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SYMMETRIC MASSEY PRODUCTS AND A HIRSCH FORMULA IN HOMOLOGY(1),(2)

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Abstract. A Hirsch formula is proved for the singular chains of a second loop space and is applied to show that the symmetric Massey produce $\langle x\rangle^p$ is defined for x an odd dimensional mod p homology class of a second loop space with p an odd prime. $\langle x\rangle^p$ is then interpreted in terms of the Dyer-Lashof and Browder operations.

1. **Introduction.** When using the Eilenberg-Moore spectral sequence for odd primes p, it is often desirable to interpret the Massey product with a symmetric defining system $\langle x\rangle^p$, deg x odd, in terms of operations on x. We will show that under suitable technical hypotheses, there are operations βQ and λ defined on the homology of the cobar construction on a differential graded Hopf algebra and $\langle x\rangle^p = -\beta Q(x) + \mathrm{ad}_\lambda^{p-1}(x)(\beta x)$. We will apply this result to the homology of second and higher loop spaces where we will identify βQ with the first nontrivial Dyer-Lashof operation and λ with the first Browder operation. Then we will apply this result to the cohomology of any topological space where we will identify βQ with the last nontrivial Steenrod operation to obtain a new proof of a theorem of D. Kraines. In our study of the former situation we will show that the singular chains of the Moore loops of a topological monoid satisfy the Hirsch formula

$$(ab) \cup_1 c = (-1)^{\deg a} a(b \cup_1 c) + (-1)^{\deg b \deg c} (a \cup_1 c) b,$$

which G. Hirsch [8] proved for a, b and c singular cochains on a topological space.

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2. **A Hirsch formula in homology.** Before we turn our attention to the Hirsch formula, we recall a few definitions.

If $(Y, *)$ is a based topological space then we define the space of loops on Y as in J. Moore [17, p. 18-03] with the convention that if $(r, f) \in \Omega Y$ then f has domain

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$\boldsymbol{R}$ with $f(t)=*$ if $t\leqq 0$ or $t\geqq r$. If Y is a topological monoid then define the Pontryagin product on the singular chains of ΩY to be the composite

$$C_k(\Omega Y)\otimes C_h(\Omega Y)\xrightarrow{\eta} C_{k+h}(\Omega Y\times\Omega Y)\xrightarrow{P_*} C_{k+h}(\Omega Y)$$

where η is the shuffle map and $P((r_1,f_1),(r_2,f_2))=(r_1+r_2,f)$ with $f(t)=f_1(t)f_2(t)$. $C_*(\Omega Y)$ thus becomes a differential Hopf algebra.

DEFINITION 1. Let A be a homotopy commutative DGA-algebra. Any homotopy on A which shows that the product is homotopy commutative is called a cup-one ($\cup_1$) product. In symbols, for all $a, b\in A$

$$\partial(a\cup_1 b)+\partial(a)\cup_1 b+(-1)^{\deg a}a\cup_1\partial(b) = [a,b].$$

Let $\pi=\{e,T\}$ be the cyclic group of order two. The join of π with itself is

$$J^2\pi = \{[tx,(1-t)y]\mid t\in[0,1] \text{ and } x,y\in\pi\}/\sim$$

where $\sim$ is the equivalence relation generated by identifying $0e$ with $0T$ (see J. Milnor [16, p. 430]). $J^2\pi$ has the π-action

$$z\cdot[tx,(1-t)y] = [tzx,(1-t)zy] \quad \text{for } x,y,z\in\pi \text{ and } t\in[0,1].$$

There are singular 1-simplexes τ_1 and τ_2 on $J^2\pi$ defined by $\tau_1(t,1-t)=[(1-t)e,te]$ and $\tau_2(t,1-t)=[(1-t)T,te]$ for $t\in[0,1]$. Let $\tau=\tau_2-\tau_1\in C_*(J^2\pi)$.

DEFINITION 2 (E. DYER AND R. LASHOF [5, p. 37]). If Y is a topological monoid and $X=\Omega Y$ then the following composite defines a cup-one product on the singular chains of X:

$$C_*(X)\otimes C_*(X)\xrightarrow{\iota} C_*(J^2\pi)\otimes_\pi C_*(X)\otimes C_*(X)\xrightarrow{1\otimes\eta} C_*(J^2\pi)\otimes_\pi C_*(X\times X)$$
$$\xrightarrow{\eta} C_*(J^2\pi\times_\pi X\times X)\xrightarrow{\theta_*} C_*(X)$$

where ι is the inclusion map of degree one defined by $\iota(a\otimes b)=\tau\otimes a\otimes b$, η is the shuffle map and $\theta: J^2\pi\times X\times X\to X$ is the π-equivariant map defined by

$$\theta([(1-t)x,te],(r_0,f_0),(r_1,f_1))(s) = f^t_{x(0)}(s)f^t_{x(1)}(s),$$

$f^t_0=f_0$, $f^t_1(s)=f_1(s-tr_0)$, $x\in\pi$, $t\in[0,1]$, $(r_0,f_0)\in X$, $(r_1,f_1)\in X$ and $s\in[0,r_0+r_1]$. Note that X is given the trivial π-action while $X\times X$ and $C_*(X)\otimes C_*(X)$ have a π-action by permuting their factors.

The object of this section is to prove the following theorem:

THEOREM 3. *If a, b and c are singular chains on the Moore loops of a topological monoid and $\cup_1$ denotes the cup-one product of E. Dyer and R. Lashof* [5] *then*

$$(ab)\cup_1 c = (-1)^{\deg a}a(b\cup_1 c)+(-1)^{\deg b\cdot\deg c}(a\cup_1 c)b.$$

The following definition and theorem from S. Eilenberg and S. Mac Lane [7] will be used in the proof of Theorem 3 to manipulate the shuffle product.

DEFINITION 4. (a) Let Σ_n be the symmetric group on n letters which acts on the set $\{0, \dots, n-1\}$. Then

$$\operatorname{sh}(n_1, \dots, n_t)$$
$$= \left\{\sigma \in \Sigma_n \mid \sigma(u) < \sigma(v) \text{ if } \sum_{i=1}^{j-1} n_i \leqq u < v < \sum_{i=1}^{j} n_i \text{ for some } 1 \leqq j \leqq t\right\}$$

where $\sum_{i=1}^{t} n_i = n$ and $n_i \geqq 0$, $1 \leqq i \leqq t$.

(b) Let X be a topological space. For $1 \leqq j \leqq t$ define the natural degeneracy operators $\rho_j \colon \operatorname{sh}(n_1, \dots, n_t) \to \operatorname{Hom}(C_{n_j}(X), C_n(X))$ as follows: Let

$$\sigma \in \operatorname{sh}(n_1, \dots, n_t)$$

and let

$$\{m_1, \dots, m_{n-n_j}\} = \{0, \dots, n-1\} - \left\{\sigma\left(h + \sum_{i=1}^{j-1} n_j\right) \,\middle|\, 0 \leqq h < n_j\right\}$$

with $m_k < m_{k+1}$ for $1 \leqq k < n - n_j$. Then

$$\rho_j(\sigma) = S_{m_{n-n_j}} \circ \cdots \circ S_{m_1}.$$

THEOREM 5. *Let $X_1, \dots, X_t$ be topological spaces. Define the shuffle map*

$$\eta_t \colon [C_*(X_1) \otimes \cdots \otimes C_*(X_t)]_n \to C_n(X_1 \times \cdots \times X_t)$$

by

$$\eta_t(a_1 \otimes \cdots \otimes a_t) = \sum_{\sigma \in \operatorname{sh}(n_1, \dots, n_t)} (\operatorname{sgn} \sigma)[\rho_1(\sigma)(a_1)] \times \cdots \times [\rho_t(\sigma)(a_t)]$$

where $a_i \in C_{n_i}(X_i)$ for $1 \leqq i \leqq t$ and $n = \sum_{i=1}^{t} n_i$. Then η_t is a natural chain equivalence. Furthermore,

$$\eta_t(a_1 \otimes \cdots \otimes a_t) = \sum_{i=2}^{t} \sum_{\sigma \in \operatorname{sh}(n_1 + \cdots + n_{i-1}, n_i)} \eta_2((\eta_2 \cdots \eta_2(\eta_2(a_1 \otimes a_2) \otimes a_3) \cdots) \otimes a_t).$$

