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THE STRUCTURE OF CERTAIN UNITARY REPRESENTATIONS OF INFINITE SYMMETRIC GROUPS

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Abstract. Let S be an infinite set, β an infinite cardinal number, and $G_\beta(S)$ the group of those permutations of S whose support has cardinal number less than β. If T is any nonempty set, S^T is the set of functions from T to S. The canonical representation Λ_β^T of $G_\beta(S)$ on $L^2(S^T)$ is the direct sum of factor representations. Factor representations of types I_∞, II_1, and II_∞ occur in this decomposition, depending upon S, β, and T; the type II_1 factor representations are quasi-equivalent to the left regular representation.

Let $G_\beta(S)$ have the topology of pointwise convergence on S. $G_\beta(S)$ is a topological group but is not locally compact. Every continuous representation of $G_\beta(S)$ is the direct sum of irreducible representations. Let Γ be a nontrivial continuous irreducible representation of $G_\beta(S)$. Then Γ is continuous iff Γ is equivalent to a subrepresentation of Λ_β^T for some nonempty finite set T iff there is a nonempty finite subset Z of S such that the restriction of Γ to the subgroup of those permutations which leave Z pointwise fixed contains the trivial representation of this subgroup.

I. **Introduction.** The representations of the symmetric group on a finite set have been studied in great detail. Little has been done toward analyzing and classifying the representations of the symmetric groups on an infinite set. Thoma [7] has found all type II_1 factor representations of the group of finite permutations of a countably infinite set; he did this by explicitly constructing the characters of the group.

If S is an infinite set and β is a cardinal number, let $G_\beta(S)$ be the group of all permutations p of S such that the cardinal number of $\{s \in S : p(s) \neq s\}$ is less than β.

Below we analyze the canonical representation Λ_β^T of $G_\beta(S)$ on $L^2(S^T)$, where T is any nonempty set and S^T is the set of functions with domain T and range contained in S. Λ_β^T is the direct sum of factor representations of $G_\beta(S)$. The factor representations in the decomposition are of types I_∞, II_1, and II_∞, depending upon S, T, and β; the type II_1 factor representations are all quasi-equivalent to the left regular representation of $G_\beta(S)$.

Assume now that T is finite and nonempty. Then the von Neumann algebra generated by $\Lambda_\beta^T(G_\beta(S))$ is the same for all infinite cardinal numbers β. Λ_β^T is the direct sum of finitely many infinite-dimensional irreducible representations of $G_\beta(S)$. The set of equivalence classes of irreducible subrepresentations of the

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representation Λ_β^T of $G_\beta(S)$, as the cardinal number of T varies over the positive integers, is in a canonical one-to-one correspondence with the set of equivalence classes of irreducible representations of the symmetric group on n symbols, as n ranges over the positive integers.

Let $G_\beta(S)$ have the topology of pointwise convergence on S. $G_\beta(S)$ is a topological group but is not locally compact. We will say that a representation of $G_\beta(S)$ is continuous if it is continuous with respect to this topology.

Λ_β^T is continuous if T is finite. We show that any continuous representation of $G_\beta(S)$ is the direct sum of irreducible representations. Any nontrivial continuous irreducible representation of $G_\beta(S)$ is equivalent to a subrepresentation of Λ_β^T for some nonempty finite set T.

We further characterize the continuous irreducible representations of $G_\beta(S)$ by showing that the following are equivalent if Γ is an irreducible representation of $G_\beta(S)$:

1. Γ is continuous.
2. $\exists S_1 \subseteq S$ such that S_1 is finite and the restriction of Γ to the subgroup of $G_\beta(S)$ of those permutations which leave S_1 pointwise invariant contains the trivial representation of this subgroup.
3. There is a nonnegative integer n such that condition 2 holds for any subset S_2 of S if S_2 has cardinal number $\geqq n$.

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II. Lemmas and notation. The following lemmas will be needed. Lemma 3 is well known; we include it for completeness.

LEMMA 1. *Let G be a group. Let H_i, $i=1, 2$, be a Hilbert space. Let Γ_i, $i=1, 2$, be a representation of G on H_i. Let $Z \subseteq G$. Assume*

1. $\Gamma_1 \sim \Gamma_2$.
2. $\exists v \in H_1$, $v \neq 0$, *such that* $\Gamma(g)v = v$ *if* $g \in Z$.

Then $\exists w \in H_2$, $w \neq 0$, *such that* $\Gamma_2(g)w = w$ *if* $g \in Z$.

Proof. Since $\Gamma_1 \sim \Gamma_2$, there exist cardinal numbers α_1, α_2 such that $\alpha_1\Gamma_1 \cong \alpha_2\Gamma_2$. $\alpha_i\Gamma_i$ acts on $\bigoplus_{a\in\alpha_i} H_{ia}$, where $H_{ia} = H_i$ for all $a \in \alpha_i$. Let V be a unitary operator which implements the equivalence between $\alpha_1\Gamma_1$ and $\alpha_2\Gamma_2$. If $b \in \alpha_i$, for $i=1, 2$, let P_{ib} be projection onto the subspace H_{ib} of $\bigoplus_{a\in\alpha_i} H_{ia}$.

