TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 163, January 1972

CONJUGACY SEPARABILITY OF CERTAIN FUCHSIAN GROUPS

BY
P. F. STEBE

Abstract. Let G be a group. An element g is c.d. in G if and only if given any element h of G, either it is conjugate to h or there is a homomorphism ξ from G onto a finite group such that $\xi(g)$ is not conjugate to $\xi(h)$. Following A. Mostowski, a group is conjugacy separable or c.s. if and only if every element of the group is c.d. Let F be a Fuchsian group, i.e. let F be presented as

$$F = (S_1, \dots, S_n, a_1, \dots, a_{2r}, b_1, \dots, b_t;$$
$$S_1^{e_1} = \cdots = S_n^{e_n} = S_1 \dots S_n a_1 \dots a_{2r} a_1^{-1} \dots a_{2r}^{-1} b_1 \dots b_t = 1).$$

In this paper, we show that every element of infinite order in F is c.d. and if $t \neq 0$ or $r \neq 0$, F is c.s.

1. **Introduction.** A. Mostowski [4] defined conjugacy separability and proved that the conjugacy problem can be solved for a finitely presented c.s. group. The author [5] has shown that a free product of c.s. groups is again a c.s. group, and that elements of infinite order in a finite extension of a free group are c.d. In another reference [6], the author has shown that an element conjugate to a cyclically reduced element of length greater than one in a free product of two free groups with a cyclic amalgamated subgroup is c.d.

In addition to the results above, a few more results about conjugacy separability and related concepts will be used in the proof of the main theorem. These results are contained in the sequence of lemmas to follow.

A pair of elements of a group is distinguished if and only if there is a homomorphism from the group onto a finite group such that the images of the given pair of elements are not conjugate. Clearly an element g of a group is conjugacy distinguished if and only if given any element h of the group either g is conjugate to h or the pair $\{g, h\}$ is distinguished.

LEMMA 1.1. *Let G be a group. Let g and h be two nonconjugate elements of G. Let θ be a homomorphism defined on G. If $\theta(g)$ is not conjugate to $\theta(h)$ and either $\theta(g)$ or $\theta(h)$ is c.d. in $\theta(G)$, then the pair $\{g, h\}$ is distinguished.*

Proof. Since $\theta(g) \nsim \theta(h)$ and either $\theta(g)$ or $\theta(h)$ is c.d. in $\theta(G)$, there is a homomorphism φ from $\theta(G)$ onto a finite group such that $\varphi\theta(g) \nsim \varphi\theta(h)$. Since $\varphi\theta(G)$ is finite and $\varphi\theta(g) \nsim \varphi\theta(h)$, the pair $\{g, h\}$ is distinguished in G.

Received by the editors March 12, 1970 and, in revised form, September 14, 1970.

AMS 1970 *subject classifications.* Primary 20H10, 20E25.

Key words and phrases. Group, conjugacy separable group, Fuchsian group, conjugacy problem.

COROLLARY 1.2. *Let G be a group and let g and h be elements of G. If θ is a homomorphism defined on G, $\theta(g) \nsim \theta(h)$ and $\theta(G)$ is c.s., the pair $\{g, h\}$ is distinguished.*

LEMMA 1.3. *Let G be a group. Let a, b, c, and d be elements of G. If $a \sim b$, $c \sim d$ and the pair $\{b, c\}$ is distinguished then the pair $\{a, d\}$ is distinguished.*

Proof. Let $a=b^x$, $d=c^y$. Let ξ be a homomorphism from G onto a finite group such that $\xi(b) \nsim \xi(c)$. Since $\xi(a)=\xi(b)^{\xi(x)}$, $\xi(d)=\xi(c)^{\xi(y)}$, $\xi(a) \sim \xi(b)$ and $\xi(d) \sim \xi(c)$. It follows that $\xi(a) \nsim \xi(d)$. Thus the pair $\{a, d\}$ is distinguished.

COROLLARY 1.4. *Let G be a group. Let g and h be elements of G. If $g \sim h$ and g is c.d. in G, then h is c.d. in G.*

LEMMA 1.5. *Let G be a group. Let γ be an automorphism of G. If g and h are elements of G and the pair $\{\gamma(g), \gamma(h)\}$ is distinguished, then the pair $\{g, h\}$ is distinguished.*

Proof. Let ξ be a homomorphism from G onto a finite group such that $\xi\gamma(h) \nsim \xi\gamma(g)$. Since $\xi\gamma(G)$ is finite, the pair $\{g, h\}$ is distinguished.

COROLLARY 1.6. *Let G be a group. Let γ be an automorphism of G. Let g be an element of G. If $\gamma(g)$ is c.d., then g is c.d.*

Proof. Let h be an element of G not conjugate to g. Clearly, $\gamma(g)$ is not conjugate to $\gamma(h)$. Since $\gamma(g)$ is c.d., the pair $\{\gamma(g), \gamma(h)\}$ is distinguished. By Lemma 1.5, the pair $\{g, h\}$ is distinguished. Thus g is c.d. in G.

In the following sections, references to the book by W. Magnus, A. Karass, and D. Solitar [2] will be made by writing M.K.S.

2. **Certain automorphisms.** In this section, we will investigate certain automorphisms of a free group to obtain three key results to be used in the subsequent arguments. The main results, Corollary 2.6, Lemma 2.12, and Lemma 2.14 will be used to show that the groups presented as

$$F = (a_1, b_1, \ldots, a_n, b_n; (a_1, b_1), \ldots, (a_n, b_n) = 1)$$

are c.s.

First let F be a free group of rank 2 freely generated by its elements a and b. Let t be an integer: Let α_t be the automorphism of F defined by the assignments $a \to b^t a$, $b \to b$. Let β_t be the automorphism defined by interchanging a and b in the definition of α_t. F is the free product of its subgroups generated by a and b. In the following, we think of the syllables of a word in F according to the definition of syllables of a word in the free product of the subgroups generated by a and b.

LEMMA 2.1. *Let F, α_t, and β_t be as above. Let g be an element of F and let n_1 and n_2 be nonzero integers. If $\alpha_{n_1}(g) \sim g$ and $\beta_{n_2}(g) \sim g$ then g is an element of F'.*

Proof. Let $g \equiv a^u b^v \bmod F'$. Then $\alpha_{n_1}(g) \equiv a^u b^{v+un_1} \bmod F'$. Since $\alpha_t(g) \sim g$, $\alpha_t(g) \equiv g \bmod F'$. Since F/F' is a free abelian group generated by the images of a and b, $v = v + un_1$. Since $n_1 \neq 0$, $u = 0$. By symmetry, $v = 0$ and $g \in F'$.

We adopt the commutator notation $(x, y) = x^{-1}y^{-1}xy$.

LEMMA 2.2. *Let F be the free group freely generated by its elements a and b. Let g be an element of F represented by the word $a^{m_1}b^{n_1} \cdots a^{m_r}b^{n_r}$. Let $m_i \neq 0$ for $i > 1$ and $n_i \neq 0$ for $r > i$. Let g be an element of F' and let every m_i and n_i be 0, 1 or -1. If $m_i m_{i+1} \leqq 0$ and $n_i n_{i+1} \leqq 0$, then $g = (a^u, b^v)^z$ where u and v are $+1$ or -1 and z is an integer.*

Proof. The proof of this lemma is a straightforward mathematical induction.

LEMMA 2.3. *Let F, a, and b be as above. Let g be an element of F and let g be represented by the word $a^{m_1}b^{n_1} \cdots a^{m_r}b^{n_r}$ where $m_i \neq 0$ for $i > 1$ and $n_i \neq 0$ for $r > i$. If t is an integer such that $|t| > 2 \max_{1 \leq i \leq r} |n_i|$ and either some $|m_i| > 1$, or for some i, $m_i m_{i+1} > 0$, then $\alpha_t(g)$ contains a syllable b^u, neither the initial nor final syllable of $\alpha_t(g)$, with u a nonconstant linear function of t. If $m_1 < 0$ and $n_r \neq 0$, then $\alpha_t(g)$ is cyclically reduced.*

Proof. The proof of this lemma is by mathematical induction. To aid in the proof, we add that $\alpha_t(g)$ also has a property P_r defined as follows:

P_r. The syllable length of $\alpha_t(g)$ is greater than the number of nonzero m_i. The final syllable of $\alpha_t(g)$ is b^{n_r} if $n_r \neq 0$, $m_r \geqq 0$; $b^{n_r - t}$ if $n_r \neq 0$, $m_r < 0$; a if $n_r = 0$, $m_r > 0$; a if $n_r = 0$, $m_r > 0$; and b^{-t} if $n_r = 0$, $m_r < 0$.

