is an admissible condenser, (P_nD, P_nE_0, P_nE_1) is the P_n symmetrized condenser, and if these condensers have respective capacities c and c_n, then $c_n \leqq c$.

4. **Applications for Bieberbach-Eilenberg functions.** Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $\mathscr{U} = \{|z| < 1\}$. $f(z)$ is a Bieberbach-Eilenberg function if $f(z)\cdot f(w) \neq 1$ for any $|z| < 1$ and $|w| < 1$. $\mathscr{B}$ will denote the class of all such functions. We will show that with a suitable choice of the sequences $\{\alpha_k\}$ and $\{\beta_k\}$, P_n symmetrization preserves the class $\mathscr{B}$. First we prove the following lemma the elegant proof of which was supplied by the referee who greatly simplified the authors' original proof.

Lemma. *Let $f(z) \in \mathscr{B}$ and $D = f(\mathscr{U})$. Then*

$$R(\phi)\cdot R(-\phi) \leqq 1 \qquad (0 \leqq \phi < 2\pi)$$

where $R(\phi)$ is defined by (2.2).

Proof. Suppose that D contains the disk $|w| < \rho$, so that D lies in $|w| < 1/\rho$. Let E_1, E_2 denote the sets of real t, such that $\rho < t < 1/\rho$ and $te^{i\phi} \in D$, $te^{-i\phi} \in D$ respectively. Let E_1^* consist of all t for which $t^{-1} \in E_1$. Then by the hypothesis E_1^* and E_2 are disjoint. Thus

$$\int_{E_1}\frac{dt}{t}+\int_{E_2}\frac{dt}{t} = \int_{E_1^*}\frac{dt}{t}+\int_{E_2}\frac{dt}{t} \leqq \int_{\rho}^{1/\rho}\frac{dt}{t} = 2\log\frac{1}{\rho}.$$

Hence

$$\log R(\phi)+\log R(-\phi) = \log \rho+\int_{E_1} \frac{dt}{t}+\log \rho+\int_{E_2} \frac{dt}{t} \leqq 0$$

and the proof is complete.

THEOREM 4.1. *Let $f(z) \in \mathscr{B}$ and $D=f(\mathscr{U})$. Let n be a positive integer and $A=\{\alpha_k\}_{k=1}^{2n}$, $B=\{\beta_k\}_{k=1}^{2n}$ be two sequences of real numbers with $|\alpha_k|=1$, $\alpha_{n+k}=-\alpha_k$ and $\beta_{n+k}=-\beta_k$ $(1 \leqq k \leqq n)$. If $P_{2n}=P_{2n}(A, B)$ and $g(z)=a_1 z+a_2 z^2+\cdots$ $(a_1>0)$ is the conformal map of $\mathscr{U}$ onto $P_{2n}D$, then*

(i) $|g'(0)| \geqq |f'(0)|$,

(ii) $|g(z)|<1$ *for* $z \in \mathscr{U}$ *(in particular* $g(z) \in \mathscr{B}$*)*,

(iii) $g(\mathscr{U})$ *is starlike.*

Proof. From Definition 2.1

$$P_{2n}D = \{w \mid w = re^{i\phi}, 0 \leqq r < R^{(2n)}(\phi)\}$$

where

$$\begin{aligned} R^{(2n)}(\phi) &= \left[\prod_{k=1}^{n} R(\alpha_k\phi+\beta_k) \prod_{k=1}^{n} R(-\alpha_k\phi-\beta_k)\right]^{1/2n} \\ &= \left[\prod_{k=1}^{n} R(\alpha_k\phi+\beta_k)R(-\alpha_k\phi-\beta_k)\right]^{1/2n}. \end{aligned}$$

By the lemma $R^{(2n)}(\phi) \leqq 1$. This proves (ii). (iii) is immediate and (i) follows from (3.1) and Theorem 3.1.

REMARK. Another consequence of the lemma is that Marcus' method of radial symmetrization, S_1, (see Remark 4 of §2) preserves Bieberbach-Eilenberg domains. Indeed if $D=f(\mathscr{U})$ where $f \in \mathscr{B}$ then

$$S_1 D = \{w \mid w = re^{i\phi}, 0 \leqq r < R(\phi)\}.$$

$S_1 D$ is a starlike domain. Using this fact, it is easy to see that since $R(\phi)\cdot R(-\phi) \leqq 1$, the function $g(z)$ that maps $\mathscr{U}$ 1-1 onto $S_1 D$ and satisfies $g(0)=0$ belongs to $\mathscr{B}$.