Let $v \in H_1$ be chosen as in hypothesis 2. Let w in $\bigoplus_{a\in\alpha_i} H_{ia}$ be chosen so that $P_{10}w = v$ and $P_{1a}w = 0$ if $a \in \alpha_1$ and $a \neq 0$. Note that $\bigoplus_{a\in\alpha_2} P_{2a}$ is the identity operator on $\bigoplus_{a\in\alpha_2} H_{2a}$. Therefore there is an element c in α_2 such that $P_{2c}Vw \neq 0$. Let $x = P_{2c}Vw$. Let $g \in Z$. Then $\Gamma_2(g)x = \Gamma_2(g)P_{2c}Vw = P_{2c}\Gamma_2(g)Vw = P_{2c}V\Gamma_1(g)w = P_{2c}Vw = x$.

LEMMA 2. *Let G be a group, H a Hilbert space, and Γ a representation of G on H. Let J be a closed subspace of H, P be projection onto J, and $G_J=\{g \in G: \Gamma(g)J=J\}$. Assume*

a. $\exists X \subseteq G: H=\bigoplus_{x \in X} \Gamma(x)J$.

b. *If $g \in G$, then $\Gamma(g)J=J$ or $\Gamma(g)J \perp J$.*

c. $P \in \Gamma(G)''$.

Then $T \to TP$ is an algebraic $*$-isomorphism from $\Gamma(G)'$ onto $\Gamma(G)'|J$.

$$\Gamma(G)'|J = \Gamma(G_J)'|J.$$

Proof. $T \to TP$ is an algebraic $*$-homomorphism from $\Gamma(G)'$ onto $\Gamma(G)'|J$. $T=\bigoplus_{x \in X} \Gamma(x)TP\Gamma(x^{-1})|\Gamma(x)J$. Therefore $T=0$ if $TP=0$, so that $T \to TP$ is one-to-one.

If $g \in G-G_J$, then $P\Gamma(g)P=0$. Therefore $\Gamma(G)''|J=\Gamma(G_J)''|J$.

LEMMA 3. *Let H be a Hilbert space, G a group, and Γ a representation of G on H. Let Q be a subset of G, $J=\{v \in H : \Gamma(g)v=v$ if $g \in Q\}$, and P be projection onto J. Then $P \in \Gamma(G)''$.*

Proof. Let $v \in J$ and $T \in \Gamma(G)'$. Let g be an arbitrary member of Q. Then $\Gamma(g)Tv=T\Gamma(g)v=Tv$. Therefore $Tv \in J$. The conclusion follows immediately from the double commutant theorem.

Notation. Let S be an arbitrary infinite set. S will remain fixed for the remainder of this paper. If β is an infinite cardinal number, G_β is the set of permutations p of S such that the cardinal number of $\{s \in S: p(s) \neq s\}$ is less than β. Note that G_β is the group of all permutations of S if β is sufficiently large.

Let T be any set. Then $|T|$ will denote the cardinal number of T. $L^2(T)$ is the Hilbert space of all complex valued functions f with domain T such that $\|f\| < \infty$, where $\|f\|^2 = \sum_{t \in T} |f(t)|^2$.

Let Z be any set. T^Z will denote the set of functions with domain Z and range contained in T. $T^{Zi}=\{f \in T^Z : f$ is 1-1$\}$. $T^{Zb}=\{f \in T^Z : f$ is 1-1 and onto$\}$.

A partition of T is a collection of nonempty disjoint subsets of T whose union is T. $\mathscr{P}(T)$ will denote the set of partitions of T.

$G(T)$ denotes the group of all permutations of T. If β is an infinite cardinal number, $G_\beta(T)$ denotes $\{\pi \in G(T) : |\{t \in T: \pi(t) \neq t\}| < \beta\}$.

Recall [3, pp. 21–31] that a cardinal number α is also an ordinal number, and that $\alpha=\{$ordinal numbers $\gamma : 0 \leqq \gamma < \alpha\}$.

III. **The representation of G_β on $L^2(S^\alpha)$.** Let α be a nonzero cardinal number and β be an infinite cardinal number. Let Ω_β^α be the representation of G_β on $L^2(S^\alpha)$ defined by $\Omega_\beta^\alpha(g)v(f)=v(g^{-1}f)$, for $g \in G_\beta$, $v \in L^2(S^\alpha)$, and $f \in S^\alpha$. If $g \in G_\beta$, let $\Lambda_\beta^\alpha(g)$ be the restriction of $\Omega_\beta^\alpha(g)$ to $L^2(S^{\alpha i})$; Λ_β^α is a representation of G_β.

The following theorem reduces the study of the representations Ω_β^α of G_β to that of the representations Λ_β^α of G_β.

THEOREM 1. *Let α and β be cardinal numbers such that $\alpha>0$ and $\beta\geqq\aleph_0$. Then $\Omega_\beta^\alpha\simeq\bigoplus_{\mathscr{P}\in\mathscr{P}(\alpha)}\Lambda_\beta^{|\mathscr{P}|}\simeq\bigoplus_{1\leqq\gamma\leqq|S|}\zeta(\gamma)\Lambda_\beta^\gamma$, where $\zeta(\gamma)=|\{\mathscr{P}\in\mathscr{P}(\alpha):|\mathscr{P}|=\gamma\}|$ if γ is a cardinal number and $1\leqq\gamma\leqq|S|$.*

Proof. The equivalence of $\bigoplus_{\mathscr{P}\in\mathscr{P}(\alpha)}\Lambda_\beta^{|\mathscr{P}|}$ with $\bigoplus_{1\leqq\gamma\leqq|S|}\zeta(\gamma)\Lambda_\beta^\alpha$ is obvious.

