TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 164, February 1972

CYCLIC VECTORS AND IRREDUCIBILITY FOR PRINCIPAL SERIES REPRESENTATIONS. II(1)

BY
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Abstract. This paper is a continuation of the author's paper *Cyclic vectors and irreducibility for principal series representations*. In this paper the nonunitary principal series is studied. Using a theorem of Kostant, a sufficient condition is found for irreducibility of nonunitary principal series representations.

1. **Introduction.** This paper is a continuation of the analysis of Wallach [6] of the principal series of representations using results of Kostant [4], [5]. In this paper we apply our techniques to get conditions leading to irreducibility for nonunitary principal series representations.

Our main new tool is Theorem 2.1 of this paper (which we think is new) which says (in essence) that if a representation (π, H) of a semisimple group G (not necessarily unitary) has the same character as an infinitesimally irreducible representation of G, (π_1, H') then (π, H) and (π_1, H') are infinitesimally equivalent. We apply this result to the nonunitary principal series by using Harish-Chandra's computation of the characters of the nonunitary principal series (see [2], and Theorem 3.1 in this paper).

We are indebted to Professor B. Kostant for giving us access to the manuscript of [5].

2. **A result on characters.** Let G be a connected semisimple Lie group with finite center. Let K be a maximal compact subgroup of G. Let $\hat{K}$ be the set of all equivalence classes of irreducible finite-dimensional representations of K. A representation (π, H) of G on a separable Hilbert space (a representation will mean a continuous representation, that is, the map $g, f \mapsto \pi(g)f$ of $G \times H$ to H is continuous) is said to be admissible if

(1) as a K-representation, (π, H) is unitary,

(2) as a K-representation, H splits into a direct sum $H = \sum_{\gamma\in\hat{K}} H_\gamma$ where H_γ is a direct sum of m_γ copies of an element of γ in $\hat{K}$ and $m_\gamma \leqq Cd_\gamma$ where d_γ is the dimension of any element of γ.

Received by the editors March 12, 1971.

AMS 1969 *subject classifications*. Primary 2260, 2257.

Key words and phrases. Representation, Hilbert space, character, universal enveloping algebra, irreducible, admissible, Haar measure, trace class, module, root, Weyl group, Peter-Weyl Theorem.

(1) Supported by NSF Grant GP-27959.

If (π, H) is admissible and if $C_0^\infty(G)$ denotes the space of all complex valued C^∞ functions on G with compact support then if $f \in C_0^\infty(G)$ and if dx is Haar measure on G define

$$\pi(f) = \int_G f(x)\pi(x)\,dx.$$

Then $\pi(f)$ as an operator on H is of trace class (see Harish-Chandra [1]). That is, if $\{\psi_n\}$ is an orthonormal basis of H then

$$\Theta_\pi(f) = \sum_{n=1}^{\infty} \langle \pi(f)\psi_n, \psi_n\rangle$$

is finite and independent of the basis $\{\psi_n\}$ ($\langle\ ,\ \rangle$ is the Hilbert space inner product on H).

Θ_π is called the character of (π, H).

Following Harish-Chandra we define, for $\gamma \in \hat{K}$, the function

$$\phi_\gamma^\pi(g) = \operatorname{tr} E_\gamma^\pi \pi(g) E_\gamma^\pi$$

where $E_\gamma^\pi : H \to H_\gamma$ is the orthogonal projection. We note that if χ_γ is the character of $\gamma \in \hat{K}$ then

$$(*)\qquad E_\gamma^\pi v = d(\gamma) \int_K \chi_\gamma(k)\pi(k)v\,dk,$$

where dk is normalized Haar measure on K.

LEMMA 2.1. *If (π_1, H^1) and (π_2, H^2) are admissible representations of G so that $\Theta_{\pi_1} = \Theta_{\pi_2}$ then, for each $\gamma \in \hat{K}$, $\phi_\gamma^{\pi_1} = \phi_\gamma^{\pi_2}$.*

Proof. Suppose for some $\gamma \in \hat{K}$, $\phi_\gamma^{\pi_1} - \phi_\gamma^{\pi_2} = \phi \neq 0$. Then there is $f \in C_0^\infty(G)$ so that $\int_G \phi(g) f(g)\,dg \neq 0$. Setting $f'(x) = d_\gamma \int_K \chi_\gamma(k) f(k^{-1}x)\,dk$ a computation shows that $\Theta_{\pi_1}(f') - \Theta_{\pi_2}(f') = \int \phi(g) f(g)\,dg$. This contradiction yields the result.