Suppose $r = 1$. If $m_1 = 0$, then $g = b^{n_1}$ and $\alpha_t(g) = b^{n_1}$, so that both the lemma and P_1 are true. If $g = a^{m_1}$ and $m_1 > 0$, then $\alpha_t(g) = b^t a b^t a \cdots b^t a$ where there are m_1 factors $b^t a$. If $g = a^{m_1}$ and $m_1 < 0$, then $\alpha_t(g) = a^{-1}b^{-t} \cdots a^{-1}b^{-t}$ where there are m_1 factors $a^{-1}b^{-t}$. Thus the lemma and P_1 are true if $g = a^{m_1}$. If $g = a^{m_1}b^{n_1}$ and $m_1 > 0$, $\alpha_t(g) = b^t a \cdots b^t a b^{n_1}$, where there are m_1 factors $b^t a$. If $g = a^{m_1}b^{n_1}$ and $m_1 < 0$, $\alpha_t(g) = a^{-1}b^{-t} \cdots a^{-1}b^{-t}b^{n_1}$, where there are m_1 factors $a^{-1}b^{-t}$. Thus both the lemma and P_1 are valid for $g = a^{m_1}b^{n_1}$.

Assume the lemma and P_r for $1 \leqq r \leqq k-1$. Let $g = a^{m_1}b^{n_1} \cdots a^{m_k}b^{n_k}$, where $m_j \neq 0$ for $1 < j \leqq k$ and $n_j \neq 0$ for $1 \leqq j < k$. Let $g_1 = a^{m_1}b^{n_1} \cdots a^{m_{k-1}}b^{n_{k-1}}$. Now $\alpha_t(g) = \alpha_t(g_1)\alpha_t(a^{m_k})b^{n_k}$. We consider two cases.

Case 1. Let m_k be positive. We have $\alpha_t(g) = s_1 \cdots s_v b^t a b^t a \cdots b^t a b^{n_k}$. Here $s_1 \cdots s_v$ are the syllables of $\alpha_t(g_1)$ in order and there are m_k factors $b^t a$. By hypothesis $s_v = b^{n_{k-1}}$ for $m_{k-1} \geqq 0$ and $s_v = b^{n_{k-1}-t}$ for $m_{k-1} < 0$. Since $|t| \geqq 2 \max |n_i| > 2|n_{k-1}|$, $s_v b^t$ is never the identity. Thus the syllables of $\alpha_t(g)$ are $s_1 \cdots s_{v-1}(s_v b^t) a b^t a \cdots b^t a$ or $s_1 \cdots s_{v-1}(s_v b^t) a b^t a \cdots b^t a b^{n_k}$. The syllable length of $\alpha_t(g)$ is at least $v+1$ so that P_{k-1} implies that the syllable length of $\alpha_t(g)$ is greater than the number of nonzero m_i. Thus P_k is true in this case. If $m_k > 1$, there is a syllable b^t at neither extreme position. If $m_k m_{k-1}$ is positive, v is greater than one and $s_v b^t = b^{n_{k-1}+t}$ is a syllable in neither extreme position. If $m_i m_{i-1} > 0$ or $|m_k| > 1$ for some $i \leqq k-1$, one of $s_2, \ldots, s_{v-1}$ is b^u where u is a nonconstant linear function of t. If $m_1 < 0$, $\alpha_t(g_1)$ is cyclically reduced by inductive hypothesis since n_{k-1} is not zero. Since the last syllable of $\alpha_t(g_1)$ is a power of b, the first syllable of $\alpha_t(g_1)$ is a power of a. Thus s_1 is a power of a for $m_1<0$ and the last syllable of $\alpha_t(g)$ is b^{n_k} for $n_k\neq 0$. Thus $m_1<0$ and $n_k\neq 0$ imply that $\alpha_t(g)$ is cyclically reduced.

Case 2. Let m_k be negative. We have

$$\alpha_t(g) = s_1\cdots s_v a^{-1}b^{-t}\cdots a^{-1}b^{-t}\cdots a^{-1}b^{-t}b^{n_k}.$$

As before, $s_1,\ldots,s_v$ are the syllables of $\alpha_t(g_1)$ in order. Since $n_{k-1}\neq 0$, P_{k-1} implies that s_v is a power of b. Thus $s_1\cdots s_v a^{-1}b^{-t}\cdots a^{-1}b^{n_k-t}$ are the syllables of $\alpha_t(g)$ in order. The syllable length of $\alpha_t(g)$ is at least $v+1$. As in Case 1, P_k is true. If m_k is greater than one, b^{-t} is a syllable of $\alpha_t(g)$ in neither extreme position. If $m_k m_{k-1}$ is positive, m_{k-1} is negative. Thus $v\geqq 2$ and P_{k-1} implies that $s_v=b^{n_{k-1}-t}$. If $m_i m_{i-1}$ is positive or $|m_i|$ is greater than one for some $i\leqq k-1$, one of $s_2\cdots s_{v-1}$ equals b^u where u is a nonconstant linear function of t. If $m_1<0$, s_1 is a power of a by the same argument used in Case 1. Since $|t|>2\max|n_i|$, the last syllable of $\alpha_t(g)$ is a nonidentity power of t. Thus $\alpha_t(g)$ is cyclically reduced.

Since m_k cannot be zero for $k>1$, the lemma is proven.

LEMMA 2.4. *Let F, α_t, and β_t be as above. Let g be an element of F. If $\alpha_t(g)=g$ for all elements t of an infinite set of integers T_1 and $\beta_t(g)=g$ for all elements t of an infinite set of integers T_2, then $g=(a,b)^z$ for an integer z.*

Proof. All but finitely many elements of T_1 are greater than twice any exponent appearing in the reduced word representing g. By Lemma 2.3, $\alpha_t(g)=g$ contains a syllable b^u with $|u|$ arbitrarily large for $|t|$ sufficiently large unless every syllable of g equal to a power of $a^{\pm 1}$ and the signs of the exponents of a alternate in the reduced word representing g. By the symmetry between a and b, every syllable of g equal to a power of b is $b^{\pm 1}$ and the signs of the exponents of b alternate in g. By Lemma 2.1, g is an element of F'. By Lemma 2.2, $g=(a^u,b^v)^z$ for integral z and u and v equal to ± 1. An easy computation shows that $u=v=1$.

LEMMA 2.5. *Let F, a, b, α_t and β_t be as above. If $g\in F$, $\alpha_t(g)\sim g$ for an infinite set of integers T_1 and $\beta_t(g)\sim g$ for an infinite set of integers T_2, then $g\sim(a,b)^n$ for an integer n.*

Proof. Let g_1 be a cyclically reduced conjugate of g. Since, by Lemma 2.1, $g\in F'$ it follows that $g_1\in F'$. If $g_1=1$, $g_1=(a,b)^0$ and $g\sim(a,b)^0$. We can restrict our attention to g_1 of syllable length greater than 1. Thus g_1 has a cyclically reduced conjugate g_2 such that the first syllable of g_2 is a^m with $m<0$. The element g_2 may be obtained by cyclically permuting the syllables of g_1. Since $g_2\sim g$, $\alpha_t(g_2)\sim\alpha_t(g)$ so that $\alpha_t(g_2)\sim g_2$ for all $t\in T_1$. By Lemma 2.3, $\alpha_t(g_2)$ is cyclically reduced and hence, for all sufficiently large $t\in T_1$, is obtained by cyclically permuting the syllables of g_2. By Lemma 2.3 the syllables of g_2 are a^{+1}, a^{-1} or b^r and the signs of the exponents on a alternate in g_2. Since g_2 is obtained by cyclically permuting the syllables of g_1, the syllables of g_1 are a^{+1}, a^{-1} or b^r and the signs of the exponents of a alternate in g_1. By symmetry, the syllables of g_1 equal to powers of b are b^{+1} or b^{-1} and the signs of exponents of b alternate in g_1. By Lemma 2.2, $g_1=(a^u, b^v)^z$ where z is an integer and u and v are independently $+1$ or -1. Thus $g_1 \sim (a, b)^n$ for an integer n and so $g \sim (a, b)^n$.