That is, the following diagram commutes:

$$\begin{array}{ccc}
C_*(X_1) \otimes \cdots \otimes C_*(X_t) & \xrightarrow{\qquad \eta_t \qquad} & C_*(X_1 \times \cdots \times X_t) \\
\Big\downarrow {\scriptstyle \eta_2 \otimes 1} & & \Big\uparrow {\scriptstyle \eta_2} \\
C_*(X_1 \times X_2) \otimes C_*(X_3) \otimes \cdots \otimes C_*(X_t) & \xrightarrow{\eta_2} \cdots \xrightarrow{\eta_2} & C_*(X_1 \times \cdots \times X_{t-1}) \otimes C_*(X_t)
\end{array}$$

We now apply all of the preceding definitions and Theorem 5 to evaluate $a(b \cup_1 c)$, $(a \cup_1 c)b$ and $(ab) \cup_1 c$.

LEMMA 6. *Let Y be a topological monoid with $X=\Omega Y$ and $a\colon \Delta^\alpha \to X$, $b\colon \Delta^\beta \to X$ and $c\colon \Delta^\gamma \to X$ singular simplexes. Then*

(i)
$$a(b \cup_1 c) = \sum_{\sigma \in \mathrm{sh}(\alpha,\beta,\gamma,1)} (-1)^{\beta+\gamma}(\mathrm{sgn}\ \sigma)P_*\{[\rho_1(\sigma)(a)] \times \theta_*[\rho_4(\sigma)(\tau) \times \rho_2(\sigma)(b) \times \rho_3(\sigma)(c)]\},$$

(ii)
$$(a \cup_1 c)b = \sum_{\sigma \in \mathrm{sh}(\alpha,\beta,\gamma,1)} (-1)^{\alpha+\beta+\gamma+\beta\gamma}(\mathrm{sgn}\ \sigma) \cdot P_*\{\theta_*[\rho_4(\sigma)(\tau) \times \rho_1(\sigma)(a) \times \rho_3(\sigma)(c)] \times [\rho_2(\sigma)(b)]\},$$

(iii)
$$(ab) \cup_1 c = \sum_{\sigma \in \mathrm{sh}(\alpha,\beta,\gamma,1)} (-1)^{\alpha+\beta+\gamma}(\mathrm{sgn}\ \sigma) \cdot \theta_*\{\rho_4(\sigma)(\tau) \times P_*[(\rho_1(\sigma)(a)) \times (\rho_2(\sigma)(b))] \times \rho_3(\sigma)(c)]\}.$$

Proof. (i)

$$a(b \cup_1 c) = (P_* \circ \eta) \circ (1 \otimes \theta_*) \circ (1 \otimes \eta) \circ (1 \otimes 1 \otimes \eta) \circ (1 \otimes \iota)[a \otimes (b \otimes c)]$$

by Definition 2. Hence

$$a(b \cup_1 c) = \sum_{\mu \in \mathrm{sh}(\alpha,\beta+\gamma+1)} \sum_{\nu \in \mathrm{sh}(1,\beta+\gamma)} \sum_{\xi \in \mathrm{sh}(\beta,\gamma)} (\mathrm{sgn}\ \mu)(\mathrm{sgn}\ \nu)(\mathrm{sgn}\ \xi) \cdot P_*\{[\rho_1(\mu)(a)] \times \theta_*[\rho_2(\mu) \circ \rho_1(\nu)(\tau) \times \rho_2(\mu) \circ \rho_2(\nu) \circ \rho_1(\xi)(b) \times \rho_2(\mu) \circ \rho_2(\nu) \circ \rho_2(\xi)(c)]\}$$

by Definition 4. Hence

$$a(b \cup_1 c) = \sum_{\mu \in \mathrm{sh}(\beta+\gamma+1,\alpha)} \sum_{\nu \in \mathrm{sh}(\beta+\gamma,1)} \sum_{\xi \in \mathrm{sh}(\beta,\gamma)} (-1)^{\alpha\beta+\alpha\gamma+\alpha+\beta+\gamma}(\mathrm{sgn}\ \mu)(\mathrm{sgn}\ \nu)(\mathrm{sgn}\ \xi) \cdot P_*\{[\rho_2(\mu)(a)] \times \theta_*[\rho_1(\mu) \circ \rho_2(\nu)(\tau) \times \rho_1(\mu) \circ \rho_1(\nu) \circ \rho_1(\xi)(b) \times \rho_1(\mu) \circ \rho_1(\nu) \circ \rho_2(\xi)(c)]\}$$

since there is a one-to-one correspondence $F\colon \mathrm{sh}\,(m, n_1, \ldots, n_t) \to \mathrm{sh}\,(n_1, \ldots, n_t, m)$ with $\mathrm{sgn}\ F(\sigma)=(-1)^{m(n_1+\cdots+n_t)}\ \mathrm{sgn}\ \sigma$ defined by

$$\begin{aligned} F(\sigma)(k) &= \sigma(k+m) && \text{if } 0 \leqq k < n_1+\cdots+n_t, \\ &= \sigma(k-n_1-\cdots-n_t) && \text{if } n_1+\cdots+n_t \leqq k < n_1+\cdots+n_t+m. \end{aligned}$$

Hence

$$a(b \cup_1 c) = \sum_{\sigma \in \mathrm{sh}(\beta,\gamma,1,\alpha)} (-1)^{\alpha\beta+\alpha\gamma+\alpha+\beta+\gamma}(\mathrm{sgn}\ \sigma) \cdot P_*\{[\rho_4(\sigma)(a)] \times \theta_*[\rho_3(\sigma)(\tau) \times \rho_1(\sigma)(b) \times \rho_2(\sigma)(c)]\}$$

by Theorem 5. Thus,

$$a(b \cup_1 c) = \sum_{\sigma \in \mathrm{sh}(\alpha,\beta,\gamma,1)} (-1)^{\beta+\gamma}(\mathrm{sgn}\ \sigma) \cdot P_*\{[\rho_1(\sigma)(a)] \times \theta_*[\rho_4(\sigma)(\tau) \times \rho_2(\sigma)(b) \times \rho_3(\sigma)(c)]\}$$

by using the above one-to-one correspondence F again.

(ii) and (iii) are proved in the same way as (i).

Proof of Theorem 3. By the previous lemma, it suffices to show that if

$$\sigma \in \mathrm{sh}\,(\alpha, \beta, \gamma, 1)$$

then

$$\begin{aligned}\theta_*\{\rho_4(\sigma)(\tau) \times P_*[(\rho_1(\sigma)(a)) \times (\rho_2(\sigma)(b))] \times \rho_3(\sigma)(c)\} \\ = P_*\{[\rho_1(\sigma)(a)] \times \theta_*[\rho_4(\sigma)(\tau) \times \rho_2(\sigma)(b) \times \rho_3(\sigma)(c)]\} \\ +P_*\{\theta_*[\rho_4(\sigma)(\tau) \times \rho_1(\sigma)(a) \times \rho_3(\sigma)(c)] \times [\rho_2(\sigma)(b)]\}.\end{aligned}$$