If $f\in S^\alpha$, let $\mathscr{P}(f)=\{f^{-1}(s):s$ is in the range of $f\}$. $\mathscr{P}(f)\in\mathscr{P}(\alpha)$. Let $\theta(f)$ be the unique element of $S^{\mathscr{P}(f)i}$ such that $\theta f^{-1}(s)=s$ for all s in the range of f. θ is a one-to-one mapping from S^α onto $\bigcup_{\mathscr{P}\in\mathscr{P}(\alpha)}S^{\mathscr{P}i}$.

Let U be the mapping from $L^2(S^\alpha)$ to $\bigoplus_{\mathscr{P}\in\mathscr{P}(\alpha)}L^2(S^{\mathscr{P}i})$ defined by $Uv(h)=v(\theta^{-1}h)$ for $v\in L^2(S^\alpha)$ and $h\in\bigcup_{\mathscr{P}\in\mathscr{P}(\alpha)}S^{\mathscr{P}i}$. U is unitary and implements an equivalence between Ω_β^α and $\bigoplus_{\mathscr{P}\in\mathscr{P}(\alpha)}\Lambda_\beta^{\mathscr{P}}$.

Notation. If $T\subseteq S$, let P_T^α be projection onto the subspace $L^2(T^{\alpha b})$ of $L^2(S^{\alpha i})$.

If $0<\alpha\leqq|S|$, define a representation Φ of $G(\alpha)$ on $L^2(S^{\alpha i})$ by $\Phi(\pi)v(f)=v(f\pi)$, for $\pi\in G(\alpha)$, $v\in L^2(S^{\alpha i})$, and $f\in S^{\alpha i}$.

LEMMA 4. *Let α and β be cardinal numbers such that $\alpha>0$ and $\beta\geqq\aleph_0$. Let $T\subseteq S$. Assume $|T|=\alpha$ and $|S-T|\geqq\aleph_0$. Then $P_T^\alpha\in\Lambda_\beta^\alpha(G_\beta)''$.*

Proof. $P_T^\alpha L^2(S^{\alpha i})=\{v\in L^2(S^{\alpha i}):\Lambda_\beta^\alpha(g)v=v$ for all $g\in G_\beta(S-T)\}$. Therefore $P_T^\alpha\in\Lambda_\beta^\alpha(G_\beta)''$ by Lemma 3.

LEMMA 5. *Let $|S|\geqq\alpha>0$.*

1. *Let $T\subseteq S$ and $|T|=\alpha$. Then $\Phi|L^2(T^{\alpha b})\simeq$ the right regular representation of $G(\alpha)$.*

2. *Φ is quasi-equivalent to the right regular representation of $G(\alpha)$. If $\alpha\geqq\aleph_0$, then Φ is a type II_1 factor representation of $G(\alpha)$.*

3. *Let β be an infinite cardinal number. Assume $\beta>\alpha$. Then $\Lambda_\beta^\alpha(G_\beta)'=\Phi(G(\alpha))''$.*

4. *Let β be an infinite cardinal number and assume $\beta>\alpha$. Then $\Lambda_\beta^\alpha(G_\beta(T))|L^2(T^{\alpha b})$ is equivalent to the left regular representation of $G_\beta(T)$ if $T\subseteq S$ and $|T|=\alpha$.*

5. *Let β be an infinite cardinal number and assume $\beta\leqq\alpha$. Then $\Lambda_\beta^\alpha(G_\beta(T))|L^2(T^{\alpha b})$ is quasi-equivalent to the left regular representation of $G_\beta(T)$ if $T\subseteq S$ and $|T|=\alpha$.*

Proof. 1. Note that $L^2(T^{\alpha b})$ is invariant under $\Phi(G(\alpha))$.

Let $f\in T^{\alpha b}$. Let an operator W from $L^2(T^{\alpha b})$ to $L^2(G(\alpha))$ be defined by $Wv(p)=v(fp)$, for $v\in L^2(T^{\alpha b})$ and $p\in G(\alpha)$. W is unitary and implements an equivalence between $\Phi|L^2(T^{\alpha b})$ and the right regular representation of $G(\alpha)$.

2. The first statement is immediate. The second follows from [4, pp. 483–486].

3. Assume $\beta>\alpha$. Let $T\subseteq S$ and assume $|T|=\alpha$. Let $Y\in\Lambda_\beta^\alpha(G_\beta)'$. By Lemma 4 and Lemma 2, to show that $Y\in\Phi(G(\alpha))''$, it suffices to show that $Y|L^2(T^{\alpha b})\in\Phi(G(\alpha))''|L^2(T^{\alpha b})$.

Let W and f be defined as in the proof of part 1 of this lemma. If $p\in G_\beta(T)$, then $W\Lambda_\beta^\alpha(p)W^{-1}$ is the image under the left regular representation of $G(\alpha)$ of the permutation $f^{-1}pf$ of α. Because $\beta>\alpha$, every permutation of α is of the form $f^{-1}pf$ for some $p\in G_\beta(T)$. Therefore $W\Lambda_\beta^\alpha(G_\beta(T))W^{-1}$ generates the same von Neumann algebra as the left regular representation of $G(\alpha)$. However, the ring generated by the left regular representation of $G(\alpha)$ is the commutant of the ring generated by the right regular representation of $G(\alpha)$.