COROLLARY 2.6. *Let F be the free group freely generated by its elements a and b. If $g \in F$, either $g \sim (a, b)^n$ for an integer n or there is an automorphism δ of F such that $\delta((a, b))=(a, b)$ and $\delta(g) \nsim g$.*

Proof. Since $\alpha_t((a, b))=(a, b)$ and $\beta_t((a, b))=(a, b)$, the result follows from Lemma 2.5.

Note that $\alpha_t=\alpha_1^t$ and $\beta_t=\beta_1^t$, and that if g is an element of F and $\alpha_t(g) \sim g$ for an integer t then $\alpha_{nt}(g) \sim g$ for all n.

LEMMA 2.7. *Let F, a, b, α_t, and β_t be as above. If $g \in F$, S is a finite subset of F, $\alpha_t(g) \in S$ for all t in an infinite set of integers T_1 and $\beta_t(g) \in S$ for all t in an infinite set of integers T_2, then $g=(a, b)^n$ for an integer n.*

Proof. Since $\alpha_t(g)$ ranges over the finitely many elements of S for $t \in T_1$, there exist integers n and z in T_1 such that $n \neq z$ and $\alpha_n(g)=\alpha_z(g)$. Thus $\alpha_{n-z}(g)=g$ for $n-z \neq 0$. By the above remarks, $\alpha_t(g)=g$ for all α_t in an infinite cyclic subgroup of the group of α_t. Thus there is an infinite set of α_t fixing g. The symmetric statement holds for β_t. By Lemma 2.4, $g=(a, b)^z$, for an integer z.

THEOREM 2.8. *Let F, a, and b be as above. The element g is conjugate to $(a, b)^n$ for an integer n if and only if $\gamma(g) \sim g$ or g^{-1} for all automorphisms γ of F.*

Proof. According to a theorem of Nielson, M.K.S. [2, Theorem 3.9], $\gamma((a, b)) \sim (a, b)^{\pm 1}$ for all γ in the automorphism group of F. If $g \sim (a, b)^n$ then $\gamma(g) \sim g^{\pm 1}$ for all automorphisms γ of F.

Let $g \in F$ and let $\gamma(g) \sim g$ or g^{-1} for all γ in the automorphism group of F. If $\alpha_1(g) \sim g$ then $\alpha_t(g) \sim g$ for all integers t. If $\alpha_1(g) \sim g^{-1}$, then $\alpha_2(g) \sim g$ so that $\alpha_t(g) \sim g$ for all even integers t. We may replace α by β in the above statements. By Lemma 2.5, g is conjugate to $(a, b)^n$ for some integer n.

In the following lemmas, we will consider free groups of rank greater than 2. Let F_i be a free group of rank 2 freely generated by its elements a_i and b_i. Set $F=F_1 * F_2$. Let $\alpha_{i,t}$ and $\beta_{i,t}$ be automorphisms defined as α_t and β_t on F_i. If γ is an automorphism of F_i, define the automorphism γ^* of F by setting $\gamma^*(a_i)=\gamma(a_i)$, $\gamma^*(b_i)=\gamma(b_i)$, $\gamma^*(a_j)=a_j$ for $j \neq i$, and $\gamma^*(b_j)=b_j$ for $j \neq i$. In the following lemmas, the syllables of a reduced word w representing an element of F are the syllables of w as a word representing an element of the free product $F=F_1 * F_2$. Finally, we let W be the subgroup of F generated by the commutators (a_1, b_1) and (a_2, b_2).

LEMMA 2.9. *If g is an element of F and $\delta^*(g) \sim g$ for each $\delta^*=\alpha^*_{1,t}$, $\beta^*_{1,t}$, $\alpha^*_{2,t}$, or $\beta^*_{2,t}$, then g is an element of W.*

Proof. Clearly we may assume that g is cyclically reduced, for if g_1 is a cyclically reduced conjugate of g, $\delta^*(g_1) \sim g_1$ for every automorphism δ^* mentioned in the hypothesis. The proof is divided into several cases.

Case 1. Let g have syllable length 1, so that $g \in F_i$ for some i. Without loss of generality, assume that $g \in F_1$. Since $\delta^*(g) \sim g$ for $\delta = \alpha_{1,t}$ or $\beta_{1,t}$ and δ^* acts as δ on F_1 for these values of δ, we have $\delta(g) \sim g$ for $\delta = \alpha_{1,t}$ and $\delta = \beta_{1,t}$. By Lemma 2.5, $g \sim (a_1, b_1)^n$ for some integer n.

Case 2. Let g have syllable length greater than 1. Since $\alpha^*_{i,t}$ and $\beta^*_{i,t}$ transform F_j onto F_j for $j=1, 2$, $\alpha^*_{i,t}(g)$ and $\beta^*_{i,t}(g)$ are cyclically reduced. Thus the syllables of every $\alpha^*_{i,t}(g)$ and $\beta^*_{i,t}(g)$ are cyclic permutations of the syllables of g. If s is a syllable of g and $s \in F_i$, there can be only finitely many distinct elements $\alpha_{i,t}(s)$ and $\beta_{i,t}(s)$. By Lemma 2.7, every syllable of g in F_i equals $(a_i, b_i)^n$ for some integer n.

LEMMA 2.10. *Let c be the element $(a_1, b_1)(a_2, b_2)$ of F. Let h be an element of F, and let*

$$h = \prod_{i=1}^{r} (a_1, b_1)^{n_i} c^{m_i}$$

where $n_i \neq 0$, $m_i \neq 0$ for all i, $1 \leqq i \leqq r$. Let γ be the automorphism of F given by the assignments: $a_1 \to a_1$, $b_j \to b_j$, all j and $a_2 \to a_2^{b_2}(a_1, b_1)$. The initial syllable of $\gamma(h)$ is always $(a_1, b_1)^{n_1}$ and the final syllable is always b_2 or $(a_2, b_2)^{-1}b_2$.

Proof. One has

$$\begin{aligned}
\gamma(a_1, b_1) &= (a_1, b_1),\\
\gamma((a_2, b_2)) &= (a_1, b_1)^{-1}b_2^{-1}a_2^{-1}b_2 \cdot b_2^{-1} \cdot b_2^{-1}a_2b_2(a_1, b_1) \cdot b_2,\\
\gamma((a_2, b_2)) &= (a_1, b_1)^{-1}b_2^{-1}(a_2, b_2)(a_1, b_1) \cdot b_2
\end{aligned}$$

and

$$\gamma(c) = b_2^{-1}(a_2, b_2)(a_1, b_1)b_2 = b_2^{-1}(a_1, b_1)^{-1}c(a_1, b_1)b_2.$$

Thus

$$\gamma(a_1, b_1)^{n_i}c^{m_i} = (a_1, b_1)^{n_i}b_2^{-1}(a_1, b_1)^{-1}c^{m_i}(a_1, b_1)b_2.$$

Now if the initial syllable of every $\gamma((a_1, b_1)^{n_i}c^{m_i})$ is $(a_1, b_1)^{n_i}$ and the final syllable is b_2 or $(a_2, b_2)^{-1}b_2$ the result holds for h. We thus consider two cases.