Since $\tau = \tau_2 - \tau_1$ this follows from the following three identities:

$$\text{(1)}\quad \begin{aligned}\theta_*\{\rho_4(\sigma)(\tau_1) \times P_*[(\rho_1(\sigma)(a) \times (\rho_2(\sigma)(b))] \times \rho_3(\sigma)(c)\} \\ = P_*\{[\rho_1(\sigma)(a)] \times \theta_*[\rho_4(\sigma)(\tau_1) \times \rho_2(\sigma)(b) \times \rho_3(\sigma)(c)]\},\end{aligned}$$

$$\text{(2)}\quad \begin{aligned}\theta_*\{\rho_4(\sigma)(\tau_2) \times P_*[(\rho_1(\sigma)(a)) \times (\rho_2(\sigma)(b))] \times \rho_3(\sigma)(c)\} \\ = P_*\{\theta_*[\rho_4(\sigma)(\tau_2) \times \rho_1(\sigma)(a) \times \rho_3(\sigma)(c)] \times [\rho_2(\sigma)(b)]\},\end{aligned}$$

$$\text{(3)}\quad \begin{aligned}P_*\{[\rho_1(\sigma)(a)] \times \theta_*[\rho_4(\sigma)(\tau_2) \times \rho_2(\sigma)(b) \times \rho_3(\sigma)(c)]\} \\ = P_*\{\theta_*[\rho_4(\sigma)(\tau_1) \times \rho_1(\sigma)(a) \times \rho_3(\sigma)(c)] \times [\rho_2(\sigma)(b)]\}.\end{aligned}$$

We will only prove (1) since the proofs of (2) and (3) are similar to the proof of (1). We will show that the two singular $(\alpha+\beta+\gamma+1)$-simplexes in (1) are the same by evaluating them on a point $(t_0, \ldots, t_{\alpha+\beta+\gamma+1})$ of $\Delta^{\alpha+\beta+\gamma+1}$ and then by evaluating the resulting two points of ΩY on $s \in [0, \infty)$.

$$\begin{aligned}\theta_*\{\rho_4(\sigma)(\tau_1) \times P_*[(\rho_1(\sigma)(a)) \times (\rho_2(\sigma)(b))] \times \rho_3(\sigma)(c)\}(t_0, \ldots, t_{\alpha+\beta+\gamma+1})(s) \\ = \theta[\tau_1(\bar{t}, 1-\bar{t}), a(t'_0, \ldots, t'_\alpha)\cdot b(t''_0, \ldots, t''_\beta), c(t'''_0, \ldots, t'''_\gamma)](s) \\ = [a(t'_0, \ldots, t'_\alpha)(s)][b(t''_0, \ldots, t''_\beta)(s)][c(t'''_0, \ldots, t'''_\gamma)^{\bar{t}}(s)]\end{aligned}$$

by Definitions 2 and 4 where

$$\begin{aligned}\rho_1(\sigma)(t_0, \ldots, t_{\alpha+\beta+\gamma+1}) &= (t'_0, \ldots, t'_\alpha), \\ \rho_2(\sigma)(t_0, \ldots, t_{\alpha+\beta+\gamma+1}) &= (t''_0, \ldots, t''_\beta), \\ \rho_3(\sigma)(t_0, \ldots, t_{\alpha+\beta+\gamma+1}) &= (t'''_0, \ldots, t'''_\gamma)\end{aligned}$$

and

$$\rho_4(\sigma)(t_0, \ldots, t_{\alpha+\beta+\gamma+1}) = (\bar{t}, 1-\bar{t}).$$

$$\begin{aligned}P_*\{[\rho_1(\sigma)(a)] \times \theta_*[\rho_4(\sigma)(\tau_1) \times \rho_2(\sigma)(b) \times \rho_3(\sigma)(c)]\}(t_0, \ldots, t_{\alpha+\beta+\gamma+1})(s) \\ = \{a(t'_0, \ldots, t'_\alpha)\cdot \theta[\tau_1(\bar{t}, 1-\bar{t}), b(t''_0, \ldots, t''_\beta), c(t'''_0, \ldots, t'''_\gamma)]\}(s) \\ = [a(t'_0, \ldots, t'_\alpha)(s)][b(t''_0, \ldots, t''_\beta)(s)][c(t'''_0, \ldots, t'''_\gamma)^{\bar{t}}(s)].\end{aligned}$$

These two calculations prove (1).

As in G. Hirsch [8], we have the following corollary to the Hirsch formula.

COROLLARY 7. *Assume that* Y *is a topological monoid and let* $x, y \in H_*(\Omega Y; Z_2)$. *If* $xy=0$ *then* $\langle x, y, x\rangle$ *is defined and*

$$\langle x, y, x\rangle = yQ^{1+\deg x}(x) \quad modulo\ xH_*(\Omega Y; Z_2).$$

3. **Symmetric Massey products.** Henceforth p will denote an odd prime.

DEFINITION 8. Let A be a GDA-algebra over Z_p with $x \in H_{2n-1}(A)$. The symmetric Massey product $\langle x\rangle^k$ is said to be defined if there exist elements $a_1, \ldots, a_{k-1} \in A$, called a defining system of $\langle x\rangle^k$, such that $\{a_1\}=x$ and $d(a_t)=\sum_{i=1}^{t-1} a_i a_{t-i}$ if $2 \leqq t \leqq k-1$. If $\langle x\rangle^k$ is defined then it equals the set of all homology classes $\{\sum_{i=1}^{k-1} a_i a_{k-i}\}$ where $a_1, \ldots, a_{k-1}$ varies over all defining systems of $\langle x\rangle^k$. If $\langle x\rangle^k$ is defined and equal to a single homology class then $\langle x\rangle^k$ is said to be defined with zero indeterminacy.

LEMMA 9. *Let* A *be a DGA-algebra over* Z_p *which has a cup-one product that satisfies the Hirsch formula. If* x *is an odd dimensional homology class of* A *then* $\langle x\rangle^p$ *is defined with zero indeterminacy.*

The proof of Lemma 9 is a direct generalization of the proof of Theorems 15 and 17 in D. Kraines [10] where Lemma 9 is proved for the special case when x is an odd dimensional cohomology class.

DEFINITION 10. Let A be a DGA-coalgebra over a commutative ring R. Define the cobar construction FA on A as the tensor algebra $T(sIA)$ on the augmentation ideal sIA where the s indicates that the degree of each element is decreased by one. Elements of $T(sIA)$ will be denoted $[a_1|\cdots|a_k]$ for $a_1, \ldots, a_k \in sIA$, and $[\ ]$ will denote the identity element of $T(sIA)$. FA has three differentials d, d_1 and d_2 where

$$d = d_1 + d_2,$$

$$d_1[a_1|\cdots|a_k] = \sum_{i=1}^{k} [\bar{a}_1|\cdots|\bar{a}_{i-1}|d(a_i)|a_{i+1}|\cdots|a_k],$$

$$d_2[a_1|\cdots|a_k] = \sum_{i=1}^{k} \sum [\bar{a}_1|\cdots|\bar{a}_{i-1}|\bar{a}_i'|a_i''|a_{i+1}|\cdots|a_k],$$

$a_1, \ldots, a_k \in sIA$, $\bar{a}_i = (-1)^{1+\deg a_i} a_i$ and $\bar{\psi}(a_i) = \sum a_i' \otimes a_i''$. FA is a differential algebra under d. By J. F. Adams [2, p. 36] and J. P. May [14], if A is a differential Hopf algebra then FA has a cup-one product which satisfies the Hirsch formula.

DEFINITION 11. Let A be a connected DGA-Hopf algebra over Z_p. Define the primitive elements of A by $PA = \{a \in A \mid \psi(a) = a \otimes 1 + 1 \otimes a\}$. For $a \in A$, let $\bar{\psi}(a) = \psi(a) - a \otimes 1 - 1 \otimes a$. There is a natural inclusion of complexes $[PA] \to FA$. When this map induces an isomorphism in homology we will write $H_*([PA]) = H_*(FA)$.

There are two important examples of connected DGA-Hopf algebras A over Z_p with $H_*([PA]) = H_*(FA)$. They are $A = C_*(\Omega S^2 X)$ for X a connected topological space and $A = C^*(K(Z_p, n))$, $n \geqq 2$. In fact, by D. Kraines [11] these two examples have the stronger property that every decomposable d_2-cycle of FA is a d_2-boundary.

