A THEOREM OF COMPLETENESS FOR FAMILIES OF COMPACT ANALYTIC SPACES

BY

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Abstract. A sufficient condition is given for a family of compact analytic spaces to be complete. This condition generalizes to analytic spaces the Theorem of Completeness of Kodaira and Spencer [6]. It contains, as a special case, the rigidity theorem proved by Schuster in [11].

Introduction. If X is a family of deformations of a compact complex manifold, X_0, parametrized by a germ of complex manifold, $(S, 0)$, Kodaira and Spencer have proved in [6] that X contains all small deformations of X_0 (i.e. is (locally) complete) provided that the Kodaira-Spencer map $\rho_0: T_0(S) \to H^1(X_0, \Theta_0)$ is surjective. This condition is equivalent to the statement that X contains all deformations of X_0 parametrized by the analytic space, D, whose underlying topological space is a point, and whose structure sheaf is the ring of dual numbers. The paper [6] appeared before Kuranishi's Completeness Theorem [7], [8]. A very simple proof can be given of the Kodaira-Spencer Theorem if we use Kuranishi's result. We give this proof in §1.

If X_0 has singularities, it is quite plausible that an analog of the Kuranishi space exists; but, at the moment, no proof exists. A formal analog of the Kuranishi space can be shown to exist, however. In this paper we will prove a generalization of some results of Artin [1] which will allow us to prove the completeness theorem using this formal analog.

The Theorem of Completeness is useful in answering questions of whether a certain property of X_0 is shared by all small deformations of X_0—for it allows one to recognize that a family of spaces having the special property is complete. We have used the theorem in this way in our paper [13]. Some applications are given in §4.

I am indebted to Michael Artin for a sketch of the proof of Theorem 2.2.

1. **Quasi-representable functors.** By a family of compact complex analytic spaces, we mean a pair of analytic spaces X, S together with a flat, proper morphism $\pi: X \to S$. If X_0 is a given compact analytic space, a family of deformations of X_0 is a family, (X, π, S), of compact spaces together with a distinguished point $s_0 \in S$

and a fixed isomorphism $i: X_0 \to X_{s_0}$ (X_{s_0} is the fibre of π over s_0). An isomorphism of families of deformations is required to be compatible with the given isomorphisms i. Since all considerations are local with respect to S, we identify a family with its restriction to any open neighborhood of the distinguished point s_0. Thus, following Grothendieck [4, 16–06], we define a contravariant functor, $\mathscr{F}$, from the category of germs of analytic spaces to the category of sets by

$$\mathscr{F}(S, 0) = \text{set of isomorphism classes of germs of families of}$$
$$\text{deformations of } X_0 \text{ parametrized by } (S, 0).$$

If $(S', 0) \to (S, 0)$ is a morphism, the map $\mathscr{F}(S) \to \mathscr{F}(S')$ is given by the fibre product. We call $\mathscr{F}$ the deformation functor of X_0.

An analytic algebra is a local C-algebra which is the quotient of a ring $C\{x_1, \ldots, x_n\}$ of convergent power series. The category $\mathscr{A}$ of analytic algebras (the morphisms being C-algebra homomorphisms) is dual to the category of germs of analytic spaces [4, 13–02]. Thus, alternatively, $\mathscr{F}$ may be considered as a covariant functor, F, from the category $\mathscr{A}$ to the category of sets.

In general, the functor F will not be representable—even if X_0 is a manifold (see [14]). However, Kuranishi has essentially proved the following (see [7], [8] and [3]):

THEOREM 1.1 (KURANISHI). *If X_0 is a compact complex manifold, F the deformation functor of X_0, then there is a pair (A, ξ) with $A \in \mathscr{A}$, $\xi \in F(A)$ such that $h_A(B) \to F(B), f \mapsto F(f)(\xi)$ is*

(i) *surjective for all $B \in \mathscr{A}$,*

(ii) *bijective for $B = C[t]/(t^2)$,*

where $h_A(B) = \text{Hom}_{\mathscr{A}}(A, B)$.

Notation. We will always use D to denote $C[t]/(t^2)$ or the corresponding germ of analytic space.