If (π, H) is admissible let H_F be the algebraic direct sum of the H_γ. Then H_F is the space of all $f \in H$ so that $\pi(K)f$ is contained in a finite-dimensional subspace of H.

Let $\mathfrak{g}$ be the lie algebra of G and let $U(\mathfrak{g})$ be the complexified universal enveloping algebra of $\mathfrak{g}$. If $X \in \mathfrak{g}, f \in H_F$, define $\pi(X)f = (d/dt)\pi(\exp(tX))f|_{t=0}$. Then $\pi(X)H_F \subset H_F$ and defines a representation of $U(\mathfrak{g})$ on H_F. We say that (π, H) is infinitesimally irreducible if the representation (π, H_F) is an irreducible representation of $U(\mathfrak{g})$. If (π_1, H^1) and (π_2, H^2) are admissible representations of G then π_1 is said to be infinitesimally equivalent with π_2 if the representations (π_1, H_F^1) and (π_2, H_F^2) are equivalent representations of $U(\mathfrak{g})$.

THEOREM 2.1. *Let (π_1, H^1) and (π_2, H^2) be admissible representations of G. Suppose that (π_1, H^1) is infinitesimally irreducible and that $\Theta_{\pi_1} = \Theta_{\pi_2}$. Then (π_1, H^1) and (π_2, H^2) are infinitesimally equivalent.*

Proof. For each $v \in H_F^i$ define $A_v^i: H_F^i \to C^\infty(G)$ by

$$A_v^i(w)(g) = \langle \pi_i(g)^{-1}w, v\rangle.$$

Then $A_v^i(\pi_i(X)w)(g)=(d/dt)A_v^i(w)(\exp(-tX)g)|_{t=0}$. Thus if we let $U(\mathfrak{g})$ act on $C^\infty(G)$ by $X\cdot f(g)=(d/dt)f(\exp(-tX)g)|_{t=0}$ for $X \in \mathfrak{g}$, A_v^i is a homomorphism of $U(\mathfrak{g})$-representations.

Let $\gamma \in \hat{K}$ be so that $\phi_\gamma^{\pi_1} \neq 0$. Let $C_\gamma^i = \dim H_\gamma^i$. Then Lemma 2.1 implies that $C_\gamma^1 = C_\gamma^2$ (indeed $C_\gamma^i = \phi_\gamma^{\pi_i}(e)$). Let $C_\gamma = C_\gamma^1$.

Let $v_1^i, \dots, v_{C_\gamma}^i$ be an orthonormal basis of H_γ^i, $i=1, 2$. Let $\eta: G \to G$ be defined by $\eta(g)=g^{-1}$. Then

$$\phi_\gamma^{\pi_i} \circ \eta = \sum_{j=1}^{C_\gamma} A_{v_j^i}^i(v_j^i).$$

Now $A_{v_j^1}^1(H_F^1)$ is equivalent with H_F^1 as a $U(\mathfrak{g})$-module for $j=1, \dots, C_\gamma$, and $\phi_\gamma^{\pi_1} \circ \eta \in \sum_{j=1}^{C_\gamma} A_{v_j^1}^1(H_F^1)$. Let $j_1, \dots, j_k$ be so that

(a) $\phi_\gamma^{\pi_1} \circ \eta \in \sum_{i=1}^{k} A_{v_{j_i}^1}^1(H_F^1)$,

(b) no subset of $\{j_1, \dots, j_k\}$ satisfies (a).

It is then clear that if $V_i = A_{v_{j_i}^1}^1(H_F^1)$ then the sum $V_1 + \cdots + V_k$ is direct (indeed V_j is an irreducible $U(\mathfrak{g})$-module). Set $V = V_1 \oplus \cdots \oplus V_k$. Let $P_j: V \to V_j$ be the corresponding projection.