Case 1. $m_i < 0$. In this case

$$\begin{aligned}
\gamma((a_1, b_1)^{n_i}c^{m_i}) = (a_1, b_1)^{n_i}b_2^{-1}(a_1, b_1)^{-1}&\boldsymbol{(a_2, b_2)^{-1}(a_1, b_1)^{-1}}\\
&\cdots \boldsymbol{(a_2, b_2)^{-1}(a_1, b_1)^{-1}}(a_1, b_1)b_2.
\end{aligned}$$

In the expression there are $|m_i| \geqq 1$ factors of the form $(a_2, b_2)^{-1}(a_1, b_1)^{-1}$ printed in boldface. Thus we have

$$\begin{aligned}
\gamma((a_1, b_1)^{n_i}c^{m_i}) = (a_1, b_1)^{n_i}b_2^{-1}(a_1, b_1)^{-1}&\boldsymbol{(a_2, b_2)^{-1}(a_1, b_1)^{-1}}\\
&\cdots (a_2, b_2)^{-1}(a_1, b_1)^{-1} \cdot (a_2, b_2)^{-1}b_2
\end{aligned}$$

where there are $|m_i|-1\geqq 0$ factors of the form $(a_2, b_2)^{-1}(a_1, b_1)^{-1}$. In this case the first syllable is $(a_1, b_1)^{n_i}$ and the last syllable is $(a_2, b_2)^{-1}b_2$.

Case 2. $m_i>0$. In this case

$$\gamma((a_1, b_1)^{n_i}c^{m_i}) = (a_1, b_1)^{n_i}b_2^{-1}(a_1, b_1)^{-1}\mathbf{(a_1, b_1)(a_2, b_2)} \cdots \mathbf{(a_1, b_1)(a_2, b_2)}(a_1, b_1)b_2$$

where there are $m_i\geqq 1$ factors of the form $(a_1, b_1)(a_2, b_2)$ printed in boldface. Thus

$$\gamma((a_1, b_1)^{n_i}c^{m_i}) = (a_1, b_1)^{n_i}b_2^{-1}(a_2, b_2)\mathbf{(a_1, b_1)(a_2, b_2)} \cdots \mathbf{(a_1, b_1)(a_2, b_2)}(a_1, b_1)b_2$$

where there are $m_i-1\geqq 0$ factors of the form $(a_1, b_1)(a_2, b_2)$ printed in boldface. The syllables are just the factors shown with the exception that $b_2^{-1}(a_2, b_2)$ is a single syllable. In this case the initial syllable is $(a_1, b_1)^{n_i}$ and the final syllable is b_2.

LEMMA 2.11. *Let $c=(a_1, b_1)(a_2, b_2)$. If g is conjugate to an element of W either there is an automorphism δ of F such that $\delta(g)$ is not conjugate to an element of W and $\delta(c)$ is conjugate to c or g is conjugate to a power of c.*

Proof. Note that if g is conjugate to an element of W, g is conjugate to a cyclically reduced element of F in W.

Let h be a cyclically reduced conjugate of g in W. Since $(a_2, b_2)=(a_1, b_1)^{-1}c$, h is in the subgroup generated by c and (a_1, b_1). If h is a power of c, there is nothing to prove. Let h be a power of (a_1, b_1), say $h=(a_1, b_1)^n$. Let α be the automorphism of F determined by the assignments

$$b_i \to b_i \quad \text{for all } i,$$
$$a_2 \to a_2, \qquad a_1 \to a_1^{b_1}(a_2, b_2).$$

Now $\alpha(c)$ is conjugate to c and $\alpha((a_1, b_1))=(a_2, b_2)^{-1}c^{b_1}$. For $n\neq 0$, clearly $\alpha((a_1, b_1)^n)$ is not conjugate to an element of W. The remaining case is the case of h not a power of c or (a_1, b_1). In this case there is a conjugate h' of h of the form:

$$h' = \prod (a_1, b_1)^{n_i}c^{m_i}$$

with $n_i\neq 0$ and $m_i\neq 0$. By Lemma 2.10, $\gamma(h')$ has first syllable $(a_1, b_1)^{n_1}$ and final syllable b_2, and so is cyclically reduced. Thus $\gamma(h')$ is not conjugate to an element of W. It follows that $\gamma(g)$ is not conjugate to an element of W. Now $\gamma(c)=c^{(a_1,b_1)b_2}$ so that $\gamma(c')$ is conjugate to c.

In the next three lemmas we turn our attention to the group

$$G = (a_0, a_1, a_2, b_0, b_1, b_2; (a_0, b_0)(a_1, b_1)(a_2, b_2) = 1).$$

LEMMA 2.12. *Let G be the group in the above remark. Let U_i be the subgroup of G generated by a_j, b_j for $j\neq i$. Let V_i be the subgroup of G generated by a_i and b_i. The subgroups U_i and V_i are freely generated by the given generators. For each i, G is the free product of U_i and V_i with a cyclic subgroup C_i amalgamated. If $g\in G$ and g is conjugate to an element of C_i for every i, then g is the identity.*

Proof. The defining relation of G is equivalent to the relation

$$(a_i, b_i)^{-1} = (a_{i+1}, b_{i+1})(a_{i+2}, b_{i+2})$$

where subscripts are taken modulo 3. Thus G is the free product of U_i and V_i with a cyclic amalgamated subgroup generated by (a_i, b_i). Since G is a one relator group, a theorem of W. Magnus, M.K.S. [2, Theorem 4.10] implies that U_i is free of rank 4 and is freely generated by the given generators, and that V_i is free of rank 2 and is freely generated by the given generators.

Let g be an element of G and let g be conjugate to an element of C_i for every i, $i=0, 1, 2$. There are integers n_i such that $g\sim(a_i, b_i)^{n_i}$ for $i=0, 1, 2$. Thus $(a_1, b_1)^{n_1} \sim(a_2, b_2)^{n_2}$ in G. Now $(a_1, b_1)^{n_1}$ and $(a_2, b_2)^{n_2}$ are elements of U_0 and neither element is in the amalgamated subgroup C_0. By a theorem of D. Solitar, M.K.S. [2, Theorem 4.6] $(a_1, b_1)^{n_1}$ is conjugate to $(a_2, b_2)^{n_2}$ in the subgroup U_0. Since U_0 is freely generated by a_1, b_1, a_2, b_2, both n_1 and n_2 must be zero and g must be the identity.

LEMMA 2.13. *Let G, U_i, V_i, and C_i be as in Lemma 2.12. Let W_i be the subgroup of U_i generated by (a_{i+1}, b_{i+1}) and (a_{i+2}, b_{i+2}), where subscripts are taken modulo 3. Let g be an element of G. If g is conjugate to an element of U_i for each i, then g is conjugate to an element of W_i for each i.*

Proof. Let $\alpha^*_{i,t}$ be defined by the assignments $a_i \to b_i^t a_i$, $a_j \to a_j$ for $j\neq i$, $b_j \to b_j$ for all j. Let $\beta^*_{i,t}$ be defined as $\alpha^*_{i,t}$ except that the symbols a and b are interchanged. Clearly, $\alpha^*_{i,t}$ and $\beta^*_{i,t}$ are automorphisms of G.

By hypothesis there exists an $h_i \in U_i$ such that $g\sim h_i$. Thus $\alpha^*_{i,t}(g)\sim g$ and $\beta^*_{i,t}(g)\sim g$ for all i and all t since $\alpha^*_{i,t}$ and $\beta^*_{i,t}$ fix U_j for $j\neq i$. Since each h_j is conjugate to g, we have $\alpha^*_{i,t}(h_j)\sim h_j$ and $\beta^*_{i,t}(h_j)\sim h_j$ for all i and all t. Thus, using subscripts modulo 3, we have $\alpha^*_{i+1,t}(h_i)\sim h_i$ and $\alpha^*_{i+2,t}(h_i)\sim h_i$ in G. But $\alpha^*_{i+1,t}(h_i)$ and $\alpha^*_{i+2,t}(h_i)$ are elements of U_i. Hence, by a theorem of D. Solitar, M.K.S. [2, Theorem 4.6], either h_i is conjugate to an element of C_i or $\alpha^*_{i+1,t}(h_i)\sim h_i$ and $\alpha^*_{i+2,t}(h_i)\sim h_i$ in U_i. Now these relations also hold if α is replaced by β. By Lemma 2.9, h_i and hence g is conjugate to an element of W_i.