For $A \in \mathscr{A}$, $h_A(D)$ is naturally isomorphic to the Zariski tangent space of the analytic germ corresponding to A. If X_0 is a complex manifold, $F(D)$ may be identified with $H^1(X_0, \Theta_0)$ (see [4] and [14]). $h_A(D) \to F(D)$ is, with these identifications, the Kodaira-Spencer map. Thus, for arbitrary X_0, $h_A(D) \to F(D)$ plays the role of the Kodaira-Spencer map. $F(D)$ has, in fact, a vector space structure (see [10]), as does $h_A(D)$; the map is a vector space homomorphism.

DEFINITION 1.2. A covariant functor, F, from $\mathscr{A}$ to (Sets) is called *quasi-representable* if there is a pair (A, ξ), as in Theorem 1.1, such that (i), (ii) hold. The pair (A, ξ) is said to *quasi-represent F*. If only (i) is satisfied, (A, ξ) is called *complete*.

Notation. If $A \in \mathscr{A}$, let $A^{(n)} = A/\mathfrak{M}_A^{n+1}$. If $f: A \to B$ is a morphism in $\mathscr{A}$, let $f^{(n)}: A^{(n)} \to B^{(n)}$ denote the induced morphism. If $\xi \in F(A)$, F a covariant functor, let $\xi^{(n)}$ denote the image of ξ in $F(A^{(n)})$.

We will now collect some results about morphisms in $\mathscr{A}$.

PROPOSITION 1.3. *If $f: A \to B$ is a morphism in $\mathscr{A}$ such that $f^{(1)}$ is surjective, then f is surjective.*

Proof. Take a basis for $\mathfrak{M}_A/\mathfrak{M}_A^2$ and pick representatives in $\mathfrak{M}_A$. We obtain an epimorphism $C\{x\} \to A$ which is an isomorphism mod $\mathfrak{M}_A^2$. Thus, for $A \in \mathscr{A}$, we may find a regular $\tilde{A} \in \mathscr{A}$ and an epimorphism $\alpha: \tilde{A} \to A$ such that $\alpha^{(1)}$ is an isomorphism.

In the situation above, the morphism f may be extended to a morphism $\tilde{f}: \tilde{A} \to \tilde{B}$. Since $f^{(1)}$ is surjective, so is $\tilde{f}^{(1)}$. It follows from the Implicit Function Theorem that $\tilde{f}$ is surjective ($\tilde{A}$, $\tilde{B}$ are convergent power series rings). Hence f is surjective.

PROPOSITION 1.4. *If A, $B \in \mathscr{A}$, $f: A \to B$, $g: B \to A$ morphisms such that $f^{(1)}$, $g^{(1)}$ are surjective, then f, g are isomorphisms.*

Proof. Proposition 1.3 shows that f and g are surjective. Thus $g \circ f: A \to A$ is surjective. It must therefore be an isomorphism (if $K = \ker(g \circ f)$, $K \subset (g \circ f)^{-1}(K)$ $\subset (g \circ f)^{-2}(K) \subset \cdots$ would be a nonstationary chain unless $K = 0$). Thus f is an isomorphism. Similarly g is an isomorphism.

PROPOSITION 1.5. *If $f: A \to B$ is a morphism in $\mathscr{A}$, B regular, $f^{(1)}$ injective, then f has a left inverse and, moreover, A is regular.*

Proof. As in the proof of Proposition 1.3, we have $\alpha: \tilde{A} \to A$. $f^{(1)} \circ \alpha^{(1)}$ is injective, so, by the Implicit Function Theorem, $f \circ \alpha$ has a left inverse $g: B \to \tilde{A}$. Since α is surjective, $\alpha \circ g$ is a left inverse for f. Moreover, $f \circ \alpha$ is injective, so α is injective and hence an isomorphism.

DEFINITION 1.6. If F is a covariant functor from $\mathscr{A}$ to (Sets), a *pair* (A, ξ) consists of an $A \in \mathscr{A}$ and $\xi \in F(A)$. A *morphism* of pairs $(A, \xi) \to (B, \eta)$ is a morphism $f: A \to B$ such that $\eta = F(f)(\xi)$.
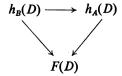
THEOREM 1.7. *If F is quasi-representable by a pair (A, ξ), the pair is unique up to (noncanonical) isomorphism.*

Proof. Let (A, ξ), (B, η) quasi-represent F. By condition (i) there are morphisms $f: (A, \xi) \to (B, \eta)$, $g: (B, \eta) \to (A, \xi)$. Condition (ii) implies that $f^{(1)}$ and $g^{(1)}$ are bijective. By Proposition 1.4, f and g are isomorphisms.