As an application of Theorem 4.1, we consider a question that was raised by Fekete for the class $\mathscr{S}$ (i.e. the class of analytic and univalent functions in $\mathscr{U}$ with the usual normalization) [2] and solved for this class by Marcus [4]. For the class $\mathscr{B}$ the problem takes the following form. Let $\mathscr{B}(\rho)$ $(0<\rho \leqq 1)$ denote those functions $f(z) \in \mathscr{B}$ with $r(f(\mathscr{U}))=\rho$. ($r(f(\mathscr{U}))$ denotes the mapping radius of $f(\mathscr{U})$ with respect to 0.) Given n rays issuing from the origin at equal angles $2\pi/n$, let $L=L(f)$ denote the linear measure of the intersection of these rays with $f(\mathscr{U})$. What is the minimum value of L over the class $\mathscr{B}(\rho)$? In this direction we have the following result.

THEOREM 4.2. *Let $f(z) \in \mathscr{B}(\rho)$ and let $l_k=l_k(f)$ denote the linear measure of the intersection of $f(\mathscr{U})$ with the ray* $\arg w=2k\pi/n$ $(0 \leqq k \leqq n-1)$. *Then*

$$L(f) = \sum_{k=1}^{n} l_k \geqq n\left[\prod_{k=1}^{n} l_k\right]^{1/n} \geqq n[2-\rho^n-2(1-\rho^n)^{1/2}]^{1/n}/\rho \tag{4.1}$$

and there is a function in $\mathscr{B}(\rho)$ for which equality holds throughout (4.1).

Proof. It is easy to show (see for example [5, p. 224]) that the function $h(z)$ defined by

$$h(z)/(1+h(z))^2 = \rho^n z/(1+z)^2 \qquad (0 < \rho < 1)$$

maps $\mathscr{U}$ conformally onto $\{|w|<1\}$ minus the interval $[\lambda(\rho), 1)$ where

$$\lambda(\rho) = (2-\rho^n-2(1-\rho^n)^{1/2})/\rho^n.$$

Clearly $h(0)=0$ and $h'(0)=\rho^n$ so that $h(z)\in\mathscr{B}(\rho^n)$. We define

$$\mathscr{s}(z) = [h(z^n)]^{1/n}.$$

$\mathscr{s}(z)\in\mathscr{B}(\rho)$ and maps $\mathscr{U}$ conformally onto $\{|w|<1\}$ minus the n segments

$$[[\lambda(\rho)]^{1/n}e^{2k\pi i/n}, e^{2k\pi i/n}) \qquad (0 \leqq k \leqq n-1).$$

For the function $\mathscr{s}(z)$, equality holds throughout (4.1). It remains to show that if $f\in\mathscr{B}(\rho)$ then

$$L(f) \geqq L(\mathscr{s}). \tag{4.2}$$

Let $f\in\mathscr{B}(\rho)$ and $D=f(\mathscr{U})$. We will prove (4.2) by successively applying three special cases of our method of symmetrization. At each stage we will show that the corresponding L has been decreased. First we apply Marcus' method of radial symmetrization to D, obtaining S_1D, where

$$S_1D = \{w \mid w = re^{i\phi}, 0 \leqq r < R(\phi), 0 \leqq \phi < 2\pi\}.$$

By the result of Marcus cited in (3.2), $r(D)\leqq r(S_1D)$. Let $g_1(z)$ be the conformal map of $\mathscr{U}$ onto S_1D with $g_1(0)=0$ and $g_1'(0)>0$. (In the sequel we will refer to the conformal map, $\tau(z)$, of $\mathscr{U}$ onto a domain H with the normalization $\tau(0)=0$, $\tau'(0)>0$ as the *associated mapping function of* H.) $g_1(z)$ satisfies $\rho_1=g_1'(0)=r(S_1D)\geqq\rho$. By the remark following Theorem 4.1, $g_1(z)\in\mathscr{B}$. For each k, $0\leqq k\leqq n-1$, $l_k(g_1)=R(2k\pi/n)$. Marcus has shown [4, p. 625] that for an arbitrary domain D, $R(\phi)\leqq l(\phi)$, where $l(\phi)$ denotes the linear measure of $D\cap\{\arg w=\phi\}$. Hence, $l_k(g_1)\leqq l_k(f)$ and $L(g_1)\leqq L(f)$. Define

$$f_1(z) = (\rho/\rho_1)g_1(z).$$

Since $g_1(z)$ is starlike and $\rho/\rho_1\leqq 1$, $f_1(z)\in\mathscr{B}$. In fact, $f_1(z)\in\mathscr{B}(\rho)$ since $f'(0)=\rho$. Also, $\rho/\rho_1\leqq 1$ implies

$$L(f_1) \leqq L(g_1) \leqq L(f). \tag{4.3}$$

Let $f_1(\mathscr{U})=D_1$. We now apply the symmetrization of Theorem 4.1 to D_1 with $n=1$, $\alpha_1=1$, $\alpha_2=-1$, $\beta_1=0=\beta_2$.