Proceeding similarly we note that $\phi_\gamma^{\pi_2} \circ \eta \in \sum_{j=1}^{C_\gamma} A_{v_j^2}^2(\pi_2(U(\mathfrak{g}))v_j^2)$. Let $p_1, \dots, p_l$ be so that

(a) $\phi_\gamma^{\pi_2} \circ \eta \in \sum_{i=1}^{l} A_{v_{p_i}^2}^2(\pi^2(U(\mathfrak{g})v_{p_i}^2)$,

(b) no subset of $\{p_1, \dots, p_l\}$ satisfies (a).

Let

$$W_i = A_{v_{p_i}^2}^2(\pi_2(U(\mathfrak{g}))v_{p_i}^2), \qquad i = 1, \dots, l.$$

Then W_i is a $U(\mathfrak{g})$-submodule of $C^\infty(G)$ and is equivalent to a subquotient of H_F^2. Let $W = W_1 + \cdots + W_l$. Let $W^j = \sum_{k=j}^{l} W_k$. Let $Q_j: W^j \to W^j/W^{j+1}$ be the natural map.

By hypothesis $W \cap V \neq (0)$. Suppose that $\operatorname{Ker} Q_1|_{W\cap V} = (0)$. Then $W \cap V$ as a $U(\mathfrak{g})$-module is equivalent to a subquotient of W_1, hence $W \cap V$ is equivalent to a subquotient of a subquotient of H_F^2. Now for some j, $P_j(W \cap V) \neq (0)$. Thus $P_j(W \cap V) = V_j$ since V_j is irreducible. But V_j is equivalent to H_F^1. Thus H_F^1 is equivalent to a quotient of a subquotient of a subquotient of H_F^2. But as K-representations H_F^1 and H_F^2 are equivalent. Hence H_F^1 is equivalent to H_F^2 as a $U(\mathfrak{g})$-representation. Thus if $\operatorname{Ker} Q_1|_{W\cap V} = (0)$ the theorem is proved.

Otherwise $Z_1 = \operatorname{Ker} Q_1|_{W \cap V} \neq (0)$. $Z_1 \subset W^2 \cap V$. If $\operatorname{Ker} Q_2|_{Z_1} = 0$ then applying to the argument above we see that H_F^1 and H_F^2 are equivalent as $U(\mathfrak{g})$-representations. Otherwise $Z_2 = \operatorname{Ker}(Q_2|_{Z_1}) \neq 0$, $Z_2 \subset W^3 \cap V$. Continuing this process we either prove the theorem or after $l-1$ stages we find $W_l \cap V \neq (0)$. But then for some j, $P_j(W_l \cap V) \neq (0)$. Thus a quotient of $W_l \cap V$ is equivalent to H_F^2. But

$W_l \cap V$ is equivalent to a subquotient of H_F^1. Thus noting that as K-representations H_F^1 and H_F^2 are equivalent we have completely proved the theorem.

Note. The condition (1) of admissibility can be removed and the formula $(*)$ can be used to define E_γ^π for a representation satisying (2).

COROLLARY 2.1 (to the proof of Theorem 2.1). *Let (π_1, H^1) and (π_2, H^2) be admissible representations of G. Suppose that*

(1) *(π_1, H^1) is infinitesimally irreducible,*

(2) *as K-representations (π_1, H^1) and (π_2, H^2) are equivalent,*

(3) *there is $\gamma \in \hat{K}$ so that $\phi_\gamma^{\pi_1} \neq 0$ and $\phi_\gamma^{\pi_1} = \phi_\gamma^{\pi_2}$.*

Then (π_1, H^1) and (π_2, H^2) are infinitesimally equivalent.

3. **The principal series.** Let G, K be as in §2. Let $G=KAN$ be an Iwasawa decomposition of G. Let M be the centralizer of A in K. Let M^* be the normalizer of A in K. Then M^*/M defines a finite group of automorphisms of A, the Weyl group, W, of A. Let $\hat{M}$ be the set of all equivalence classes of irreducible finite-dimensional representations of M. If $\xi \in \hat{M}$ we fix an element (ξ, H_ξ) of ξ. Let $\mathfrak{a}$ be the Lie algebra of A, and $\mathfrak{a}^*$, $\mathfrak{a}_C^*$ be the spaces of linear real-valued and complex-valued forms on $\mathfrak{a}$. If $s \in W$, and $m^* \in s$, $\xi \in \hat{M}$, $m \in M$ we define $\xi^s(m) = \xi(m^{*-1}mm^*)$. If $\nu \in \mathfrak{a}_C^*$ we define $s\nu(H) = \nu(\mathrm{Ad}\,(m^*)^{-1}H)$.