We will now show how the Theorem of Completeness follows from Kuranishi's Theorem.

THEOREM 1.8. *If F is quasi-representable, (B, η) a pair with B regular such that $h_B(D) \to F(D)$ is surjective, then (B, η) is complete.*

Proof. Let (A, ξ) quasi-represent F. We have an $f: (A, \xi) \to (B, \eta)$, so the diagram

$$h_B(D) \longrightarrow h_A(D)$$
$$\searrow \qquad \swarrow$$
$$F(D)$$

is commutative. $h_B(D) \to F(D)$ is surjective while $h_A(D) \to F(D)$ is bijective. Thus, $h_B(D) \to h_A(D)$ is surjective or, what is the same, $f^{(1)}$ is injective. By Proposition 1.5, f is left invertible. Let g be a left inverse. We have $g: (B, \eta) \to (A, \xi)$ and hence a commutative diagram

$$h_A(C) \longrightarrow h_B(C)$$
$$\searrow \qquad \swarrow$$
$$F(C)$$

for any $C \in \mathscr{A}$. Since $h_A(C) \to F(C)$ is surjective, so is $h_B(C) \to F(C)$.

2. **Solutions of analytic equations.** In Artin's paper [1, Theorem 1.5a], it has been shown that if

$$
\begin{array}{ccc}
A & & B \\
\nwarrow & & \nearrow \\
v & \searrow \; \swarrow & w \\
& C &
\end{array}
$$

is a diagram in $\mathscr{A}$, and if there is a morphism $\bar{u}: \hat{B} \to \hat{A}$ with $\bar{u}\hat{w} = \hat{v}$, then there is also a $u: B \to A$ making the diagram commute (here $\hat{}$ denotes $\mathfrak{M}$-adic completion). Moreover, for any c, u can be chosen such that $\hat{u} \equiv \bar{u}$ mod $\mathfrak{M}^c$. The hypothesis means that we are given a compatible sequence of maps $u_n: B^{(n)} \to A^{(n)}$ such that $u_n w^{(n)} = v^{(n)}$. The purpose of this section is to prove Theorem 2.4 which removes from us the obligation of finding a *compatible* sequence of maps: it is enough that there exists a u_n for each n.

The proof of this and related theorems depends on the following lemma:

LEMMA 2.1. *Let K be an algebraically closed, uncountable field,*

$$(*) \qquad \qquad \cdots \to V_n \xrightarrow{\phi_n} V_{n-1} \to \cdots \to V_0$$

a chain of affine varieties over K and regular maps. If $V_n \neq \varnothing$ for all n, then

$$\text{proj lim } V_n \neq \varnothing.$$

Proof. Adjoin to the prime field the coefficients of the defining equations for the V_n and maps ϕ_n. We obtain a field $k \subset K$ over which K has infinite transcendence degree (K is a universal domain). k is the ground field for specializations, etc. Consideration of the irreducible components of the V_n shows that we may reduce to the case in which the V_n are irreducible (pigeon-hole principle). Let g_n be a generic point for V_n and P_n its image in V_0. P_{n+1} is a specialization of P_n. Thus there is an N so that the P_n with $n \geq N$ are generic specializations of each other. Since P_n comes from V_n, so does P_N if $n \geq N$. Thus P_N is in the image of all the maps $V_n \to V_0$. Replace $(*)$ by

$$(**) \qquad \qquad \cdots \to V_n' \to \cdots \to V_1'$$

where V_n' is the subvariety of V_n consisting of points whose image in V_0 is P_N. The previous argument produces a $v_1 \in V_1$ which is in the image of all the maps $V_n \to V_1$ and whose image in V_0 is $v_0 = P_N$. It should now be clear how to obtain, by induction, a point in proj lim V_n.