LEMMA 2.14. *Let G and U_i be as above. Let $g \in G$. If g is conjugate to an element of U_i for every i, either g is the identity or there is an automorphism δ of G such that $\delta(g)$ is not conjugate to an element of U_i for some i.*

Proof. Since g satisfies the hypotheses of Lemma 2.13, g is conjugate to an element w_0 of W_0. Suppose $w_0 \nsim ((a_1, b_1)(a_2, b_2))^n$. Lemma 2.11 implies that there is an automorphism α of U_0 such that $\alpha(w_0)$ is not conjugate in U_0 to an element of W_0 and $\alpha((a_1, b_1)(a_2, b_2))=y^{-1}(a_1, b_1)(a_2, b_2)y$ for some element y of U_0. We will show that $\alpha(w_0)$ is not conjugate in G to an element w of W_0. Suppose, to obtain a contradiction, that there is an element g_1 of G and an element w of W_0 such that $\alpha(w_0)=w^{g_1}$. Since $\alpha(w_0)$ is not conjugate to w in U_0, it follows from M.K.S. [2, Theorem 4.6, (ii)] that w is in a conjugate of the amalgamated subgroup C_0.

Thus there is an element g_2 of G' and an element h_1 of C_0 such that $\alpha(w_0)=h_1^{g_2}$. By M.K.S. [2, Theorem 4.6, (i)], there is an element h_2 of C_0 and an element u of U_0 such that $\alpha(w_0)=h_2^u$. But h_2 is an element of W_0 so that a contradiction has been obtained. Let β be defined by the equation $\beta(x)=y\alpha(x)y^{-1}$. Now β is an automorphism of U_1 and $\beta((a_1, b_1)(a_2, b_2))=(a_1, b_1)(a_2, b_2)$. We extend β to an automorphism δ of G by setting $\delta=\beta$ on U_0 and $\delta(a_0)=a_0$, $\delta(b_0)=b_0$. Thus δ is an automorphism of G with $\delta(w_0)$ and hence $\delta(g)$ not conjugate in G to an element of W_0. The construction fails only if g is conjugate to an element of C_0.

If g is not the identity, Lemma 2.12 implies that g is not conjugate to an element of C_i for some i. Thus for some i we can obtain an automorphism δ of G by the above construction such that $\delta(g)$ is not conjugate to an element of W_i. By Lemma 2.13, $\delta(g)$ is not conjugate to an element of U_i for some i.

3. **Fuchsian groups.** In this section we will obtain the principal results of the paper.

LEMMA 3.1. *The group* $G=(a, b, c, d; (a, b)=(c, d))$ *is c.s.*

Proof. Let g and h be a pair of nonconjugate elements of G. We will show that the pair $\{g, h\}$ is distinguished in G. By Lemma 1.3, we can assume that g and h are cyclically reduced. Since G is the free product of two free groups with a cyclic amalgamated subgroup, it follows from Theorem 1 of [6] that every cyclically reduced element of length greater than one is c.d. in G. Thus we may assume that g and h have syllable length 1. The proof is divided into two cases.

Case 1. Both g and h are in the same factor of G, say the factor F of G generated by a and b. Let θ be the homomorphism of G onto the factor of G generated by a and b given by the assignments $\theta(a)=a$, $\theta(b)=b$, $\theta(c)=a$ and $\theta(d)=b$. F is c.s. by Theorem 1 of [5]. Since $\theta(g)$ and $\theta(h)$ are not conjugate in F, it follows from Corollary 1.2 that the pair $\{g, h\}$ is distinguished in G.

Case 2. The elements g and h are in different factors of G. Let θ be defined as in Case 1. Without loss of generality, assume that g is in the factor F of G generated by a and b. If $\theta(g)$ is not conjugate to $\theta(h)$, the pair $\{g, h\}$ is distinguished in G as above. Suppose that $\theta(g)$ and $\theta(h)$ are conjugate in F. Now g is not in the amalgamated subgroup. Suppose g is conjugate to an element g' of the amalgamated subgroup. Since g is not conjugate to h, g' is not conjugate to h and g' and h are in the same factor of G. By Case 1, the pair $\{g', h\}$ is distinguished in G. By Lemma 1.3, the pair $\{g, h\}$ is distinguished in G. Thus we may assume that g is not conjugate to an element of the amalgamated subgroup. By Corollary 2.6, there is an automorphism δ of F such that $\delta(g)$ is not conjugate to g but $\delta((a, b))=(a, b)$. Let φ act as θ on the factor of h and as $\theta\delta$ on F. Then $\varphi(g)\nsim\theta(g)$, $\theta(h)=\varphi(h)$ so that $\varphi(h)\nsim\varphi(g)$ in F. Thus the pair $\{g, h\}$ is distinguished.

LEMMA 3.2. *The group* $G=(a_0, a_1, a_2, b_0, b_1, b_2; (a_0, b_0)(a_1, b_1)(a_2, b_2)=1)$ *is c.s.*

Proof. Let us use the notation U_i, V_i, and C_i as defined in Lemma 2.12. Let g be an element of G. We will show that g is c.d. in G. If g is conjugate to a cyclically reduced element of syllable length greater than one in some decomposition of G into the free product of U_i and V_i with C_i amalgamated, g is c.d. in G by Theorem 1 of [6]. Thus we may assume that for each i, g is conjugate to an element of U_i or V_i. The proof is divided into two cases.

Case 1. For some i, g is conjugate to an element of V_i but g is not conjugate to an element of C_i. Without loss of generality, suppose that g is conjugate to an element g' of V_0. Let h be any element of G not conjugate to g. We will show that g and h are distinguished in G. We may assume that h is conjugate to an element h' of syllable length 1 in the decomposition of G into the free product of U_0 and V_0 with C_0 amalgamated. Let θ be the natural homomorphism from G onto $G'=(a_0, a_1, b_0, b_1; (a_0, b_0)(a_1, b_1)=1)$ induced by adding the relations $a_2=b_2=1$ to the relation of G. If $h' \in V_0$, $\theta(h') \nsim \theta(g')$ since θ acts as an isomorphism on V_0. If $h' \in U_0$, $\theta(h') \in \theta(U_0)$ so that, by M.K.S. [2, Theorem 4.6], $\theta(h) \sim \theta(g)$ is possible only if $\theta(h')$ and $\theta(g')$ are each conjugate to an element of the amalgamated subgroup of G'. But the amalgamated subgroup of G' is $\theta(C_0)$. Since θ is an isomorphism on V_0, $\theta(g)$ and hence $\theta(g')$ is not conjugate to an element of $\theta(C_0)$, so that $\theta(g') \nsim \theta(h')$. Since G' is c.s. by Lemma 3.1, it follows from Lemma 1.1 that the pair $\{g', h'\}$ is distinguished in G. By Lemma 1.3, the pair $\{g, h\}$ is distinguished in G. Since h is arbitrary, g is c.d. in G.

Case 2. Suppose $g \neq 1$ but g is conjugate to an element of U_i for every i. By Lemma 2.14, there is an automorphism δ of G such that $\delta(g)$ is not conjugate to an element of U_i for some i. By the above paragraphs, $\delta(g)$ is c.d. in G. By Corollary 1.6, g is c.d. in G.

Thus every nonidentity element of G is c.d. in G. If $g=1$, let $h \nsim g$. Then $h \neq 1$, so h is c.d. in G. Thus the pair $\{g, h\}$ is distinguished. It follows that 1 is c.d. in G and G is c.s.

THEOREM 3.3. *The groups* $G_n=(a_1, \ldots, a_n, b_1, \ldots, b_n; (a_1, b_1)\cdots(a_n, b_n)=1)$ *are c.s.*

Proof. The result has been shown for $n=2$ and $n=3$. Suppose the result holds for all $n<k$ and let $k \geqq 4$. Let G be G_k.