$$P_2D_1 = \{w \mid w = re^{i\phi}, 0 \leqq r < R^{(2)}(\phi), 0 \leqq \phi < 2\pi\}$$

where

$$R^{(2)}(\phi) = (R(\phi)R(-\phi))^{1/2}. \tag{4.4}$$

By Theorem 3.1, $r(P_2D_1)\geqq r(D_1)$. Let $g_2(z)$ be the associated mapping function of P_2D_1. From Theorem 4.1 we have that $|g_2(z)|<1$, $g_2(z)\in\mathscr{B}$ and $\rho_2=g_2'(0)=r(P_2D_1)\geqq r(D_1)=\rho$. From (4.4) it follows that

$$l_k(g_2) = l_{n-k}(g_2) = (l_k(f_1)l_{n-k}(f_1))^{1/2} \qquad (0 \leqq k \leqq n-1)$$

and hence by the arithmetic-geometric mean inequality

$$L(g_2) = \sum_{k=0}^{n-1} l_k(g_2) \leqq \sum_{k=0}^{n-1} l_k(f_1) = L(f_1).$$

Define

$$f_2(z) = (\rho/\rho_2)g_2(z).$$

Repeating the argument given above for $f_1(z)$, we see that $f_2(z)\in\mathscr{B}(\rho)$ and

$$L(f_2) \leqq L(f_1). \tag{4.5}$$

Let $f_2(\mathscr{U})=D_2$. We now apply Szegö's method of n-fold symmetrization to D_2 (see Remark 3 of §2). Let D_2^* be the resulting domain and $g_3(z)$ the associated mapping function. Since D_2 is contained in $\{|w|<1\}$, D_2^* is contained in $\{|w|<1\}$ and therefore $g_3\in\mathscr{B}$. By the theorem of Szegö [6] (which is included in Theorem 3.1)

$$\rho_3 = g_3'(0) = r(D_2^*) \geqq r(D_2) = \rho.$$

D_2^* is n-fold symmetric and hence $l_k(g_3)=l_j(g_3)$ $(0\leqq k,j\leqq n-1)$. Further,

$$l_k(g_3) = (l_1(f_2)\cdots l_n(f_2))^{1/n} \qquad (0 \leqq k \leqq n-1).$$

By the arithmetic-geometric mean inequality we have

$$L(g_3) = \sum_{k=0}^{n-1} l_k(g_3) \leqq \sum_{k=0}^{n-1} l_k(f_2) = L(f_2).$$

If we set $f_3(z)=(\rho/\rho_3)g_3(z)$ then

$$L(f_3) \leqq L(f_2). \tag{4.6}$$

Let $D_3=f_3(\mathscr{U})$ and let D_4 be $\{|w|<1\}$ minus the n segments $[l(f_3)e^{2k\pi i/n}, e^{2k\pi i/n})$ where $l(f_3)=l_k(f_3)$ $(0\leqq k\leqq n-1)$. Let $g_4(z)$ be the associated mapping function of D_4. $g_4\in\mathscr{B}$ and $g_4(\mathscr{U})=D_4\supset D_3$, so by the principle of subordination,

$$\rho_4 = g_4'(0) = r(D_4) \geqq r(D_3) = \rho.$$

Define

$$f_4(z) = (\rho/\rho_4)g_4(z).$$

$f_4(z)\in\mathscr{B}(\rho)$ and again,

$$L(f_4) \leqq L(g_4) = L(f_3). \tag{4.7}$$

From the definition of $g_4(\mathscr{U})$ it follows that $f_4(z)$ maps the unit disk onto the disk $\{|w| < \rho/\rho_4 \leqq 1\}$ slit along the n segments $[(\rho/\rho_4)l(f_3)e^{2k\pi i/n}, (\rho/\rho_4)e^{2k\pi i/n})$ $(0 \leqq k \leqq n-1)$. If $L(\mathscr{s}) > L(f_4)$ then it follows that $f_4(\mathscr{U})$ is properly contained in $\mathscr{s}(\mathscr{U})$. But this is impossible since $r(f_4(\mathscr{U})) = \rho = r(\mathscr{s}(\mathscr{U}))$. Thus $L(\mathscr{s}) \leqq L(f_4)$ and combining (4.3), (4.5), (4.6) and (4.7) we see that (4.2) is established. Since $f \in \mathscr{B}(\rho)$ was arbitrary, the proof is complete.

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