DEFINITION 12. Let A be a connected DGA-Hopf algebra over Z_p. Define two operations $Q\colon H_{2n-1}([PA]) \to H_{2np-1}(FA)$ for $n \geqq 1$ and $\lambda\colon H_r([PA]) \otimes H_s([PA]) \to H_{r+s+1}(FA)$ for $r \geqq 0$ and $s \geqq 0$ by $Q\{[a]\} = \{[a^p]\}$, $\lambda(1 \otimes x) = \lambda(x \otimes 1) = 0$ and $\lambda(\{[b]\} \otimes \{[c]\}) = \{[bc - (-1)^{(r+1)(s+1)}cb]\}$ where $a \in PA_{2n}$, $x \in H_*(FA)$, $b \in PA_{r+1}$ with $r > 0$, $c \in PA_{s+1}$ with $s > 0$, $d(a) = 0$, $d(b) = 0$ and $d(c) = 0$.

We will show in Lemma 15 that λ is well defined and if $A = A' \otimes Z_p$ with A' a Z-free connected DGA-Hopf algebra then βQ is well defined. Thereafter, we will prove the following theorem.

THEOREM 13. *Let A' be a Z-free connected DGA-Hopf algebra with $A = A' \otimes Z_p$ and assume that every decomposable d_2-cycle of FA is a d_2-boundary. If $x \in H_{2n-1}(FA)$, $n \geqq 1$, then $\langle x \rangle^p$ is defined with zero indeterminacy and*

$$\langle x \rangle^p = -\beta Q(x) + \mathrm{ad}_\lambda^{p-1}(x)(\beta x).$$

Recall the use of the notation ad_λ:

$$\mathrm{ad}_\lambda^1(x)(y) = \lambda(x \otimes y) \quad \text{and} \quad \mathrm{ad}_\lambda^k(x)(y) = \lambda(x \otimes \mathrm{ad}_\lambda^{k-1}(x)(y)) \quad \text{if } k \geqq 2.$$

Similarly in any algebra over Z_p we may define $\mathrm{ad}^k(x)(y)$ by substituting the commutator operation for λ. Thus,

$$\begin{aligned} \mathrm{ad}^{p-1}(x)(y) &= [x, [x, \ldots, [x, y] \ldots] \quad (x \text{ taken } p-1 \text{ times}) \\ &= \sum_{i=0}^{p-1} x^{p-i-1} y x^i \end{aligned}$$

if $\deg x$ or $\deg y$ is even by N. Jacobson [9, p. 186]. Observe that if $x = \{[a]\}$, $y = \{[b]\}$, $a, b \in PA$, $d(a) = 0$ and $d(b) = 0$ then $\mathrm{ad}_\lambda^{p-1}(x)(y) = \{[\mathrm{ad}^{p-1}(a)(b)]\}$.

LEMMA 14. *Let A be a connected DGA-Hopf algebra over Z_p. If $a \in PA_{2n}$ with $d(a) = 0$ then $\langle\{[a]\}\rangle^p \in H_{2np-2}(FA)$ is defined with zero indeterminacy and*

$$\langle\{[a]\}\rangle^p = \left\{\sum_{i=1}^{p-1} \frac{(i, p-i)}{p} [a^i | a^{p-i}]\right\}.$$

Proof. By Lemma 9, $\langle\{[a]\}\rangle^p$ is defined with zero indeterminacy. Let $a_i = (-1)^{i+1}(1/i!)[a^i]$ if $1 \leqq i \leqq p-1$. Then $a_1, \ldots, a_{p-1}$ is a defining system for $\langle\{[a]\}\rangle^p$, and

$$\sum_{i=1}^{p-1} a_i a_{p-i} = \sum_{i=1}^{p-1} \frac{(i, p-i)}{p} [a^i | a^{p-i}].$$

LEMMA 15. *Let A' be a Z-free connected DGA-algebra with $A = A' \otimes Z_p$. Then βQ and λ are well defined.*

Proof. To prove that βQ is well defined, we must show that if $b \in PA_{2n+1}$, $a \in PA_{2n}$ and $d(a) = 0$ then $\beta\{[a^p]\} = \beta\{[(a + d(b))^p]\}$. By N. Jacobson [9, p. 187], $(a + d(b))^p - a^p - d(b)^p$ is in the sub-Lie algebra S of PA generated by a and $d(b)$. Clearly, every element of S except for Z_p-multiples of a is a boundary. Hence

$[(a+d(b))^p]$ is homologous to $[a^p]+[d(b)^p]$. Therefore $\beta\{[(a+d(b))^p]\}=\beta\{[a^p]\}$ since

$$\begin{aligned}\beta\{[d(b)^p]\} &= -\left\{\sum_{i=1}^{p-1}\frac{(i,p-i)}{p}[d(b)^i|d(b)^{p-i}]\right\}\\ &= -\langle\{[d(b)]\}\rangle^p \quad \text{by Lemma 14}\\ &= -\langle\{d[b]\}\rangle^p = 0.\end{aligned}$$

λ is well defined since if $a\in PA_r$, $b\in PA_s$, $c\in PA_{r+1}$, $e\in PA_{s+1}$ with $r, s\geqq 2$, $d(a)=0$ and $d(b)=0$ then

$$\begin{aligned}&[(a+d(c))(b+d(e))-(-1)^{rs}(b+d(e))(a+d(c))]\\ &\quad= [ab-(-1)^{rs}ba]\\ &\qquad+d([cb-(-1)^{r(s+1)}bc]+(-1)^s[ae-(-1)^{s(r+1)}ea]+[cd(e)-(-1)^{r(s+1)}d(e)c]).\end{aligned}$$

Proof of Theorem 13. We will compute $\beta Q(x)$ by using the hypothesis that every decomposable d_2-cycle is a d_2-boundary and then relate the result to $\langle x\rangle^p$ by applying Lemma 14. Since $H_*([PA])=H_*(FA)$, write $x=\{[a]\}$ with $a\in PA_{2n}$ and $d(a)=0$. Choose any $a'\in A'_{2n}$ which reduces to $a \bmod p$. Then $d(a')=pb'$ and $\bar{\psi}(a')=p\sum_{i=1}^t a'_i\otimes b'_i$ for some $b', a'_i, b'_i\in A'$. Let b', a'_i, b'_i reduce mod p to $b, a_i, b_i\in A$ respectively. Then $\beta\{[a]\}=\{[b]+\sum_{i=1}^t[\bar{a}_i|b_i]\}$. $\sum_{i=1}^t[\bar{a}_i|b_i]$ is a decomposable d_2-cycle which by hypothesis must bound. Hence there exists $c\in A_{2n}$ with $\bar{\psi}(c)=\sum_{i=1}^t a_i\otimes b_i$. Thus, $\beta\{[a]\}=\{[b-d(c)]\}$. Choose $c'\in A'_{2n}$ which reduces to c mod p. Let $a''=a'-pc'$ and $b''=b'-d(c')$. Then a'' and b'' reduce mod p to a and b respectively, $d(a'')=pb''$ and $\bar{\psi}(a'')=p^2e'$ for some $e'\in(A'\otimes A')_{2n}$.

$$d_1[a''^p] = p[\mathrm{ad}^{p-1}(a'')(b'')]$$

since

$$d(a''^p) = \sum_{i=1}^{p} a''^{i-1}d(a'')a''^{p-i} = p\sum_{i=1}^{p} a''^{i-1}b''a''^{p-i} = p\,\mathrm{ad}^{p-1}(a'')(b'')$$

by N. Jacobson [9, p. 186].

$$d_2[a''^p] = -p\sum_{i=1}^{p-1}\frac{(i,p-i)}{p}[a''^i|a''^{p-i}] \quad \text{modulo } p^2A'\otimes A'$$

since

$$\begin{aligned}\psi(a''^p) &= (a''\otimes 1+1\otimes a''+p^2e')^p\\ &= (a''\otimes 1+1\otimes a'')^p \quad \text{modulo } p^2A'\otimes A'\\ &= \sum_{i=1}^{p-1}(i,p-i)a''^i\otimes a''^{p-i} \quad \text{modulo } p^2A'\otimes A'.\end{aligned}$$

Hence

$$\begin{aligned}\beta Q(x) = \beta\{[a^p]\} &= \{[\mathrm{ad}^{p-1}(a)(b)]\}-\left\{\sum_{i=1}^{p-1}\frac{(i,p-i)}{p}[a^i|a^{p-i}]\right\}\\ &= \mathrm{ad}_\lambda^{p-1}(x)(\beta x)-\langle x\rangle^p\end{aligned}$$

by Lemma 14 and Definition 12.