THEOREM 2.2. *Let* $f_i(x, y) \in C\{x, y\}$, $i = 1, \ldots, m$, *where* $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_N)$. *Suppose that for each* k *we have* $y_\nu^{(k)}(x) \in C\{x\}$ *with no constant term such that*

$(1)_k$ $$f(x, y^{(k)}(x)) \equiv 0 \mod \mathfrak{M}^{k+1}.$$

Then there are $y_\nu(x) \in C\{x\}$ *with no constant term such that* $f(x, y(x)) = 0$. ($\mathfrak{M}$ *denotes the maximal ideal of* $C\{x\}$.)

Proof. Observe that in the expansion of $y_\nu^{(k)}$, only the coefficients of x^I with $|I| \leq k$ (multi-index notation!) make any contribution to the problem. Thus, we may write

(2) $$y_\nu^{(k)} = \sum_{|I| \leq k} y_{\nu}^{(k)}{}_I x^I.$$

The $y_\nu^{(k)}{}_I$ may be taken as coordinates for a point in an affine space C^{N_k}. $(1)_k$ imposes polynomial relations on the coordinates of this point. Hence there is an affine variety V_k in C^{N_k} whose points correspond, via (2), to solutions of $(1)_k$. The projection map $C^{N_{k+1}} \to C^{N_k}$ takes V_k to V_{k-1}. We obtain a sequence of affine varieties over C as in Lemma 2.1. By the Lemma, there is a point in proj lim V_k which, clearly, corresponds to a formal solution of $f(x, y) = 0$. Now apply Artin's Theorem [1, Theorem 1.2].

We will need a more general result:

THEOREM 2.3. *Let* $\mathfrak{A}_1, \ldots, \mathfrak{A}_m, \mathfrak{B}_1, \ldots, \mathfrak{B}_N$ *be proper ideals in* $C\{x\}$. *Let*

$$f_i(x, y) \in C\{x, y\}, \qquad i = 1, \ldots, m.$$

Let $u_1, \ldots, u_N \in C\{x\}$. *Suppose that for each* k *we have* $y_\nu^{(k)} \in C\{x\}$ *with no constant term such that*

$(1)_k$ $$f_i(x, y^{(k)}(x)) \in \mathfrak{A}_i + \mathfrak{M}^{k+1}, \qquad y_\nu^{(k)} \equiv u_\nu \mod (\mathfrak{B}_\nu + \mathfrak{M}^{k+1}).$$

Then there are $y_\nu \in C\{x\}$ *with no constant term such that*

$(1)_\infty$ $$f_i(x, y(x)) \in \mathfrak{A}_i, \qquad y_\nu \equiv u_\nu \mod \mathfrak{B}_\nu.$$

Proof. Let $a_{i\mu} \in C\{x\}$ generate $\mathfrak{A}_i$, $b_{\nu\tau} \in C\{x\}$ generate $\mathfrak{B}_\nu$. As before, we write $y_\nu^{(k)} = \sum_{|I| \leq k} y_\nu^{(k)}{}_I x^I$. The conditions $(1)_k$ are equivalent to the existence of $c_{i\mu}^{(k)}$, $d_{\nu\tau}^{(k)} \in C$, $z_{i\mu}^{(k)}$, $w_{\nu\tau}^{(k)} \in C\{x\}$ with no constant term, such that

$(2)_k$
$$f_i(x, y^{(k)}(x)) - \sum (c_{i\mu}^{(k)} + z_{i\mu}^{(k)}(x))a_{i\mu} \equiv 0 \quad (\mathfrak{M}^{k+1}),$$
$$y_\nu^{(k)}(x) - u_\nu(x) - \sum (d_{\nu\tau}^{(k)} + w_{\nu\tau}^{(k)}(x))b_{\nu\tau} \equiv 0 \quad (\mathfrak{M}^{k+1}).$$

We take $c_{iu}^{(k)}$, $d_{vt}^{(k)}$, $y_{v}^{(k)}{}_I$, $z_{iu}^{(k)}{}_I w_{vt}^{(k)}{}_I$ as coordinates of a point in an affine space and proceed exactly as in the proof of Theorem 2.2. The result is a formal solution of the equations $(2)_\infty$ and hence, by Artin's Theorem, there is a convergent solution.