4. Let $T \subseteq S$ and $|T| = \alpha$. Clearly $L^2(T^{\alpha b})$ is invariant under $\Lambda_\beta^\alpha(G_\beta(T))$. Let $f \in T^{\alpha b}$. Let an operator W be defined from $L^2(T^{\alpha b})$ to $L^2(G_\beta(T))$ by $Wv(p) = v(pf)$, for $v \in L^2(T^{\alpha b})$ and $p \in G_\beta(T)$. W is unitary and implements an equivalence between $\Lambda_\beta^\alpha(G_\beta(T))|L^2(T^{\alpha b})$ and the left regular representation of $G_\beta(T)$.

5. Assume $\alpha \geqq \beta \geqq \aleph_0$. Let $T \subseteq S$ and $|T| = \alpha$. Let W be defined as in the proof of part 4 of this lemma. Let Γ_1 be the restriction to $G_\beta(T)$ of the left regular representation of $G(T)$. W implements an equivalence between $\Lambda_\beta^\alpha(G_\beta(T))|L^2(T^{\alpha b})$ and Γ_1. Let Γ_2 be the right regular representation of $G(T)$.

Let RC be the set of right cosets of $G_\beta(T)$ in $G(T)$. Let C_1 be the right coset which contains the identity. Pick an element p_C from each member of RC. Then $\Gamma_1 = \bigoplus_{C \in RC} \Gamma_1|L^2(C) = \bigoplus_{C \in RC} \Gamma_2(p_C^{-1})\Gamma\Gamma_2(p_C)|L^2(C) \sim \Gamma_1|L^2(C_1) =$ the left regular representation of $G_\beta(T)$.

Notation. Let α and β be cardinal numbers such that $\alpha \geqq 0$ and $\beta \geqq \aleph_0$. Let $S^{\#\alpha i} = \{f \in S^{|S|i} : |S - f(S)| = \alpha\}$. Note that $L^2(S^{\#\alpha i})$ is a subspace of $L^2(S^{|S|i})$ and is invariant under $\Lambda_\beta^{|S|}(G_\beta)$. If $g \in G_\beta$, let $\Lambda_\beta^{\#\alpha}(g)$ be the restriction of $\Lambda_\beta^{|S|}(g)$ to $L^2(S^{\#\alpha i})$.

Note that if $T \subseteq S$, $|T| = |S|$, and $|S - T| = \alpha$, then $T^{|S|b} \subseteq S^{\#\alpha i}$.

In the following theorem, we will express the representations Λ_β^α of G_β as the direct sum of factor representations. The reader is referred to [8, Chapters III and IV] for a discussion of maximal symmetry types and the properties of the regular representation of the symmetric group on α symbols, where α is a positive integer.

THEOREM 2. *Let α and β be cardinal numbers, where $\alpha > 0$ and $\beta \geqq \aleph_0$.*

1. *Assume $\aleph_0 > \alpha$. Then*

a. *$\Lambda_\beta^\alpha(G_\beta)''$ is the same von Neumann algebra for all infinite cardinal numbers β.*

b. *Λ_β^α is the direct sum of irreducible representations of G_β. The irreducible subrepresentations of Λ_β^α are in a canonical one-to-one correspondence with the irreducible subrepresentations of the left regular representation of the full symmetric group on α symbols. This one-to-one correspondence and its inverse both preserve equivalence.*

2. *Assume $|S| > \alpha \geqq \aleph_0$.*

a. *If $\beta > \alpha$, then Λ_β^α is a type* II_∞ *factor representation of G_β.*

b. *If $\beta \leqq \alpha$, then Λ_β^α is the direct sum of infinitely many disjoint type* II_∞ *factor representations.*

3. *Assume $\alpha = |S|$.*

a. $\Lambda_\beta^\alpha = \bigoplus_{0 \leqq \gamma \leqq |S|} \Lambda_\beta^{\#\gamma}$.

b. *If $\beta > \gamma \geqq \aleph_0$, then $\Lambda_\beta^{\#\gamma}$ is a type* II_∞ *factor representation.*

c. If $|S| \geqq \gamma \geqq \beta$, *then $\Lambda_\beta^{\#\gamma}$ is the direct sum of infinitely many disjoint type* II_∞ *factor representations.*

d. *If $\aleph_0 > \gamma \geqq 0$, then $\Lambda_\beta^{\#\gamma}$ is quasi-equivalent to the left regular representation of G_β.*

4. a. *If* $0<\alpha,\ \alpha'\leqq|S|$, *then* $\Lambda_\beta^\alpha \mathrel{\delta} \Lambda_\beta^{\alpha'}$ *if* $\alpha\neq\alpha'$.

b. *If* $0\leqq\gamma\leqq|S|$ *and* $\aleph_0\leqq\gamma'\leqq|S|$ *then* $\Lambda_\beta^{\#\gamma} \mathrel{\delta} \Lambda_\beta^{\#\gamma'}$.

c. *If* $0\leqq\gamma,\ \gamma'\leqq\aleph_0$ *then* $\Lambda_\beta^{\#\gamma}\sim\Lambda_\beta^{\#\gamma'}$.

Proof. 1a. By Lemma 5, part 3, $\Lambda_\beta^\alpha(G_\beta)''=\Phi(G(\alpha))'$.

1b. By Lemma 5, part 3, $\Lambda_\beta^\alpha(G_\beta)'=\Phi(G(\alpha))''$. By Lemma 5, part 2, Φ is quasi-equivalent to the right regular representation of $G(\alpha)$. Since α is finite, $G(\alpha)$ is a finite group and $\Phi(G(\alpha))''$ is generated by its minimal projections. Consequently, Λ_β^α is the direct sum of certain of its irreducible subrepresentations.