Let $m=[k/2]$, so $m \geqq 2$. Let G be presented as the free product of the groups F_1 generated by $a_1, \ldots, a_m, b_1, \ldots, b_m$ and F_2 generated by $a_{m+1}, \ldots, a_k, b_{m+1}, \ldots, b_k$, with amalgamation of the subgroup generated by $(a_1, b_1)\cdots(a_m, b_m) \in F_1$ identified with the subgroup generated by $(b_k, a_k)\cdots(b_{m+1}, a_{m+1}) \in F_2$. Let $g_1 \in G$. We will show that g_1 is c.d. in G. If g_1 is conjugate to a cyclically reduced element h of length greater than 1 in G, h is c.d. in G by Theorem 1 of [6]. Thus g_1 is c.d. in G. If g_1 is conjugate to a cyclically reduced element h of length 1, let g_2 be any element of G not conjugate to g_1. If g_2 is conjugate to a cyclically reduced element of length greater than one, g_2 is c.d. in G and there is a homomorphism φ from G onto a finite group such that $\varphi(g_1)$ and $\varphi(g_2)$ are not conjugate. Thus we need only consider the case when g_2 is conjugate to a cyclically reduced element of length 1. We consider two subcases:

Case 1. The elements g_1 and g_2 are conjugate to elements h_i, $i=1, 2$, of the same factor of G. Let a_j, b_j be a pair of generators of the other factor of G. Let ξ be the homomorphism from G onto G/N, where N is the normal closure of a_j, b_j in G. Now $G/N \simeq G_{k-1}$ and ξ is an isomorphism from the factor of G containing the h_i onto G_{k-1}.

I claim that $\xi(h_1)$ is not conjugate to $\xi(h_2)$ in $\xi(G)$ so that $\xi(g_1)$ is not conjugate to $\xi(g_2)$ in $\xi(G)$. If both h_1 and h_2 are conjugate to elements of the amalgamated subgroup of G, we can replace them by c^{k_1} and c^{k_2} where $k_1 \neq k_2$ and c is the generator of the amalgamated subgroup of G. Let H be a free group freely generated by x and y. Let η be the homomorphism of G onto H defined as follows: $a_j \to 1$, $b_j \to 1$ for the j selected in the determination of ξ, $a_v \to x$, $b_v \to y$ for a_v, b_v generators of the factor of h_1 and h_2, $b_u \to x$, $a_u \to y$ for a_u, b_u generators of the factors of G not containing h_1 except for $u=j$. Then $\eta=\varphi\xi$ for a homomorphism φ and $\eta(c)=(a, b)$. But $(a, b)^{k_1} \nsim (a, b)^{k_2}$ in H for $k_1 \neq k_2$ so that $\xi(h_1) \nsim \xi(h_2)$. Suppose that at least one of h_1, h_2 is not conjugate to an element of the amalgamated subgroup. Without loss of generality, let h_1 be not conjugate to an element of the amalgamated subgroup. By M.K.S. [2, Theorem 4.6, (i)], there is an element $\hat{h}$ of the amalgamated subgroup of $\xi(G)$ and an element p of the factor of $\xi(G)$ containing $\xi(h_1)$ such that $\xi(h_1)=p^{-1}\hat{h}p$. But since ξ is an isomorphism on the factor $\xi(G)$ containing $\xi(h_1)$, $h_1=\xi^{-1}(p)^{-1}(\hat{h})\xi^{-1}(p)$, and h_1 is conjugate in G to an element of the amalgamated subgroup of G. Since $\xi(h_1)$ is not conjugate to an element of the amalgamated subgroup, $\xi(h_1) \sim \xi(h_2)$ in $\xi(G)$ implies, by M.K.S. [2, Theorem 4.6, (ii)] that there is an element p of $\xi(G)$ such that $\xi(h_1)=\xi(h_2)^p$ and p is the same factor of $\xi(G)$ as $\xi(h_1)$. Since ξ is an isomorphism on the factor of G containing h_1, $h_1=h_2^{\xi^{-1}(p)}$ in G, contrary to hypothesis.

Case 2. The elements g_1 and g_2 are conjugate to elements h_i, $i=1, 2$, in different factors of G. If either h_1 or h_2 is conjugate to an element of the amalgamated subgroup, g_1 and g_2 are conjugate to elements of the same factor. Thus, by Case 1, there is a homomorphism ξ of G onto G_{k-1} such that $\xi(g_1)$ is not conjugate to $\xi(g_2)$ in $\xi(G)$. Thus we assume that neither h_1 nor h_2 is conjugate to an element of the amalgamated subgroup.

Let ξ be defined as in Case 1, where a_j, b_j are generators of the factor of G containing h_2. Now ξ is an isomorphism of the factor of G containing h_1. If $\xi(h_1)$ is conjugate to an element of the amalgamated subgroup, then by M.K.S. [2, Theorem 4.6, (i)], $\xi(h_1)$ is conjugate in its factor to an element of the amalgamated subgroup of $\xi(G)$. Since ξ is an isomorphism on the factor of G containing h_1, h is conjugate to an element of the amalgamated subgroup of G. Since this is impossible, $\xi(h_1)$ is not conjugate to an element of the amalgamated subgroup. If $\xi(h_2) \sim \xi(h_1)$ in $\xi(G)$, then by M.K.S. [2, Theorem 4.6, (ii)], $\xi(h_2)$ is the same factor of $\xi(G)$ as $\xi(h_1)$. Since $\xi(h_2)$ is in the image of the factor of G containing h_2, this is possible only if $\xi(h_2)$ is in the amalgamated subgroup. By the above, $\xi(h_1)$ is not conjugate to an element of the amalgamated subgroup, so that $\xi(h_1) \nsim \xi(h_2)$.

In either Case 1 or Case 2 there is a homomorphism ξ from G onto G_{k-1} such that the $\xi(g_i)$ are not conjugate. Now G_{k-1} is c.s. by the inductive assumption, so there is ψ from G_{k-1} onto a finite group such that $\psi\xi(g_1)$ is not conjugate to $\psi\xi(g_2)$. Thus g_1 is c.d. for all $g_1 \in G$ and G is c.s.

COROLLARY 3.4. *Let F be a torsion free Fuchsian group, i.e., let F have a presentation*

$$(a_1, \ldots, a_{2g}, b_1, \ldots, b_t; a_1 \cdots a_{2g} a_1^{-1} \cdots a_{2g}^{-1} b_1 \cdots b_t = 1).$$

F is c.s.

Proof. If $t \geqq 1$, F is free and hence c.s. If there is no b_i, F is isomorphic to

$$(c_1, \ldots, c_g, d_1, \ldots, d_g; (c_1, d_1) \cdots (c_g, d_g)) = 1,$$

as is well known, and hence F is c.s.

LEMMA 3.5. *Let x be an element of the group $G=(a, b, c, d; (a, b)=(c, d))$. Let r be a nonzero integer. If x commutes with $(a, b)^r$, $x=(a, b)^t$ for some integer t.*

Proof. Let x have syllable length k, so that $x=u_1 \cdots u_k$ where each u_i is in a single factor of G and adjacent u_i are not in the same factor of G. We can assume without loss of generality that u_1 is in the factor generated by a and b. If k is greater than one,

$$(a, b)^r u_1 \cdots u_k = u_1 \cdots u_k (a, b)^r$$

implies that $u_1^{-1}(a, b)^r u_1$ is in the amalgamated subgroup, so that $u_1^{-1}(a, b)^r u_1 =(a, b)^r$. Now since the group generated by a and b is free, and $r \neq 0$, u_1 and (a, b) must generate a cyclic subgroup, so that u_1 commutes with (a, b). By Lemma 2.5, $u_1=(a, b)^t$ so that u_1 and u_2 are in the same factor of G. Thus $k=1$ and $x=u_1$. Then $u_1^{-1}(a, b)^r u_1=(a, b)^r$ implies as above that $x=u_1=(a, b)^t$.

LEMMA 3.6. *Let F be a torsion free Fuchsian group, i.e. let F have a presentation*

$$(a_1, \ldots, a_{2k}, b_1, \ldots, b_t; a_1 a_2, \ldots, a_{2k} a_1^{-1}, \ldots, a_{2k}^{-1} b_1, \ldots, b_t = 1).$$

If x and y are elements of F generating a subgroup S of F, then S is free of rank 1 *or* 2 *or free abelian of rank* 1 *or* 2. *If S is free abelian of rank* 2, $t=0$, $k=1$.