Note that A. Clark [4] contains germs of some of the elements of the proof of Theorem 13.

When we apply Theorem 13 to the homology of a second loop space and to the cohomology of any topological space, the major task will be to identify the operations βQ and λ with the Dyer-Lashof operation βQ^n and the first Browder operation λ_1 in homology and with the Steenrod operation $\beta \mathscr{P}^n$ and 0 in cohomology. We will first show in Theorem 17 that βQ and λ have all the properties that one expects. Then under suitable technical hypotheses we will prove a uniqueness theorem for βQ.

DEFINITION 16. Let A be a connected DGA-Hopf algebra over Z_p. Define the suspension map $\sigma: IH_n([PA]) \to H_{n+1}(A)$, $n \geqq 0$ by $\sigma\{[a]\} = \{a\}$ if $a \in PA$ with $d(a)=0$. Note that σ is well defined, the image of σ is a submodule of $PH_*(A)$ and if $H_*([PA]) = H_*(FA)$ then $IH_*(FA)^2$ is contained in the kernel of σ.

THEOREM 17. *Let $\mathscr{C}$ be the category of Z-free connected DGA-Hopf algebras A' such that every decomposable d_2-cycle of FA is a d_2-boundary where $A = A' \otimes Z_p$. Then βQ and λ satisfy the following properties on the objects of $\mathscr{C}$:*

(a) *λ is natural.*

(a)′ *βQ is natural.*

(b) *λ is linear.*

(b)′ *$\beta Q(x+y) = \beta Q(x) + \beta Q(y) + \sum_{i=1}^{p-1} \beta\, d^i(x \otimes y)$ and $\beta Q(kx) = k\beta Q(x)$ if $x, y \in H_{2n-1}(FA)$ and $k \in Z_p$ where $d^i(x \otimes y)$ is defined by*

$$i\, d^i(x \otimes y) = \sum \operatorname{ad}_\lambda^{j_1}(x)\operatorname{ad}_\lambda^{k_1}(y)\cdots\operatorname{ad}_\lambda^{j_r}(x)\operatorname{ad}_\lambda^{k_r}(y)(x),$$

the sum taken over all $(j_1, k_1, \ldots, j_r, k_r)$ such that $k_s \geqq 1, j_1 \geqq 0, j_t \geqq 1$ if $t > 1$, $\sum j_s = i-1$ and $\sum k_s = p-i$. The important points to note are:

(1) *$\beta Q(x+y) - \beta Q(x) - \beta Q(y)$ is in the image under the Bockstein of the sub-λ-algebra of $H_*(FA)$ generated by x and y,*

(2) *by* (f), *βQ is additive if A has a cup-one product such that $a, b \in PA$ implies $a \cup_1 b \in PA$.*

(c) *λ suspends to the commutator operation on $H_*(A)$.*

(c)′ *The relation Q suspends to the pth power operation on $H_*(A)$.*

(d) (External Cartan Formula) *Let $A', B', C' \in \mathscr{C}$ and assume that $f': A' \otimes B' \to C'$ is a map of DGA-Hopf algebras with $f_*: H_*(A \otimes B) \to H_*(C)$ an isomorphism. Let $\eta: FA \otimes FB \to F(A \otimes B)$ be the canonical map. Then $\eta_*^{-1} \circ F(f)_*^{-1}$ defines λ on $H_*(FA) \otimes H_*(FB)$ and*

$$\lambda((x \otimes y) \otimes (u \otimes v)) = (-1)^{\deg y(1+\deg u)}\lambda(x \otimes u) \otimes yv + (-1)^{\deg u(1+\deg y)} xu \otimes \lambda(y \otimes v)$$

where $x, u \in H_*(FA)$ and $y, v \in H_*(FB)$.

(d)″ (Internal Cartan Formula). *If $A \in \mathscr{C}$ and $x, y, u, v \in H_*(FA)$ all have positive degree then*

$$\lambda(xy \otimes uv) = (-1)^{\deg y(1+\deg u)}\lambda(x \otimes u)yv + (-1)^{\deg u(1+\deg y)}xu\lambda(y \otimes v)$$

and

$$\lambda(y \otimes uv) = \lambda(y \otimes u)v + (-1)^{\deg u \deg v}\lambda(y \otimes v)u.$$

(d)′ (Cartan Formula). *Let $A', B', C' \in \mathscr{C}$ and assume that $f'\colon A' \otimes B' \to C'$ [or $g'\colon C' \to A' \otimes B'$] is a map of DGA-Hopf algebras with $f_*\colon H_*(A \otimes B) \to H_*(C)[g_*\colon H_*(C) \to H_*(A \otimes B)]$ an isomorphism. Then $\eta_*^{-1} \circ F(f)_*^{-1}[\eta_*^{-1} \circ F(g)_*]$ defines βQ on $H_*(FA) \otimes H_*(FB)$ and [if $\lambda=0$ on $H_*(FA)$, $H_*(FB)$ and $H_*(FC)$ then]*

$$\beta Q(x \otimes y) = x^p \otimes \beta Q(y) + \beta Q(x) \otimes y^p$$

where $x \in H_*(FA)$, $y \in H_*(FB)$ and $\deg x \otimes y$ *is odd.*

(e) $\lambda(x \otimes y) = (-1)^{\deg x \cdot \deg y + \deg x + \deg y}\lambda(y \otimes x)$ *if $x, y \in H_*(FA)$.*

(f) *If A has a cup-one product such that $a, b \in PA$ implies $a \cup_1 b \in PA$ then $\lambda = 0$ on $H_*(FA)$.*

(g) $(-1)^{(1+\deg x)(1+\deg z)}\lambda(x \otimes \lambda(y \otimes z)) + (-1)^{(1+\deg y)(1+\deg x)}\lambda(y \otimes \lambda(z \otimes x)) + (-1)^{(1+\deg z)(1+\deg y)}\lambda(z \otimes \lambda(x \otimes y)) = 0$ *if $x, y, z \in H_*(FA)$.*

Proof. (a), (a)′, (b), (c), (c)′, (e) and (g) are clear from Definitions 12 and 16.

(b)′ Write $x = \{[a]\}$ and $y = \{[b]\}$ with $a, b \in PA_{2n}$, $d(a) = 0$ and $d(b) = 0$. Then

$$\beta Q(x+y) = \beta\{[(a+b)^p]\} = \beta\{[a^p]\} + \beta\{[b^p]\} + \sum_{i=1}^{p-1} \beta\{[s_i(a \otimes b)]\}$$

by N. Jacobson [9, p. 187]. $s_i(a \otimes b)$ is defined in the same way as $d^i(a \otimes b)$ by replacing λ with the commutator operation. Hence

$$\beta Q(x+y) = \beta Q(x) + \beta Q(y) + \sum_{i=1}^{p-1} \beta d^i(x \otimes y).$$

$$\beta Q(kx) = \beta\{[k^p a^p]\} = k\beta\{[a^p]\} = k\beta Q(x).$$

(d) Observe that $F(f)_*$ is an isomorphism by the naturality of the Eilenberg-Moore spectral sequence.