If $\xi \in \hat{M}$, $\nu \in \mathfrak{a}_C^*$ we define an admissible representation of G, $(\pi_{\xi,\nu}, H^{\xi,\nu})$ as follows:

(1) $H^{\xi,\nu}$ is the space of all measurable functions, f, from G to H_ξ so that

(i) $f(gman) = \xi(m)^{-1}e^{-\nu(\log a)}f(g)$ ($\log\colon A \to \mathfrak{a}$ is the universe to $\exp\colon \mathfrak{a} \to A$).

(ii) $\int_K \|f(k)\|^2\, dk < \infty$,

(2) $(\pi_{\xi,\nu}(g_0)f)(g) = f(g_0^{-1}g)$.

It is easy to see that $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is an admissible representation of G (indeed Frobenius reciprocity implies condition (2) of admissibility).

Let $\Theta_{\xi,\nu}$ be the character of $(\pi_{\xi,\nu}, H^{\xi,\nu})$.

THEOREM 3.1 (HARISH-CHANDRA [2]). *Let $\rho \in \mathfrak{a}^*$ be defined by*

$$\rho(H) = \tfrac{1}{2}\,\mathrm{tr}\,(\mathrm{ad}\, H|_{\mathfrak{n}})$$

($\mathfrak{n}$ *the Lie algebra of N). If $\xi, \xi' \in \hat{M}$, $\nu, \nu' \in \mathfrak{a}_C^*$ then $\Theta_{\xi,\nu} = \Theta_{\xi',\nu'}$ if and only if there is $s \in W$ so that $\xi^s = \xi'$, $s(\nu-\rho)+\rho = \nu'$.*

Actually in [2], Harish-Chandra has a different definition of the principal series. The proof he gives, however, goes through without change to our definition of the principal series.

We note that with our definition of the principal series the unitary principal series is the set of all $\pi_{\xi,\rho+\sqrt{(-1)}\nu}$, where $\xi \in \hat{M}$, $\nu \in \mathfrak{a}^*$, indeed,

LEMMA 3.1. *There is a nondegenerate, sesquilinear pairing, (,), between $H^{\xi,\nu}$ and $H^{\xi,2\rho-\bar{\nu}}$ for each $\xi \in \hat{M}$, $\nu \in \mathfrak{a}_C^*$ so that if $f_1 \in H^{\xi,\nu}$, $f_2 \in H^{\xi,2\rho-\bar{\nu}}$, $g \in G$, then*

$$(\pi_{\xi,\nu}(g)f_1, \pi_{\xi,2\rho-\bar{\nu}}(g)f_2) = (f_1, f_2).$$

Proof. We use the integral formula that if $g \in G$, $g = kan$ and $H(g) = \log a$ and if f is a continuous function on K then

$$\int_K f(k(xk)) \exp [2\rho(H(xk))]\, dk = \int_K f(k)\, dk \tag{1}$$

(cf. Helgason [3]). The result is an immediate consequence of (1).

4. **Extendible representations of** M. We retain the notation of §3. We assume that G is a linear Lie group. Let θ be the Cartan involution of G corresponding to K. Let $\bar{N} = \theta(N)$. If V is a finite-dimensional G-module and H is a subgroup of G define $V^H = \{v \in V \mid h \cdot v = v \text{ for all } h \in H\}$.