Proof. If $t>0$, F is free and the statement follows. If $t=0$ it is well known and easy to show that F may be presented as

$$(a_1, \ldots, a_k, b_1, \ldots, b_k; (a_1, b_1) \cdots (a_k, b_k) = 1).$$

Suppose x and y do not commute. Then $(x, y) \neq 1$ and $k \neq 1$. It follows from a theorem of K. Frederick [1] that F is residually free. Let ξ be a homomorphism from F onto a free group such that $\xi((x, y)) \neq 1$. Now $\xi(S)$ is free of rank 1 or 2 and since $(\xi(x), \xi(y)) \neq 1$, $\xi(S)$ is free of rank 2 and is freely generated by $\xi(x)$ and $\xi(y)$. Thus x and y can have no nontrivial relation and S is free of rank 2. Now F is torsion free, so that if $(x, y)=1$, S is free abelian of rank 1 or 2. Now suppose S is free abelian of rank 2. We have $t=0$ for $t \neq 0$ would imply that F and hence S is free. We wish to prove that $k=1$, so we assume $k>1$ and obtain a contradiction. According to K. Frederick [1], when $t=0$ and $g>1$, F may be imbedded in the group $G=(a, b, c, d; (a, b)=(c, d))$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be the images of x and y in G. Since the representation is faithful, $\boldsymbol{x}$ and $\boldsymbol{y}$ generate a free abelian subgroup of G of rank 2. According to M.K.S. [2, Theorem 4.5], $\boldsymbol{x}$ and $\boldsymbol{y}$ commute only in the cases:

(1) $\boldsymbol{x}=h^{-1}(a, b)^r h$ or $\boldsymbol{y}=h^{-1}(a, b)^r h$.

(2) $\boldsymbol{x}=g^{-1}vg$, $\boldsymbol{y}=g^{-1}ug$ where u and v are in the same factor and neither is in the amalgamation.

(3) $\boldsymbol{x}=g^{-1}(a, b)^{t_1}gW^{k_1}$, $\boldsymbol{y}=g^{-1}(a, b)^{t_2}gW^{k_2}$ if Cases 1 and 2 do not apply, and $g^{-1}(a, b)^{t_1}g$, $g^{-1}(a, b)^{t_2}g$ and W commute in pairs.

Now since the factors of G have only free cyclic abelian subgroups, Case 2 does not apply.

Consider Case 1. We can take, without loss of generality, $\boldsymbol{x}=h^{-1}(a, b)^r h$ so $h\boldsymbol{x}h^{-1}=(a, b)^r$ and $h\boldsymbol{y}h^{-1}$ commutes with $(a, b)^r$. By Lemma 3.5, $h\boldsymbol{y}h^{-1}=(a, b)^t$. Thus $\boldsymbol{x}$ and $\boldsymbol{y}$ generate a cyclic subgroup of G, so x and y generate a cyclic subgroup of F, contrary to hypothesis.

Consider Case 3. We have

$$\boldsymbol{x} = g^{-1}(a, b)^{t_1}gW^{k_1}, \qquad \boldsymbol{y} = g^{-1}(a, b)^{t_2}gW^{k_2},$$

and W, $g^{-1}(a, b)^{t_1}g$, $g^{-1}(a, b)^{t_2}g$ commute in pairs. If both t_i are zero, $\boldsymbol{x}$ and $\boldsymbol{y}$ generate a cyclic group. If one or more t_i is not zero, assume, without loss of generality that $t_1 \neq 0$. Now gWg^{-1} commutes with $(a, b)^{t_1}$ so that $gWg^{-1}=(a, b)^t$ by Lemma 3.5. Thus

$$W = g^{-1}(a, b)^t g, \qquad \boldsymbol{x} = g^{-1}(a, b)^{t_1+tk_1}g \quad \text{and} \quad \boldsymbol{y} = g^{-1}(a, b)^{t_2+tk_2}g.$$

Thus $\boldsymbol{x}$ and $\boldsymbol{y}$ generate a cyclic group. Thus $\boldsymbol{x}$ and $\boldsymbol{y}$ generate a cyclic group in F, contrary to hypothesis. The lemma follows.

LEMMA 3.7. *Let F be a finitely generated abelian group. Let $G \triangleright F$ be such that $G=\langle a, F\rangle$. Suppose y is an element of F. Either $y=a^{-1}xax^{-1}$ for some x in F or there is a normal subgroup N of finite index in G such that $a \not\equiv h^{-1}ayh$* mod *N for all h in G.*

Proof. Let S be the subset of F consisting of elements of the form $a^{-1}xax^{-1}$. Since F is abelian $(a^{-1}xax^{-1})^{-1}=xa^{-1}x^{-1}a=a^{-1}x^{-1}ax=a^{-1}x^{-1}a(x^{-1})^{-1}$, and $a^{-1}xax^{-1}a^{-1}zaz^{-1}=a^{-1}xa\cdot a^{-1}za\cdot x^{-1}z^{-1}=a^{-1}xzaz^{-1}x^{-1}$. Thus S is a subgroup of F. Now S is normal in G, for if $u \in G$, $u=va^r$ for $v \in F$ and, if $x \in F$, $u^{-1}(a, x)u =a^{-r}v^{-1}(a, x)va^r=a^{-r}(a, x)a^r=(a, x^{a^r})$.

Suppose y is not in S but there is an $h \in G$ such that $a \equiv h^{-1}ayh \bmod S$. Now $h = va^r$ for $v \in F$ and some integer r so that $a \equiv a^{-r}v^{-1}ayva^r \bmod S$ and since S is normal in G, $a \equiv v^{-1}ayv \bmod S$. Thus $v^{-1}a^{-1}va = v^{-1}yv \bmod S$ and $v^{-1}yv \in S$. Since S is normal, $y \in S$, contrary to hypothesis. Now let ξ be the natural homomorphism from G to G/S. One has that $\xi(y)$ is not conjugate to $\xi(ay)$ and G/S is c.s., since it is a finitely generated abelian group. There is a homomorphism ψ from G/S onto a finite group such that $\psi\xi(y)$ is not conjugate to $\psi\xi(ay)$. If N is the kernel of $\psi\xi$, G/N is finite and $a \neq h^{-1}ayh \bmod N$ for all $h \in G$.

LEMMA 3.8. *Let F be a finitely generated abelian group. Let $G \rhd F$ be such that $[G:F] < \infty$. G is c.s.*

Proof. Let $a \in G$. By Lemma 1 of [5], it is sufficient to prove that a is c.d. in $H = \langle a, F\rangle$ since $[G:H] < \infty$. Let $b \in H$. If $b \not\equiv a \bmod F$, the images of a and b are not conjugate in the finite group H/F. Thus suppose $b \equiv a \bmod F$, so that $b = ay$ for $y \in F$. According to Lemma 3.7, either $y = a^{-1}xax^{-1}$ so that $b = ay$ is conjugate to a or there is a normal subgroup N of finite index in H such that $a \nsim ay \bmod N$. Thus a is c.d., in H and hence in G.

THEOREM 3.9. *Elements of infinite order in a finite extension of a torsion free Fuchsian group are c.d.*

Proof. Let G be a group containing a normal subgroup F of finite index. Let F be a torsion free Fuchsian group. The proof that every element of infinite order in G is c.d. is split into several cases. Let

$$F = (a_1, \ldots, a_{2g}, b_1, \ldots, b_t; a_1, \ldots, a_{2g}a_1^{-1}, \ldots, a_{2g}^{-1} \cdot b_1, \ldots, b_t = 1).$$

Case 1. In Case 1, either g is greater than one or if g is 1, t is not zero. In this case, by Lemma 3.6, every two generator subgroup of F is free of rank 1 or 2.

Let a be an element of G. The subgroup H generated by F and a in G is of finite index in G. By Lemma 1 of [5], it is sufficient to show that a is c.d. in H.