THEOREM 2.4. *Let $A, B, C \in \mathscr{A}$, $v: C \to A$, $w: C \to B$ morphisms. Suppose, for each k, a morphism $u_k: B^{(k)} \to A^{(k)}$ such that $u_k w^{(k)} = v^{(k)}$. Then we can find a morphism $u: B \to A$ such that $uw = v$. Moreover, if the u_k, $k \geq c$, coincide mod $\mathfrak{M}^{c+1}$, then u can be chosen to coincide with u_c mod $\mathfrak{M}^{c+1}$.*

Proof. Let $A = C\{x\}/\mathfrak{A}$, $B = C\{y\}/\mathfrak{B}$, $C = C\{z\}/\mathfrak{C}$ with $\mathfrak{A} = (f_1, \ldots, f_a)$, $\mathfrak{B} = (g_1, \ldots, g_b)$, $\mathfrak{C} = (h_1, \ldots, h_c)$. The map w is described by $z_v \in C\{y\}$ without constant term such that $h_k(z(y)) \in \mathfrak{B}$ for all k. Similarly, v is described by $'z_v \in C\{x\}$ such that $h_k('z(x)) \in \mathfrak{A}$. A map u is described by $y_\mu \in C\{x\}$ such that $g_i(y(x)) \in \mathfrak{A}$ and the condition $uw = v$ is exactly $z_v(y(x)) \equiv 'z_v(x) \bmod \mathfrak{A}$. Thus, we must show the existence of a convergent solution of

$$g_i(y(x)) \in \mathfrak{A}, \qquad z_v(y(x)) - 'z_v(x) \in \mathfrak{A}.$$

We are given solutions mod $\mathfrak{M}^{k+1}$ for each k. So, by Theorem 2.3, there is a convergent solution. The second assertion is proved by applying Theorem 2.3 with $\mathfrak{B}_v = \mathfrak{M}^{c+1}$.

REMARK. Generalizations of the other Artin Theorems can be proved similarly.

3. **A theorem of completeness.** In §1 we showed how to obtain a completeness theorem for a quasi-representable functor F. No special assumptions were made about the nature of F. If F is not known to be quasi-representable, it is more difficult to prove a completeness theorem. In this section we will prove the completeness theorem when F is the deformation functor of a compact analytic space X_0. This proof uses (1) a sort of formal quasi-representability for F which follows from the work of Schlessinger [10]; (2) the relative form of Douady's Theorem as proved in [9] and [12]; and (3) the generalization of Artin's Theorems which we proved in §2. If the full strength of Schlessinger's results are used, it is possible to avoid the use of the results of §2 by showing directly that maps at each level can be chosen compatibly. There seemed to be no point in doing this, however.

We first prove a generalization of Proposition 1.5.

PROPOSITION 3.1. *Let $A, B \in \mathscr{A}$, B regular. If $f_n: A^{(n)} \to B^{(n)}$ is a morphism with $f_n^{(1)}$ injective, then f_n has a left inverse.*

Proof. Let $\alpha: \tilde{A} \to A$, $\beta: \tilde{B} \to B$ be as in the Proof of Proposition 1.3. We may assume $\tilde{B} = B$, $\beta = \mathrm{id}$. The map f_n extends to $\tilde{f}_n: \tilde{A} \to B$ with $\tilde{f}_n^{(1)}$ injective. Thus $\tilde{f}_n$ has a left inverse $\tilde{g}_n$. The composition $g'_n: B \to \tilde{A} \to A^{(n)}$ sends $\mathfrak{M}_B^{n+1}$ to 0. Thus g'_n defines $g_n: B^{(n)} \to A^{(n)}$. It is easy to see that g_n is a left inverse for f_n.

Let $\mathscr{A}_0$ denote the subcategory of $\mathscr{A}$ consisting of Artinian local C-algebras. Let R be a complete local C-algebra such that $R^{(n)} \in \mathscr{A}_0$ for all n. Let

$$\xi = \{\xi_n\} \in \mathrm{proj}\ \lim F(R^{(n)}).$$

For each $B \in \mathscr{A}_0$, ξ defines a map $h_R(B) \to F(B)$.

DEFINITION 3.2 [10]. If F, G are covariant functors from $\mathscr{A}_0$ to (Sets), a functorial morphism $F \to G$ is called smooth if for every epimorphism $B \to C$ in $\mathscr{A}_0$, the map $F(B) \to F(C) \times_{G(C)} G(B)$ is surjective.