Let $T\subseteq S$ be chosen so that $|T|=\alpha$. Then $\Lambda_\beta^\alpha(G(T))|L^2(T^{\alpha b})$ is equivalent to the left regular representation of $G(T)$ by Lemma 5, part 4. Let P vary over the set of minimal projections in $\Phi(G(\alpha))''$; then $\Lambda_\beta^\alpha|PL^2(S^{\alpha t})\leftrightarrow\Lambda_\beta^\alpha(G(T))|PL^2(T^{\alpha b})$ is a one-to-one correspondence between the irreducible subrepresentations of Λ_β^α and the irreducible subrepresentations of the left regular representation of the symmetric group on α symbols.

2a. By Lemma 5, part 3, $\Lambda_\beta^\alpha(G_\beta)'=\Phi(G(\alpha))''$. By Lemma 5, part 2, $\Phi(G\alpha))''$ is a type II_1 factor. Therefore Λ_β^α is a type II factor representation of G_β.

Let $T\subseteq S$ and $|T|=\alpha$. By Lemma 4, $P_T^\alpha\in\Lambda_\beta^\alpha(G_\beta)''$. For each $Z\subseteq S$ such that $|Z|=\alpha$, pick $g_Z\in G_\beta$ such that $g_Z(T)=Z$. Then $P_Z^\alpha=\Lambda_\beta^\alpha(g_Z)P_T^\alpha\Lambda_\beta^\alpha(g_Z^{-1})$. $P_Z^\alpha P_T^\alpha=0$ if $Z\neq T$. Therefore $\Lambda_\beta^\alpha(G_\beta)''$ contains infinitely many mutually perpendicular nonzero equivalent projections and consequently cannot be of finite type.

2b. Let $\mathscr{S}=\{T\subseteq S : |T|=\alpha\}$. Define an equivalence relation $\underline{\perp}$ on $\mathscr{S}$ by $T\underline{\perp}\,Q$ iff $|(T-Q)\cup(Q-T)|<\beta$ iff $\exists g\in G_\beta$ such that $g(T)=Q$. Let $\{C_\delta\colon\delta\in\Delta\}$, where Δ is an index set, be the set of equivalence classes of $\underline{\perp}$ in S. Δ is an infinite set because $\beta\leqq\alpha$. If $\delta\in\Delta$, let $H_\delta=\bigoplus_{T\in\delta}L^2(T^{\alpha b})$. Clearly $L^2(S^{\alpha t})=\bigoplus_{\delta\in\Delta}H_\delta$. H_δ is invariant under $\Lambda_\beta^\alpha(G_\beta)$. Therefore $\Lambda_\beta^\alpha=\bigoplus_{\delta\in\Delta}\Lambda_\beta^\alpha|H_\delta$.

By Lemma 4 and Lemma 2, $\Lambda_\beta^\alpha(G_\beta)'|H_\delta$ is algebraically $*$-isomorphic to $\Lambda_\beta^\alpha(G_\beta(T))'|L^2(T^{\alpha b})$. By Lemma 5, part 5, $\Lambda_\beta^\alpha|H_\delta$ is a type II factor representation. The proof that $\Lambda_\beta^\alpha|H_\delta$ is type II_∞ is similar to the last part of the proof of part 2a of this theorem and is left to the reader.

Let $T\in\mathscr{S}$ and $v\in L^2(T^{\alpha b})$. Then $\Lambda_\beta^\alpha(g)v=v$ if $g\in G_\beta(S-T)$. Let $Y\in\mathscr{S}$. Then $\Lambda_\beta^\alpha(p)v=v$ for all $p\in G_\beta(S-Y)$ iff $Y\supseteq T$. If $\delta\in\Delta$, $T_1\in\delta$, $T_2\in\delta$, $T_3\in\mathscr{S}$, and $T_1\supseteq T_3\supseteq T_2$, then clearly $T_3\in\delta$. It follows from Lemma 1 that $\Lambda_\beta^\alpha|H_\delta \mathrel{\delta} \Lambda_\beta^\alpha|H_{\delta'}$ if $\delta,\delta'\in\Delta$ and $\delta\neq\delta'$.

3. Part a is trivial. The proofs of parts b and c are similar to the proofs of 2b and 2c and are left to the reader.

3d. Let $\gamma=0$. By Lemma 5, parts 4 and 5, $U^{\#0}$ is quasi-equivalent to the left regular representation of G_β and so is a type II_1 factor representation.

Assume $0<\gamma<\aleph_0$. Let θ be any mapping from $S^{\#\gamma i}$ to $S^{\#0i}$ such that, if $f\in S^{\#\gamma i}$, then $(\theta f)(\delta)=f(\delta)$ if δ is an ordinal and $|S|>\delta\geqq\aleph_0$, and $\theta f(\delta)=f(\delta+\gamma)$ if δ is an ordinal and $\delta<\aleph_0$. θ is one-to-one. Let W be the unique operator from $L^2(S^{\#\gamma i})$ into $L^2(S^{\#0i})$ such that $W\chi(\{f\})=\gamma!^{-1/2}\sum_{g\in G(\gamma)}\Phi(g)\chi(\{\theta(f)\})$, for $f\in S^{\#\gamma i}$, where χ denotes characteristic function.

W is an isometry from $L^2(S^{\#\gamma i})$ to $L^2(S^{\#0i})$. Therefore, W implements an equivalence between $\Lambda_\beta^{\#\gamma}$ and a subrepresentation of $\Lambda_\beta^{\#0}$. Since $\Lambda_\beta^{\#0}$ is a factor representation, it is quasi-equivalent to any of its subrepresentations.