Let b be an element of H not conjugate to a. If $a \not\equiv b \bmod F$, then the images of a and b under the natural mapping from H to H/F are not conjugate in the abelian group H/F. Let a have order n modulo F. If a^n is not conjugate to b^n in H, there is a homomorphism ξ from H to a finite group U such that $\xi(a^n)$ is not conjugate to $\xi(b^n)$ in U. The existence of ξ follows from the fact that a^n is c.d. in F and hence in H. But we have $\xi(a)$ is not conjugate to $\xi(b)$ in U.

We will show that $a \equiv b \bmod F$ and a^n conjugate to b^n implies that a is conjugate to b. Now a^n conjugate to b^n implies that there is an integer r and an element x or F such that $x^{-1}a^{-r}a^na^r$, $x = b^n$ or $(a^n)^x = b^n$. If we set $a_1 = a_x$, we have $a_1^n = b^n$ and $a_1 \equiv a \equiv b \bmod F$. Thus there is a y in F such that $b = a_1y$, and $a_1^n = b^n = (a_1y)^n$. Since a_1y commutes with a_1^n, y commutes with a_1^n. By Lemma 3.6, y and a_1^n generate a free cyclic subgroup of F. There is an element f in F such that $f^k = a_1^n$, $f^m = y$, with $f \neq 1$ and $k \neq 0$ since a is of finite order in G. Let U be the subgroup of F

generated by f and f^a. U is free of rank at most 2. If U is free of rank 2, $a_1^{-1}fa_1$ and f are free generators of U. This is impossible, for the kth powers of the generators are equal, since $k \neq 0$. Thus U is free of rank 1, and is free cyclic. If an element of a free cyclic group has a kth root, that root is unique. Thus $f = a_1^{-1}fa_1$ and f and a_1 commute. Now $y = f^m$ so that y and a_1 commute. Thus $a_1^n = (a_1 y)^n$ implies that $y^n = 1$. Since y is in F and F is torsion free, $y = 1$ and $b = a_1 = a^x$. Thus a and b are conjugate in H.

Case 2. In this case there are no b_i in the presentation of F and $g = 1$. It follows that F is free abelian of rank 2. By Lemma 3.8, G is c.s. so that every element of G is c.d. in G.

THEOREM 3.10. *Let F be a Fuchsian group. That is, let F be presented*:

$$(S_1, \ldots, S_n, a_1, \ldots, a_{2g}, b_1, \ldots, b_t;$$
$$S_1, \ldots, S_n a_1, \ldots, a_{2g} a_1^{-1}, \ldots, a_{2g}^{-1} b_1, \ldots, b_t = S_1^{e_1} = \cdots = S_n^{e_n} = 1).$$

If $t > 0$ or $g > 0$, F is c.s. If $t = 0$, $g = 0$ every element of infinite order in F is c.d.

Proof. According to a theorem of J. Mennicke [4], F is a finite extension of a torsion free Fuchsian group, so that by Theorem 3.9, every element of infinite order in F is c.d.

If $t \neq 0$, F is a free product of a free group and finitely many finite cyclic groups, or if $g = 0$, a free product of finitely many finite groups. By Theorem 2 of [5], F is c.s. If $n = 0$, F is torsion free and is c.s. by Corollary 3.4. Thus we must consider only the cases in which $n \neq 0$, $t = 0$.

Let us consider the case $n > 1$, $g \neq 0$, $t = 0$. In this case, F can be written as the free product of the subgroups generated by the S_i and $a_1, \ldots, a_{2g}$ with a cyclic amalgamated subgroup, i.e.

$$F = (S_1, \ldots, S_n, a_1, \ldots, a_{2g}; S_1^{e_1} = \cdots$$
$$= S_n^{e_n} = 1, S_n^{-1}, \ldots, S_1^{-1} = a_1, \ldots, a_{2g} a_1^{-1}, \ldots, a_{2g}^{-1}).$$

The elements of finite order in F are, according to M.K.S. [2, Corollary 4.4.5], conjugates of elements of syllable length 1 and hence conjugates of elements the factor of F generated by the S_i. But the factor of F generated by the S_i is a free product so that it follows from M.K.S. [2, Corollary 4.4.1] that the elements of finite order in F are conjugates of powers of the S_i. Let g_1 and g_2 be two nonconjugate elements of F. If either g_1 or g_2 is of infinite order in F, it is c.d. in F so that there is a homomorphism ξ from F onto a finite group such that $\xi(g_1)$ is not conjugate to $\xi(g_2)$. Thus we can assume g_1 and g_2 are of finite order in F, so $g_1 = h_1^{-1} S_{n_1}^{k_1} h_1$, $g_2 = h_2^{-1} S_{n_2}^{k_2} h_2$. If $g_1 = 1$, it follows from the fact that F is residually finite that there is a homomorphism ξ from F to a finite group such that $\xi(g_1) \neq \xi(g_2)$. We assume $k_1 \neq 0$, and $|k_1| < |e_1|$. Let $\hat{F}$ be the group

$$(a, b, c; c = (a, b), (c, a) = (c, b) = 1, a^{e_1} = b^{e_1} = 1).$$

It is well known that the order of c in this group is e_1 and that c is central, so no two different powers (mod e_1) of c are conjugate. It is also clear that $\hat{F}$ is finite. If we add the relations $a_1=1$, $i>2$, $a_1^{e_1}=b_1^{e_1}=1$, $s_i=1$, $i\neq n_1$,

$$(a_1, (a_1, b_1)) = (b_1, (a_1, b_1)) = 1$$

to F, we obtain a group isomorphic to $\hat{F}$. Thus there is a homomorphism ξ from F onto $\hat{F}$ such that $\xi(S_{n_1})=c$, so that $\xi(g_1)=\xi(h_1)^{-1}c^{k_1}\xi(n_1)=c^{k_1}$ and $\xi(g_2)=\xi(h_2)^{-1}c^{k_2}\xi(h_2)^{-1}c^{k_2}\xi(h_2)=c^{k_2}$ if $n_2=n_1$ and $\xi(g_2)=1$ if $n_2\neq n_1$. If $n_2\neq n_1$, $\xi(g_2)=1$ and $\xi(g_2)$ is not conjugate to $\xi(g_1)\neq 1$. If $n_1=n_2$, we have $k_1\neq k_2$ since g_1 and g_2 are not conjugate. But in this case, c^{k_1} is not conjugate to c^{k_2}. Thus g_1 is c.d. in F for all $g_1 \in F$, so F is c.s.

If $n=1$, $g>1$, $t=0$, F is again a free product,

$$F = (S_1, a_1, \ldots, a_g, b_1, \ldots, b_g;\ S_1^{e_1} = 1, S_1(a_1, b_1) = (b_g, a_g)\cdots(b_2, a_2))$$

and the elements of finite order are conjugates of powers of S_1, so that the homomorphism into F described above will show that $\hat{F}$ is c.s.

If $n=1$, $g=1$, $t=0$, the group may be presented as $F=(a, b; (a, b)^n=1)$. By M.K.S. [2, Theorem 4.13], the elements of finite order are conjugates of powers of $(a, b)^n$ so that the homomorphism of F onto $\hat{F}$ described above will show that F is c.s.

References

1. K. N. Frederick, *The Hopfian property for a class of fundamental groups*, Comm. Pure Appl. Math. **16** (1963), 1–8. MR **26** #6948.

2. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Pure and Appl. Math., vol. 13, Interscience, New York, 1966. MR **34** #7617.

3. J. Mennicke, *Eine Bemerkung über Fuchssche Gruppen*, Invent. Math. **2** (1967), 301–305. MR **34** #7666.

4. A. W. Mostowski, *On the decidability of some problems in special classes of groups*, Fund. Math. **59** (1966), 123–135. MR **37** #292.

5. P. Stebe, *A residual property of certain groups*, Proc. Amer. Math. Soc. **26** (1970), 37–42. MR **41** #5494.

6. ———, *Conjugacy separability of certain free products with amalgamation*, Trans. Amer. Math. Soc. **156** (1971), 119–129.

Communications Research Division, Institute for Defense Analyses, Princeton, New Jersey 08540

Current Address: Department of Mathematics, City College, City University of New York, New York, New York 10031