Case 1. *x, y, u and v all have positive degree.* Let $x = \{[a]\}$, $y = \{[b]\}$, $u = \{[c]\}$ and $v = \{[e]\}$ with $a \in PA_\alpha$, $b \in PB_\beta$, $c \in PA_\gamma$, $e \in PB_\varepsilon$, $d(a) = 0$, $d(b) = 0$, $d(c) = 0$ and $d(e) = 0$. Then $F(f) \circ \eta([a] \otimes [b]) = [f(a \otimes 1) \mid f(1 \otimes b)]$ and $F(f) \circ \eta([c] \otimes [e]) = [f(c \otimes 1) \mid f(1 \otimes e)]$. Since $[f(a \otimes 1) \mid f(1 \otimes b)]$ and $[f(c \otimes 1) \mid f(1 \otimes e)]$ are decomposable d_2-cycles, there are $g, h \in C$ so that $[f(a \otimes 1) \mid f(1 \otimes b)]$ is homologous to $[d(g)]$ and $[f(c \otimes 1) \mid f(1 \otimes e)]$ is homologous to $[d(h)]$. Moreover, $d(g)$

and $d(h)$ are primitive, $\bar{\psi}(g)=(-1)^{\alpha}f(a\otimes 1)\otimes f(1\otimes b)$ and $\bar{\psi}(h)=(-1)^{\gamma}f(c\otimes 1)\otimes f(1\otimes e)$. Hence

$$\begin{aligned}
\lambda((x\otimes y)\otimes(u\otimes v)) \\
&= \eta_*^{-1}\circ F(f)_*^{-1}\{[d(g)d(h)-(-1)^{(\alpha+\beta+1)(\gamma+\varepsilon+1)}d(h)d(g)]\} \\
&= (-1)^{\gamma(\beta+1)}\{[ac-(-1)^{\alpha\gamma}ca]\otimes[b|e]\}+(-1)^{\beta(\gamma+1)}\{[a|c]\otimes[be-(-1)^{\beta\varepsilon}eb]\} \\
&\quad +\eta_*^{-1}\circ F(f)_*^{-1}\{d([gd(h)-(-1)^{(\alpha+\beta+1)(\gamma+\varepsilon+1)}hd(g)]-[g|h] \\
&\qquad +(-1)^{(\alpha+\beta+1)(\gamma+\varepsilon+1)}[h|g] \\
&\qquad -(-1)^{\alpha+\beta}[f(a\otimes 1)\mid f(1\otimes b)h] \\
&\qquad -(-1)^{\beta(\gamma+\varepsilon+1)+\alpha+\gamma+\varepsilon}[f(a\otimes 1)h\mid f(1\otimes b)] \\
&\qquad -(-1)^{(\alpha+\beta)(\gamma+\varepsilon+1)}[f(c\otimes 1)\mid f(1\otimes e)g] \\
&\qquad -(-1)^{\gamma(\alpha+\beta)}[f(c\otimes 1)g\mid f(1\otimes e)] \\
&\qquad\quad -(-1)^{\beta(\varepsilon+\gamma+1)+\alpha}[f(ac\otimes 1)\mid f(1\otimes eb)])\} \\
&= (-1)^{\gamma(\beta+1)}\lambda(x\otimes u)\otimes yv+(-1)^{\beta(\gamma+1)}xu\otimes\lambda(y\otimes v).
\end{aligned}$$

Case 2. $x=1$ *and* y, u, v *have positive degree.* With the notation of Case 1,

$$\begin{aligned}
\lambda((1\otimes y)\otimes(u\otimes v)) &= \eta_*^{-1}\circ F(f)_*^{-1}\{[f(1\otimes b)d(h)-(-1)^{\beta(\gamma+\varepsilon+1)}d(h)f(1\otimes b)]\} \\
&= (-1)^{\beta(\gamma+1)}\{[c]\otimes[be-(-1)^{\beta\varepsilon}eb]\} \\
&\quad +\eta_*^{-1}\circ F(f)_*^{-1}\{d[(-1)^{\beta}f(1\otimes b)h-(-1)^{\beta(\gamma+\varepsilon+1)}hf(1\otimes b)]\} \\
&= (-1)^{\beta(\gamma+1)}u\otimes\lambda(y\otimes v).
\end{aligned}$$

Case 3. $x=1$, $v=1$ *and* y, u *have positive degree.* With the notation of Case 1,

$$\begin{aligned}
\lambda((1\otimes y)\otimes(u\otimes 1)) &= \eta_*^{-1}\circ F(f)_*^{-1}\{[f(1\otimes b)f(c\otimes 1)-(-1)^{\beta\gamma}f(c\otimes 1)f(1\otimes b)]\} \\
&= \eta_*^{-1}\circ F(f)_*^{-1}\{[(-1)^{\beta\gamma}f(c\otimes b)-(-1)^{\beta\gamma}f(c\otimes b)]\} = 0.
\end{aligned}$$

(d)″ To prove the first assertion apply Case 1 of the proof of (d) with A, B and C all equal to A and $F(f)\circ\eta$ replaced by the multiplication μ of FA. The computation which concludes the argument must be appropriately altered to take place in $H_*(FA)$ rather than in $H_*(FA)\otimes H_*(FA)$ since μ_* is not an isomorphism. We now prove the second assertion with the notation of Case 1 of the proof of (d).

$$\begin{aligned}
\lambda(y\otimes uv) &= \{[bd(h)-(-1)^{\beta(\gamma+\varepsilon+1)}d(h)b]\}-\{d[(-1)^{\beta}bh-(-1)^{\beta(\gamma+\varepsilon+1)}hb]\} \\
&= \{[bc-(-1)^{\beta\gamma}cb\mid e]\}+(-1)^{\beta(\gamma+1)}\{[c\mid be-(-1)^{\beta\varepsilon}eb]\} \\
&= \lambda(y\otimes u)v+(-1)^{(\gamma+1)(\varepsilon+1)}\lambda(y\otimes v)u.
\end{aligned}$$

(d)′ We will show that if $\deg x$ is even and $\deg y$ is odd then $\beta Q(x\otimes y)=x^p\otimes\beta Q(y)$. $\xi\colon F(A\otimes B)\to FA\otimes FB$ and $F(f)\colon F(A\otimes B)\to FC$ [or $F(g)\colon FC\to F(A\otimes B)$] are maps of DGA-algebras which induce isomorphisms in homology. Hence by J. P. May [13, p. 540] any one of $FA\otimes FB$, $F(A\otimes B)$ or FC can be used to calculate Massey products in $H_*(F(A\otimes B))$. Let $x=\{[a]\}$ and $y=\{[b]\}$ with $a\in PA$, $b\in PB$, $d(a)=0$, and $d(b)=0$. Then $(-1)^{i+1}(1/i!)[a|\cdots|a]\otimes[b^i]$ for

$1 \leqq i \leqq p-1$ is a defining system in $FA \otimes FB$ for $\langle x \otimes y\rangle^p$. Hence $\langle x \otimes y\rangle^p = x^p \otimes \langle y\rangle^p$.

$$\begin{aligned}\beta Q(x \otimes y) &= -\langle x \otimes y\rangle^p + \mathrm{ad}_\lambda^{p-1}(x \otimes y)(\beta x \otimes y + x \otimes \beta y) \quad \text{by Theorem 13} \\ &= -x^p \otimes \langle y\rangle^p + \mathrm{ad}_\lambda^{p-1}(x \otimes y)(x \otimes \beta y) \quad \text{by (d)}\end{aligned}$$

since $y^2=0$ and $\lambda(y \otimes y)=0$ $[\lambda=0]$. Since $\lambda(y \otimes y\beta(y)) = y\lambda(y \otimes \beta(y))$ by (d)″ and $y^2=0$,

$$\begin{aligned}&\mathrm{ad}_\lambda^{p-1}(x \otimes y)(x \otimes \beta y) \\ &= x^p \otimes \mathrm{ad}_\lambda^{p-1}(y)(\beta y) + \sum_{i=1}^{p-1} x^{i-1}\lambda(x \otimes x^{p-i}) \otimes y\, \mathrm{ad}_\lambda^{p-2}(y)(\beta y) \\ &= x^p \otimes \mathrm{ad}_\lambda^{p-1}(y)(\beta y) + \sum_{i=1}^{p-1} (p-i)x^{p-2}\lambda(x \otimes x) \otimes y\, \mathrm{ad}_\lambda^{p-2}(y)(\beta y) \quad \text{by (d)}'' \\ &= x^p \otimes \mathrm{ad}_\lambda^{p-1}(y)(\beta y)\end{aligned}$$

since $\sum_{i=1}^{p-1} p-i = \frac{1}{2}p(p-1)$ which is divisible by p. Thus,

$$\beta Q(x \otimes y) = x^p \otimes (-\langle y\rangle^p + \mathrm{ad}_\lambda^{p-1}(y)(\beta y)) = x^p \otimes \beta Q(y).$$