We say that $\xi \in \hat{M}$ is extendible if there is a finite-dimensional irreducible G-module, V, so that as an M-representation $V^{\bar{N}}$ is in ξ. V is called an extension of ξ. In Wallach [6] we proved

THEOREM 3.1. *If $\xi \in \hat{M}$ there is $s \in W(A)$ so that ξ^s is extendible.*

THEOREM 4.2. *Let $\xi \in \hat{M}$, $\nu \in \mathfrak{a}_C^*$. There is at most one nonzero finite-dimensional G-invariant subspace of $H^{\xi,\nu}$. Denote this subspace by $V_{\xi,\nu}$ if it exists. Let $V_{\xi,\nu} = (0)$ if $H^{\xi,\nu}$ contains no nonzero finite-dimensional G-invariant subspace. If $V_{\xi,\nu} \neq (0)$ then $V_{\xi,\nu}$ is an extension of ξ. If $\xi \in \hat{M}$ there is $s \in W(A)$, $\lambda \in \mathfrak{a}^*$ so that $V_{\xi^s,\lambda} \neq (0)$.*

Proof. Suppose V is a nonzero finite-dimensional subspace of $H^{\xi,\nu}$. Then applying Lie and Engel's theorem, $V^{\bar{N}} \neq (0)$. If $f \in V^{\bar{N}}$ then

$$f(\bar{n}man) = \xi(m)^{-1} e^{-\nu(\log a)} f(e).$$

Thus since $\bar{N}MAN$ is open and dense in G (cf. Helgason [3]) the map $V^{\bar{N}} \to H_\xi$ given by $f \mapsto f(e)$ is injective. Furthermore, if $m \in M$ then $(\pi_{\xi,\nu}(m) \cdot f)(e) = \xi(m) \cdot f(e)$. Thus the map $V^{\bar{N}} \to H_\xi$ is surjective, hence it is bijective. This proves the uniqueness and that if $V_{\xi,\nu} \neq (0)$ then $V_{\xi,\nu}$ is an extension of ξ.

Let $\xi \in \hat{M}$; then by Theorem 4.1 there is $s \in W(A)$ so that ξ^s is extendible. Let V be an extension of ξ^s. In Wallach [6] it was shown that if V^* is the contragradient representation to V, then $V^{\bar{N}}$ is equivalent to $(V^{*N})^*$ as an MA module. If $v \in V$ define $\alpha(v)(g) \in (V^{*N})^* = H_{\xi^s}$ by $\alpha(v)(g)(\mu) = \mu(g^{-1}v)$. Let $a \in A$, $v \in V^{\bar{N}}$, $\lambda \in \mathfrak{a}^*$ be defined by $a \cdot v = e^{\lambda(\log a)} v$. Then it is easy to see that $\alpha(v)(gman) = \xi(m)^{-1} e^{-\lambda(\log a)} \alpha(v)(g)$ (see Wallach [6]). This proves the result.

If $\nu \in \mathfrak{a}_C^*$ define $1_\nu(g) = \exp[-\nu(H(g))]$ for $g \in G$. Let $1_\nu \in H^{1,\nu}$. If $\xi \in \hat{M}$ let $\Omega_\xi = \{\lambda \in \mathfrak{a}^* \mid V_{\xi,\lambda} \neq (0)\}$.

PROPOSITION 4.1. *Let $\xi \in \hat{M}$, $\lambda \in \Omega_\xi$. If $\nu \in \mathfrak{a}_C^*$ and if $1_{\nu-\lambda}$ is a cyclic vector for $H^{1,\nu-\lambda}$ and if $f \in V_{\xi,\lambda}^{\bar{N}}$, $f \neq 0$ then $1_{\nu-\lambda} \cdot f$ is a cyclic vector for $H^{\xi,\nu}$.*

For a proof of this result see Wallach [6].

The following result is a slight refinement of Theorem 3.3 of Wallach [6].

THFOREM 4.3. *Let $\xi \in \hat{M}$, $\lambda \in \Omega_\xi$. Suppose that as an M-representation the multiplicity of (ξ, H_ξ) in $V_{\xi,\lambda}$ is one. If $\nu \in \mathfrak{a}_C^*$ and if $1_{\nu-\lambda}$ and $1_{2\rho-\bar{\nu}-\lambda}$ are cyclic vectors for $H^{1,\nu-\lambda}$ and $H^{1,2\rho-\bar{\nu}-\lambda}$ then $H^{\xi,\nu}$ is irreducible.*