DEFINITION 3.3 [10]. If F is a covariant functor $\mathscr{A}_0$ to (Sets), a pair (R, ξ) as above is called a *hull* for F if

(i) $h_R \to F$ is smooth,

(ii) $h_R(D) \to F(D)$ is bijective.

If $F(C)$ is a point (which is the case for a deformation functor), it follows that

(iii) $h_R(B) \to F(B)$ is surjective for all $B \in \mathscr{A}_0$.

Thus, for F a deformation functor, (R, ξ) represents a formal family which is formally complete.
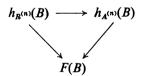
In what follows, F is a covariant functor from $\mathscr{A}$ to (Sets) such that $F(C)$ is a point. We will say F has a hull if its restriction to $\mathscr{A}_0$ has a hull.

DEFINITION 3.4. A pair (A, ξ) is called *formally complete* (for F) if $h_A(B) \to F(B)$ is surjective for all $B \in \mathscr{A}_0$.

THEOREM 3.5. *If F has a hull, (A, ξ) a pair such that A is regular and $h_A(D) \to F(D)$ is surjective, then (A, ξ) is formally complete.*

REMARK. It should be observed that only properties (ii) and (iii) of a hull are used. We do not use the full strength of the hypothesis.

Proof of Theorem 3.5. Let (R, η), $\eta = \{\eta_n\}$, be a hull for F. For each n we obtain $f_n \colon R^{(n)} \to A^{(n)}$ such that $F(f_n)(\eta_n) = \xi^{(n)}$. As in the proof of Theorem 1.8 we deduce that $f_n^{(1)}$ is injective. By Proposition 3.1, f_n has a left inverse g_n. $\eta_n = F(g_n)(\xi^{(n)})$ so that

$$h_{R^{(n)}}(B) \longrightarrow h_{A^{(n)}}(B)$$
$$\searrow \qquad \swarrow$$
$$F(B)$$

commutes. If $B \in \mathscr{A}_0$ such that $\mathfrak{M}_B^{n+1} = 0$, then $h_A(B) \to F(B)$ is surjective since $h_A(B) = h_{A^{(n)}}(B)$ for such B. For any $B \in \mathscr{A}_0$ there is an n such that $\mathfrak{M}_B^{n+1} = 0$. Hence (A, ξ) is formally complete.

THEOREM 3.6. *If F is the deformation functor for a compact analytic space X_0, then F has a hull.*

Proof. Schlessinger provides necessary and sufficient conditions for the existence of a hull [10, Theorem 2.11]. To show that a deformation functor satisfies these conditions, one follows the same plan as in Schlessinger's paper for the algebraic deformation functor. The only task is to show that the sheaves of rings define there place on $|X_0|$ the structure of analytic space. This task is technical but not difficult.

THEOREM 3.7. *If F is the deformation functor for a compact analytic space X_0, (A, ξ) a pair which is formally complete, then (A, ξ) is complete.*

Proof. It is convenient here to switch to the dual formulation. Thus (A, ξ) corresponds to a family Y/T of deformations of X_0. Let X/S be another family of deformations. We have, for each n, a morphism $f_n: S^{(n)} \to T^{(n)}$ so that the family Y_{f_n} induced by f_n is isomorphic to $X^{(n)}$ ($X^{(n)}$ is the restriction of X to $S^{(n)}$, $T^{(n)}$ the space corresponding to $A^{(n)}$, etc.).

Consider the functor

$$\mathbf{Isom}_{T \times S}(X \times_S (T \times S), \ Y \times_T (T \times S))$$

(see [12]). The maps $X \times_S (T \times S) \to T \times S$, $Y \times_T (T \times S) \to T \times S$ are flat and proper. Thus, by the relative Douady Theorem (see [2], [9] and [12]), this functor is representable by an analytic space $\mathscr{S}$ over $T \times S$. The maps $f_n: S^{(n)} \to T$, $i_n: S^{(n)} \to S$ (inclusion), provide a morphism $S^{(n)} \to T \times S$. The isomorphism $X^{(n)} \cong Y_{f_n}$ provides a morphism $\tilde{f}_n: S^{(n)} \to \mathscr{S}$ over $T \times S$. In particular, the composition $S^{(n)} \to \mathscr{S} \to T \times S \to S$ is the inclusion. Moreover, on $S^{(0)}$ all the $\tilde{f}_n$ coincide with the morphism $S^{(0)} \to \mathscr{S}$ corresponding to the isomorphism given as part of the structure of the families. By Theorem 2.4 we obtain $\tilde{f}: S \to \mathscr{S}$ such that the composition $S \to \mathscr{S} \to T \times S \to S$ is the identity. Let f denote the composition $S \to \mathscr{S} \to T \times S \to T$. Then $\tilde{f}$ corresponds to an isomorphism of the families X and Y_f.