4a. Assume $Q \subseteq S$ and $|Q| < \alpha$. Let $v \in L^2(S^{\alpha i})$, $v \neq 0$. Then $\exists g \in G_\beta(S-Q)$ such that $\Lambda_\beta^\alpha(g)v \neq v$.

$L^2(S^{\alpha i}) = \bigoplus P_T^\alpha L^2(S^{\alpha i})$, where the direct sum is taken over all subsets T of S such that $|T| = \alpha$. If Γ is any subrepresentation of Λ_β^α, then there is a subset Z of S with $|Z| = \alpha$ and a nonzero vector w in the subspace on which Γ acts such that $P_Z^\alpha w = w$. Then $\Gamma(g)w = w$ for all $w \in G_\beta(S-Z)$. The conclusion is an immediate consequence of Lemma 1.

4b. The proof of 4b is similar to the proof of 4a and is left to the reader.

4c. This is trivial.

IV. The continuous unitary representations of the infinite symmetric groups.

Give G_β the topology of pointwise convergence on S. If $s \in S$, then $\{g \in G_\beta : g(s) = s\}$ is a subbasic open neighborhood of the identity. G_β is a topological group in this topology but is not locally compact. $G_{\aleph_0}$ is dense in G_β. G_β is complete as a uniform space iff G_β is the group of all permutations of S. A representation of G_β is continuous if it is continuous with respect to this topology on G_β and the weak operator topology on the unitary group of Hilbert space.

We define the trivial representation of any group as the one-dimensional representation which maps every element of the group to the identity operator. Λ_β^0 will denote the trivial representation of G_β.

THEOREM 3. *Let β be an infinite cardinal number. Let Γ be a continuous irreducible representation of G_β on the Hilbert space H. Then there is a nonnegative integer n such that*

1. *Γ is equivalent to an irreducible subrepresentation of Λ_β^n.*
2. *There is a subset Z of S such that $|Z| = n$ and the restriction of Γ to $G_\beta(S-Z)$ contains the trivial representation of $G_\beta(S-Z)$.*
3. *If Z is any subset of S such that $|Z| \geqq n$, then the restriction of Γ to $G_\beta(S-Z)$ contains the trivial representation of $G_\beta(S-Z)$.*

Conversely, if the irreducible representation Γ of G_β satisfies condition 1, *condition* 2, *or condition* 3, *then Γ is continuous.*

THEOREM 4. *Let Γ be a continuous representation of G_β. Then Γ is the direct sum of irreducible representations of G_β.*

Two lemmas are needed to prove these theorems. If n is a positive integer, $\mathscr{S}_n$ will denote $\{Q \subseteq S : |Q| = n\}$.

LEMMA 6. *Let H_i, $i = 1, 2$, be a Hilbert space and let Γ_i be a representation of G_β on H_i. Assume*

1. *There is a positive integer n such that, for* $i=1,2$, $H_i=\bigoplus_{Q\in\mathscr{S}_n} H_{iQ}$, *where* $H_{iQ}=\{v\in H_i : \Gamma_i(g)v=v \text{ for all } g\in G_\beta(S-Q)\}$.

2. *There is a subset T of S such that* $|T|=n$ *and* $\Gamma_1(G_\beta(T))|H_{1T}\cong\Gamma_2(G_\beta(T))|H_{2T}$.

Then $\Gamma_1\cong\Gamma_2$.

Proof. In Lemma 2, let $G=G_\beta$, $H=H_1\oplus H_2$, $\Gamma=\Gamma_1\oplus\Gamma_2$, and $J=H_{1T}\oplus H_{2T}$. Then P is projection onto $H_{1T}\oplus H_{2T}$, and, by assumption 1, $G_J=\{g\in G_\beta : g(T)=T\}$.

Assumption a of Lemma 2 follows from $|T|=n<\aleph_0\leqq\beta$ and assumption 1 of this lemma. Assumption b follows immediately from assumption 1 of this lemma; c follows from Lemma 3.

Let W be a unitary operator from H_{1T} onto H_{2T} which implements the equivalence in assumption 2. Let U be the unique operator on $H_{1T}\oplus H_{2T}$ such that $Ux=Wx$ if $x\in H_{1T}$ and $Ux=0$ if $x\in H_{2T}$. Then $U\in\Gamma(G_J)'|J$. By Lemma 2, there is an operator $U_0\in((\Gamma_1\oplus\Gamma_2)(G_\beta))'$ such that $U_0|J=U$. Let $W_0=U_0|H_1$. W_0 implements an equivalence between Γ_1 and Γ_2.

LEMMA 7. *Let* Γ *be a representation of* G_β *on the Hilbert space H. Let n be a positive integer. Assume*

a. *The subspace spanned by* $\bigcup_{Q\in\mathscr{S}_n} H_Q$ *is dense in H, where*

$$H_Q = \{x\in H : \Gamma(g)x = x \text{ for all } g\in G_\beta(S-Q)\}.$$

b. $H_Z=0$ *if* $Z\subseteq S$ *and* $|Z|<n$.

Then $H=\bigoplus_{Q\in\mathscr{S}_n} H_Q$.

Proof. Let $T_1, T_2\subseteq S$, $|T_1|=|T_2|=n$, and $T_1\neq T_2$. Assume H_{T_1} is not orthogonal to H_{T_2}. Let $R=T_1\cap T_2$. Note that $|R|<n$. Pick a sequence $T_3, T_4,\ldots$ of members of $\mathscr{S}_n$ such that $T_j\cap T_k=R$ for $j\neq k$, $j,k=1,2,3,4,\ldots$. Let P_j be projection onto H_{T_j}. For $j\geqq 3$, pick $g_j\in G_\beta$ such that $g_j(T_2)=T_j$ and g_j leaves T_1 pointwise fixed.