(f) Let $x = \{[a]\}$, $y = \{[b]\} \in H_*(FA)$ with $a, b \in PA$, $d(a)=0$ and $d(b)=0$. Then $a \cup_1 b \in PA$ and $d(a \cup_1 b) = ab - (-1)^{\deg a \deg b} ba$ by Definition 1. Hence

$$\lambda(x \otimes y) = \{[ab - (-1)^{\deg a \deg b} ba]\} = \{[d(a \cup_1 b)]\} = \{d[a \cup_1 b]\} = 0.$$

A second loop space $B = \Omega^2 C$ has two homology operations defined on $H_*(B; Z_p)$: a Dyer-Lashof operation $Q^n\colon H_{2n-1}(B) \to H_{2np-1}(B)$, $n \geqq 1$, and a Browder operation $\lambda_1\colon H_j(B) \otimes H_k(B) \to H_{j+k+1}(B)$, $j \geqq 0$, $k \geqq 0$. W. Browder [3] and J. P. May [15, §6] have shown that these two operations satisfy the analogues of properties (a)–(g) and (a)′–(d)′ of Theorem 17.

The following theorem shows that βQ is uniquely determined by several of its properties.

THEOREM 18. *Let $A', B' \in \mathscr{C}$ and assume that $H_*(A) = E\{L^-\} \otimes P\{L^+\}$ as coalgebras where L^- is a set of odd degree elements and L^+ is a set of even degree elements. Let $\Delta\colon A \to B$ be a map of DGA-Hopf algebras and $f\colon A \otimes A \to B$ [or $g\colon B \to A \otimes A$] a map of DGA-Hopf algebras with $f_*\colon H_*(A \otimes A) \to H_*(B)$ [or $g_*\colon H_*(B) \to H_*(A \otimes A)$] an isomorphism [and $\lambda=0$ on $H_*(A)$ and on $H_*(B)$]. Assume that $R\colon H_{2n-1}(FX) \to H_{2np-1}(FX)$ for all $n \geqq 1$ is defined for $X=A$ and $X=B$ such that*

(1) *$R \circ F(\Delta)_* = F(\Delta)_* \circ R$.*

(2) *$R(x+y) = R(x) + R(y) + \sum_{i=1}^{p-1} d^i(x \otimes y)$ and $R(kx) = kR(x)$ if $k \in Z_p$,*

$$x, y \in H_{2n-1}(FX) \quad \textit{for } X = A \textit{ and } X = B.$$

(3) *R suspends to the pth power operation for $X=A$ and $X=B$.*

(4) *R satisfies the Cartan formula on $H_*(FA) \otimes H_*(FA)$.*

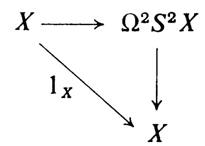
Then $\beta Q = \beta R$ on $H_*(FA)$.

Proof. Assume that $\beta Q \neq \beta R$ on $H_*(FA)$ and let $x \in H_{2n-1}(FA)$ be an element of smallest degree for which $\beta Q(x) \neq \beta R(x)$. By the naturality of the Eilenberg-Moore spectral sequence $F(f)_*: H_*(FA) \otimes H_*(FA) \to H_*(FB)$ [or $F(g)_*: H_*(FB) \to H_*(FA) \otimes H_*(FA)$] is an isomorphism, and hence we can define a coproduct on $H_*(FA)$ by $\psi = F(f)_*^{-1} \circ F(\Delta)_*$ [$\psi = F(g)_* \circ F(\Delta)_*$]. If $\psi(x) = \sum x' \otimes x''$ then $\psi \circ \beta Q(x) = \sum \beta Q(x') \otimes x''^p + \sum x'^p \otimes \beta Q(x'')$ and $\psi \circ \beta R(x) = \sum \beta R(x') \otimes x''^p + \sum x'^p \otimes \beta R(x'')$ by Theorem 17(d)$'$ and hypothesis (4) of this theorem. Hence $\beta Q(x) - \beta R(x)$ is primitive. Therefore it is indecomposable since $\deg(\beta Q(x) - \beta R(x)) = 2np-2$, and the only decomposable primitive elements of $H_*(FA)$ are pth powers. Let $Q(x)$ denote a fixed choice of the relation Q on x. Then $Q(x) - R(x)$ is in the kernel of the suspension map σ. Furthermore, $Q(x) - R(x)$ is indecomposable because its image under β is indecomposable. By J. P. May [12], $H_*(FA) = P\{sL^-\} \otimes E\{sL^+\} \otimes P\{tL^+\}$ as algebras. Hence tL^+ is a Z_p-basis of (Kernel σ) $\cap$ $QH_*(FA)$, all of whose elements lie in degrees congruent to $p-2 \bmod p$. This is a contradiction since $\deg(Q(x) - R(x)) = 2np - 1 \not\equiv p-2 \bmod p$. Hence $\beta Q = \beta R$ on $H_*(FA)$.

THEOREM 19. *Let $x \in H_{2n-1}(X)$, $X = \Omega^2 Z$, X connected and $n \geqq 1$. Then $\langle x\rangle^p$ is defined with zero indeterminacy and*

$$\langle x\rangle^p = -\beta Q^n(x) + \mathrm{ad}_{\lambda_1}^{p-1}(x)(\beta x).$$

Proof. By Theorem 3 and Lemma 9, $\langle x\rangle^p$ is defined with zero indeterminacy. The following diagram commutes and the vertical map is a morphism of second loop spaces:



Hence it suffices to evaluate $\langle x\rangle^p$ on $H_*(X) \subset H_*(\Omega^2 S^2 X)$. By J. F. Adams [1], there is a map $\phi: FC_*(\Omega S^2 X) \to C_*(\Omega^2 S^2 X)$ of differential algebras which induces an isomorphism in homology. By D. Kraines [11], the hypotheses of Theorem 13 are satisfied. Hence $\langle x\rangle^p = -\beta Q(x) + \mathrm{ad}_\lambda^{p-1}(x)(\beta x)$. It remains to show that βQ and λ correspond to βQ^n and λ_1 under the isomorphism ϕ_*. As a first step, we will prove that the Adams map ϕ_* commutes with the suspension map and the external product. That is, we will show that Figures 1 and 2 commute. Figure 1 clearly commutes. Note that

$$\eta: C_*(\Omega S^2 X) \otimes C_*(\Omega S^2 Y) \to C_*(\Omega(S^2 X \times S^2 Y))$$

is a map of DGA-Hopf algebras and hence induces an algebra homomorphism $F(\eta)$. Figure 3 shows that Figure 2 commutes on elements of $H_*(FC_*(\Omega S^2 X)) \otimes 1$. Similarly Figure 2 commutes on elements of $1 \otimes H_*(FC_*(\Omega S^2 Y))$, and hence Figure 2 commutes on all elements of $H_*(FC_*(\Omega S^2 X)) \otimes H_*(FC_*(\Omega S^2 Y))$ since

$$\begin{array}{ccc}
H_*(FC_*(\Omega S^2X)) & \xrightarrow{\ \phi_*\ } & H_*(\Omega^2S^2X) \\
\quad{\scriptstyle\sigma}\searrow & & \swarrow{\scriptstyle\sigma_*}\quad \\
& H_*(\Omega S^2X) &
\end{array}$$