Combining the previous results we obtain

THEOREM OF COMPLETENESS 3.8. *Let F be the deformation functor of a compact analytic space X_0. Let (A, ξ) be a pair such that A is regular and $h_A(D) \to F(D)$ is surjective, then (A, ξ) is complete.*

4. Applications.

1. *Rigidity theorem for analytic spaces (see [11]).*

THEOREM 4.1. *If F is the deformation functor for a compact analytic space X_0 and $F(D)$ is a point, then $F(B)$ is a point for all $B \in \mathscr{A}$.*

Proof. Let $A = C$, ξ the unique point of $F(A)$. The conditions of Theorem 3.8 are satisfied by (A, ξ). But $h_C(B)$ is a point for any $B \in \mathscr{A}$.

2. *Hypersurfaces in projective space.* Consider the family $\mathscr{V}_{n,k}$ of hypersurfaces of degree k in P^n (see [5, §14]). This family is parametrized by P^N with $N = \binom{n+k}{k} - 1$.

THEOREM 4.2. *If X_0 is a hypersurface of degree k in P^n, $\mathscr{V}_{n,k}$ is a complete family of deformations of X_0 for $n \geq 4$ and for $n = 3$, $k \leq 3$.*

Proof. If X/D is a family of deformations of X_0, $\mathscr{L}_0$ the very ample invertible sheaf defined by the imbedding of X_0 in P^n, the obstruction to extending $\mathscr{L}_0$ to X lies in $H^2(X_0, \mathscr{O}_{X_0})$. This group vanishes under the hypotheses of this theorem. If the extension exists it is very ample by [4, Theorem 2.1, 15-06]. Thus X/D is induced by the family $\mathscr{V}_{n,k}$.

REFERENCES

1. M. Artin, *On the solutions of analytic equations*, Invent. Math. **5** (1968), 277–291. MR **38** #344.

2. A. Douady, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier (Grenoble) **16** (1966), fasc. 1, 1–95. MR **34** #2940.

3. ———, *Le problème des modules pour les variétés analytiques complexes*, Séminaire Bourbaki 1964/65, exposé 277, Benjamin, New York, 1966. MR **33** #5420l.

4. A. Grothendieck, *Techniques de construction en géométrie analytique* I–X, Séminaire Henri Cartan 1960/61, 2ième éd., École Normale Supérieure, Secrétariat mathématique, Paris 1962. MR **26** #3562.

5. K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures*. I, II, Ann. of Math. (2) **67** (1958), 328–466. MR **22** #3009.

6. ———, *A theorem of completeness for complex analytic fibre spaces*, Acta. Math. **100** (1958), 281–294. MR **22** #3010.

7. M. Kuranishi, *On the locally complete families of complex analytic structures*, Ann. of Math. (2) **75** (1962), 536–577. MR **25** #4550.

8. ———, *New proof for the existence of locally complete families of complex structures*, Proc. Conference Complex Analysis (Minneapolis, 1964), Springer, Berlin and New York, 1965. MR **31** #768.

9. G. Pourcin, *Théorème de Douady au-dessus de S*, Ann. Scuola. Norm. Sup. Pisa (3) **23** (1969), 451–459. MR **41** #2053.

10. M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222. MR **36** #184.

11. H. W. Schuster, *Über die starrheit kompakter komplexer Räume*, Manuscripta Math. **1** (1969), 125–137. MR **40** #7478.

12. ———, *Zur Theorie der Deformationen kompakter komplexer Räume*, Invent. Math. **9** (1970), 284–294.

13. J. J. Wavrik, *Deformations of branched coverings of complex manifolds*, Amer. J. Math. **90** (1968), 926–960. MR **38** #1706.

14. ———, *Obstructions to the existence of a space of moduli*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 403–414. MR **40** #8089.

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