Because H_{T_1} and H_{T_2} are not orthogonal, $\exists w\in H_{T_1}$ such that $P_2w\neq 0$. Let $x=P_2w$. The unitary operator $\Gamma(g_j)$ maps H_{T_2} onto H_{T_j} for $j\geqq 3$, and consequently maps $(I-P_2)H$ onto $(I-P_j)H$ for $j\geqq 3$.

Let $j\geqq 3$. Then $P_jw=P_j\Gamma(g_j)w=P_j\Gamma(g_j)x+P_j\Gamma(g_j)(w-x)=\Gamma(g_j)x+0=\Gamma(g_j)x$. If $P_jw=P_kw$ for some j,k such that $j\neq k$, then $P_jw\in H_{(T_j\cap T_k)}=H_R$. This would contradict assumption b. Since $\|P_jw\|=\|x\|\neq 0$, for $j\geqq 3$, $\{P_jw \mid j\geqq 3\}$ has a weak limit point y. $y\neq 0$ because $|(y,w)|\geqq \liminf_j |(P_jw,w)|=\liminf_j \|P_jw\|^2 =\liminf_j \|x\|^2=\|x\|^2\neq 0$.

Let $g\in G_{\aleph_0}(S-R)$. Since g is finitely supported and $T_j\cap T_k=R$ if $j\neq k$, g leaves T_j pointwise fixed for j sufficiently large. Therefore $\Gamma(g)P_j=P_j$ for j sufficiently large. This implies that $\Gamma(g)y=y$.

By assumption a, there are scalars a_Q and vectors x_Q such that $x_Q\in H_Q$, $\|x_Q\|=1$, and $y=\sum_{Q\in\mathscr{S}_n} a_Qx_Q$. Let $p\in G_\beta(S-R)$ and $c>0$ be given. Let $\mathscr{Y}$ be a finite subset of $\mathscr{S}_n$ such that $\|\sum_{Q\in\mathscr{S}_n-\mathscr{Y}} a_Q x_Q\| < c/2$. Pick $h \in G_{\aleph_0}(S-R)$ such that gh^{-1} leaves Q pointwise fixed if $Q \in \mathscr{Y}$. Then

$$\begin{aligned}\|\Gamma(g)y-y\| &= \|\Gamma(g)\Gamma(h^{-1})y-y\| = \|\Gamma(gh^{-1})y-y\| \\ &= \left\|(\Gamma(gh^{-1})-I)\sum_{Q\in\mathscr{S}_n} a_Q x_Q\right\| \\ &\leqq \left\|(\Gamma(gh^{-1})-I)\sum_{Q\in\mathscr{Y}} a_Q x_Q\right\| + \left\|(\Gamma(gh^{-1})-I)\sum_{Q\in\mathscr{S}_n-\mathscr{Y}} a_Q x_Q\right\| \\ &< 0+2(c/2) = c.\end{aligned}$$

Since c is arbitrary, $\Gamma(g)y=y$ for all $g \in G_\beta(S-R)$. This contradicts assumption b.

Proof of Theorem 3. Assume Γ is a continuous irreducible representation of G_β on the Hilbert space H. Let $\mathscr{X}$ be the directed set of all finite subsets of S with set inclusion as the partial order relation. Let $v \in H$, $\|v\|=1$. Assume that for each $T \in \mathscr{X}$ there is a permutation $g_T \in G_\beta(S-T)$ such that $\operatorname{re}(\Gamma(g_T)v, v) \leqq \frac{1}{2}$. Then $\limsup_{T\in\mathscr{X}} \operatorname{re}(\Gamma(g_T)v, v) \leqq \frac{1}{2}$. However $\lim_{T\in\mathscr{X}} g_T = e$, where e is the group identity. Therefore, $\lim_{T\in\mathscr{X}} (\Gamma(g_T)v, v) = (\Gamma(e)v, v) = 1$, yielding a contradiction. Therefore, there is a finite subset Z of S such that $\operatorname{re}(\Gamma(g)v, v) \geqq \frac{1}{2}$ if $g \in G_\beta(S-Z)$.

Let $\mathscr{Y}$ be the directed set of all finite subsets of $S-Z$ with set inclusion as the partial order relation. If $T \in \mathscr{Y}$, let $P_T = |T|!^{-1} \sum_{g\in G(T)} \Gamma(g)$. P_T is a projection and $P_T \in \Gamma(G_\beta)''$. $P_{T_1}P_T = P_{T_1}$ if $T_1 \in \mathscr{Y}$ and $T_1 \supseteq T$. $\Gamma(g)P_T = P_T$ if $g \in G(T)$.

Let $J = (\bigcup_{Q\in\mathscr{Y}} (I-P_Q)H)^\perp$ and let P be projection onto J. Let $x \in J$. Then $(I-P_Q)x=0$ for all $Q \in \mathscr{Y}$. Let $y \in J^\perp$ and $c>0$ be given. There exist $Q_1, Q_2, \ldots, Q_m \in \mathscr{Y}$, where m is some positive integer, such that $\|P'y-y\| < c$, where P' is projection onto the subspace spanned by $\bigcup_{j=1}^m (I-P_{Q_j})H$. Let $Q' = \bigcup_{j=1}^m Q_j$. Then $\|(I-P_{Q'})y-y\| < c$. Therefore $\|P_{Q'}y\| < c$.