FIGURE 1

$$\begin{array}{ccccc}
H_*(FC_*(\Omega S^2X))\otimes H_*(FC_*(\Omega S^2Y)) & \to H_*(FC_*(\Omega S^2X)\otimes FC_*(\Omega S^2Y)) & \to H_*(F(C_*(\Omega S^2X)\otimes C_*(\Omega S^2Y))) & \xrightarrow{F(n)_*} & H_*(FC_*(\Omega(S^2X\times S^2Y))) \\
\downarrow{\scriptstyle \phi_*\otimes\phi_*} & & & & {\scriptstyle\phi_*}\downarrow \\
H_*(\Omega^2S^2X)\otimes H_*(\Omega^2S^2Y) & \longrightarrow & H_*(C_*(\Omega^2S^2X)\otimes C_*(\Omega^2S^2Y)) & \xrightarrow{\ \eta_*\ } & H_*(\Omega^2(S^2X\times S^2Y))
\end{array}$$

FIGURE 2

$$\begin{array}{ccccccc}
H_*(FC_*(\Omega S^2X))\otimes 1 & \longrightarrow & H_*(FC_*(\Omega S^2X)\otimes FC_*(*)) & \longrightarrow & H_*(F(C_*(\Omega S^2X)\otimes C_*(*))) & \longrightarrow & H_*(FC_*(\Omega(S^2X\times *))) \\
\downarrow & \text{commutes} & \downarrow & \text{commutes} & \downarrow & \text{commutes} & \downarrow \\
H_*(FC_*(\Omega S^2X))\otimes H_*(FC_*(\Omega S^2Y)) & \to & H_*(FC_*(\Omega S^2X)\otimes FC_*(\Omega S^2Y)) & \to & H_*(F(C_*(\Omega S^2X)\otimes C_*(\Omega S^2Y))) & \to & H_*(FC_*(\Omega(S^2X\times S^2Y))) \\
{\scriptstyle\phi_*\otimes 1}\ \ \text{commutes}\ \ \downarrow{\scriptstyle\phi_*\otimes\phi_*} & & & & & & {\scriptstyle\phi_*}\downarrow\ \ \text{commutes}\ \ {\scriptstyle\phi_*} \\
H_*(\Omega^2S^2X)\otimes H_*(\Omega^2S^2Y) & \longrightarrow & & & H_*(C_*(\Omega^2S^2X)\otimes C_*(\Omega^2S^2Y)) & \longrightarrow & H_*(\Omega^2(S^2X\times S^2Y)) \\
\uparrow & \text{commutes} & & & \uparrow & \text{commutes} & \uparrow \\
H_*(\Omega^2S^2X)\otimes 1 & \longrightarrow & & & H_*(C_*(\Omega^2S^2X)\otimes C_*(*)) & \longrightarrow & H_*(\Omega^2(S^2X\times *))
\end{array}$$

FIGURE 3

all the maps in Figure 2 are algebra isomorphisms. J. P. May [unpublished] has proved that the first Browder operation λ_1 is the unique operation on the category of second loop spaces which satisfies the analogues of the properties (a)–(g) of Theorem 17. Hence $\phi_* \circ \lambda \circ (\phi_*^{-1} \otimes \phi_*^{-1})$ must equal λ_1. Recall that $H_*(\Omega S^2 X) = T(H_*(SX)) = P\{L^+\} \otimes E\{L^-\}$ where $L = L^+ \cup L^-$ is the free Lie algebra generated by $H_*(SX)$. Hence by Theorem 18, βQ equals $\phi_*^{-1} \circ \beta Q^n \circ \phi_*$ on $H_*(\Omega^2 S^2 X)$.

COROLLARY 20. *Let x be a $(2n-1)$-dimensional homology class of a connected third loop space, $n \geqq 1$. Then $\langle x \rangle^p$ is defined with zero indeterminacy and $\langle x \rangle^p = -\beta Q^n(x)$.*

Proof. This result follows from Theorem 19 and the fact that the first Browder operation is zero on the homology of a third loop space.

Observe that $\mathrm{ad}_{\lambda_1}^{p-1}(x)(\beta x)$ may be nonzero. For example, let M be the Moore space with $H_{2n-1}(M; Z_p) = Z_p x$, $H_{2n-2}(M; Z_p) = Z_p \beta x$ and $\tilde{H}_k(M; Z_p) = 0$ otherwise, $k \geqq 1$. Then $\mathrm{ad}_{\lambda_1}^{p-1}(x)(\beta x) \neq 0$ in $H_*(\Omega^2 S^2 M; Z_p)$ since its suspension in $H_*(\Omega S^2 M; Z_p)$ is $\mathrm{ad}^{p-1}(\sigma_*(x))(\sigma_* \beta x)$ which is nonzero because $H_*(\Omega S^2 M; Z_p)$ contains the free Lie algebra generated by $H_*(SM; Z_p)$.

We can also use the machinery of this section to prove the following theorem of D. Kraines [10]:

THEOREM 21. *Let x be a $(2n+1)$-dimensional cohomology class of any topological space, $n \geqq 0$. Then $\langle x \rangle^p$ is defined with zero indeterminacy and $\langle x \rangle^p = -\beta \mathscr{P}^n(x)$.*

Proof. $\langle x \rangle^p$ is defined with zero indeterminacy by Lemma 9. By the naturality of $\langle\ \rangle^p$ and $\beta \mathscr{P}^n$, it suffices to show that $\langle \iota \rangle^p = -\beta \mathscr{P}^n(\iota)$ where ι is the fundamental class of $K(Z_p, 2n+1)$. Assume first that $n \geqq 1$. By S. Eilenberg and S. Mac Lane [6, p. 94] there is a map $\gamma\colon C^*(K(Z_p, 2n+1)) \to FC^*(K(Z_p, 2n))$ of differential algebras which induces an isomorphism in homology. By D. Kraines [11] the hypotheses of Theorem 13 are satisfied. Hence $\langle \iota \rangle^p = -\beta Q(\iota) + \mathrm{ad}_\lambda^{p-1}(\iota)(\beta \iota)$. We will now show that $\lambda = 0$. Let Y be a topological space with $u_i \in H^{n_i}(Y)$ represented by $\hat{u}_i\colon Y \to K(Z_p, n_i)$ for $i = 1, 2$. In $H^*(K(Z_p, n_1) \times K(Z_p, n_2))$,

$$\lambda((\iota_{n_1} \otimes 1) \otimes (1 \otimes \iota_{n_2})) = 0$$

by Theorem 17(d). Hence $\lambda(u_1 \otimes u_2) = (\hat{u}_1 \times \hat{u}_2)^* \circ \lambda((\iota_{n_1} \otimes 1) \otimes (1 \otimes \iota_{n_2})) = 0$. Thus, $\langle \iota \rangle^p = -\beta Q(\iota)$. It remains to show that βQ corresponds to $\beta \mathscr{P}^n$ under γ_*. Reasoning as in the proof of Theorem 19, one sees that γ_* commutes with the suspension map and the external product. It is well known that $H^*(K(Z_p, 2n))$ is an exterior Hopf algebra on odd degree elements tensored with a polynomial Hopf algebra on even degree elements. Hence by Theorem 18, $\beta Q(\iota) = \beta \mathscr{P}^n(\iota)$. If $n = 0$ then in $H^*(K(Z_p, 1) \times K(Z_p, 2))$, $\langle \iota_1 \rangle^p \otimes \iota_2^p = \langle \iota_1 \otimes \iota_2 \rangle^p = -\beta \mathscr{P}^1(\iota_1 \otimes \iota_2) = -\beta \iota_1 \otimes \iota_2^p$. Hence $\langle \iota_1 \rangle^p = -\beta \iota_1$.

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