Let $Q \in \mathscr{Y}$ and assume $Q \supseteq Q'$. Then $P_Q y = P_Q P_{Q'} y$ and $\|P_Q y\| < c$. Consequently, $\lim_{Q\in\mathscr{Y}} P_Q(y) = 0$ and therefore $P = \lim_{Q\in\mathscr{Y}} P_Q$.

$\operatorname{re}(Py, y) \geqq \inf_{Q\in\mathscr{Y}} (P_Q y, y) \geqq \frac{1}{2}$. Therefore $Py \neq 0$. Let $v = Py$, $T \in \mathscr{Y}$, and $g \in G(T)$. Then $\Gamma(g)v = \Gamma(g)Py = \Gamma(g)\lim_{Q\in\mathscr{Y}} (P_Q y) = \lim_{Q\in\mathscr{Y}} (\Gamma(g)P_Q y) = \lim_{Q\in\mathscr{Y}} (P_Q y) = Py = v$, since $\Gamma(g)P_Q = P_Q$ if $Q \supseteq T$. By continuity, $\Gamma(p)v = v$ if $p \in G_\beta(S-Z)$.

Without loss of generality, we can assume that $H_Q = 0$ if $Q \subseteq S$ and $|Q| < |Z|$, where $H_Q = \{x \in H : \Gamma(g)x = x \text{ for } g \in G_\beta(S-Q)\}$. Let $|Z| = n$. Conclusion 2 of the theorem has been demonstrated; conclusion 3 is an immediate consequence.

The closed subspace spanned by $\bigcup_{Q\in\mathscr{S}_n} H_Q$ is invariant under $\Gamma(G_\beta)$. Since Γ is irreducible, this subspace must be equal to H. By Lemma 7, $H = \bigoplus_{Q\in\mathscr{S}_n} H_Q$.

By Lemma 3 and Lemma 2, $\Gamma(G)' \cong \Gamma(G(Z))' | H_Z$. Since Γ is irreducible, $\Gamma(G)'$ is the set of scalar multiples of the identity operator and $\Gamma(G(Z))|H_Z$ is irreducible.

If $Z = \varnothing$ then $H_Z = H$ and $\Gamma \cong \Lambda_\beta^0$. If $Z \neq \varnothing$, let ψ be the subrepresentation of Λ_β^n which corresponds to $\Gamma(G(Z))|H_Z$ as defined in Theorem 2 part 1b. Let J be the Hilbert space on which ψ acts.

In Lemma 6, let $H_1 = J$, $H_2 = H$, $\Gamma_1 = \psi$, $\Gamma_2 = \Gamma$, $n = n$, and $T = Z$. The assumptions of Lemma 6 are satisfied, and consequently $\psi \cong \Gamma$.

We now prove the converse of the theorem when condition 2 is satisfied. The remainder of the proof will then be immediate. Assume that Γ is an irreducible representation of G_β, $Z \in \mathscr{S}_n$, and $\Gamma(G_\beta(S-Z))$ contains the trivial representation of $G_\beta(S-Z)$. This implies $H_Z \neq 0$. Without loss of generality, we can assume $H_T=0$ if $T \subseteq S$ and $|T|<|Z|$. The closed subspace generated by $\bigcup_{T\in\mathscr{S}_n} H_T$ is $\neq 0$ and is invariant under $\Gamma(G_\beta)$; consequently, this subspace is all of H. By Lemma 7, $H=\bigoplus_{T\in\mathscr{S}_n} H_T$.

Let $x \in H$ and $c>0$ be given. We can find sets $T_i \in \mathscr{S}_n$ and vectors $x_i \in H_{T_i}$, $1 \leqq i \leqq m$, m some positive integer, such that $\|x-\sum_{i=1}^m x_i\|<c/2$. Let $0=\{g \in G_\beta : g(s)=s \text{ for all } s \in \bigcup_{i=1}^m T_i\}$. 0 is an open neighborhood of the identity. If $g \in 0$, then

$$\|\Gamma(g)x-\Gamma(e)x\| = \|(\Gamma(g)-I)x\| = \left\|(\Gamma(g)-I)\left(x-\sum_{i=1}^m x_i\right)+(\Gamma(g)-I)\sum_{i=1}^m x_i\right\|$$
$$\leqq \|\Gamma(g)-I\| \left\|x-\sum_{i=1}^m x_i\right\| +0<2\cdot c/2 = c.$$

Therefore Γ is continuous.

Proof of Theorem 4. It suffices to show that Γ contains an irreducible subrepresentation. An application of Zorn's lemma will then complete the proof. Assume Γ acts on the Hilbert space H.

The proof of conclusion 2 of Theorem 3 never used the hypothesis of irreducibility. Consequently, we can assume there is a finite subset Z of S such that $H_Z \neq 0$ and $H_Q=0$ if $Q \subseteq S$ and $|Q|<|Z|$. $G(Z)$ is a finite group, and consequently $\Gamma(G(Z))|H_Z$ is the direct sum of irreducible representations of $G(Z)$. Let Γ_0 be an irreducible subrepresentation of $\Gamma(G(Z))|H_Z$. Assume Γ_0 acts on H_{0Z}. Let H_0 be the closed subspace of H generated by $\Gamma(G)H_{0Z}$. By Lemma 3 and Lemma 2, $\Gamma|H_0$ is irreducible.

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