Proof. We observe that by Frobenius reciprocity the multiplicity of the K-representation $V_{\xi,\lambda}$ in $H^{\xi,\nu}$ as a K-representation is one. Suppose now that $U \subset H^{\xi,\nu}$ is a closed invariant subspace of $H^{\xi,\nu}$. Then either $1_{\nu,\lambda} \cdot V_{\xi,\lambda} \subset U$ or $1_{2\rho-\bar{\nu}-\lambda} \cdot V_{\xi,\lambda} \subset U^\perp \subset H^{\xi,2\rho-\bar{\nu}}$ (here "$\perp$" is relative to the pairing of Lemma 3.1). Now Proposition 4.1 implies $U = H^{\xi,\nu}$ or $U^\perp = H^{\xi,2\rho-\bar{\nu}}$. Hence $U = H^{\xi,\nu}$ or $U = (0)$. Q.E.D.

5. **A theorem of Kostant.** Let us retain the notation of the previous section. Let for $\phi \in \mathfrak{a}^*$, $\mathfrak{n}_\phi = \{x \in \mathfrak{n} \mid [h, x] = \phi(h)x \text{ for all } h \in \mathfrak{a}\}$. Let $\Lambda^+ = \{\phi \in \mathfrak{a}^* \mid \mathfrak{a}_\phi \neq (0)\}$. Let $h_\phi \in \mathfrak{a}$ be defined by $\langle h_\phi, h\rangle = \phi(h)$ for $h \in \mathfrak{a}$, $\langle\ ,\ \rangle$ is the Killing form of $\mathfrak{g}$. Let $\Lambda_1^+ = \{\phi \in \Lambda^+ \mid \phi/2 \notin \Lambda^+\}$. If $\phi \in \Lambda_1^+$, set $n_\phi = 1$ if $2\phi \notin \Lambda^+$, $n_\phi = 2$ if $2\phi \in \Lambda^+$. Let $m_\phi = (\dim \phi)/2$ if $2\phi \notin \Lambda^+$ and $m_\phi = (\dim \phi)/2 + 1$ if $2\phi \in \Lambda^+$.

THEOREM 5.1 (KOSTANT [5]). *Let $\lambda \in \mathfrak{a}_C^*$ then 1_λ is not a cyclic vector for $H^{1,\lambda}$ if and only if*

$$(\lambda - \rho)(h_\phi)/\phi(h_\phi) + m_\phi = n_\phi k$$

where k is a nonpositive integer.

We apply the above theorem of Kostant combined with our previous results to the case G is a complex semisimple group. In this case K is a compact form of G. M is a maximal torus of K and $M = \exp(\sqrt{(-1)}\mathfrak{a})$. Thus if $\xi \in \hat{M}$ and if $\dot{\xi}$ is the differential of ξ then $\dot{\xi}$ induces a linear form $\dot{\xi} \in \mathfrak{a}^*$. Furthermore if $\dot{\xi}$ is negative integral $(2\dot{\xi}(h_\phi)/\phi(h_\phi)$ a nonpositive integer for each $\phi \in \Lambda^+$ $(= \Lambda_1^+))$ then $V_{\xi,\dot{\xi}}$ is the holomorphic irreducible finite-dimensional representation of G with lowest weight $\dot{\xi}$.

THEOREM 5.2. (Compare with Zelobenko [7].) *Let G be a complex semisimple Lie group. Let $\xi \in \hat{M}$, $\nu \in \mathfrak{a}_C^*$. $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is irreducible if there is $s \in W(A)$ so that $s \cdot \dot{\xi}$ is negative integral and if for each $\phi \in \Lambda^+$*

$$s(\nu - \dot{\xi} - \rho)(h_\phi)/\phi(h_\phi) + 1 \quad \textit{and} \quad s(\rho - \bar{\nu} - \dot{\xi})(h_\phi)/\phi(h_\phi) + 1$$

are not nonpositive integers.

Proof. Suppose s is so that the conditions of the theorem are satisfied. Then noting that $m_\phi = n_\phi = 1$ for $\phi \in \Lambda^+$ and that the multiplicity of the character ξ^s in $V_{\xi^s, s\cdot\dot{\xi}}$ is one, the conditions of the theorem and Theorems 5.1, 4.3 imply that $H^{\xi^s, s(\nu-\rho)+\rho}$ is irreducible. But $\Theta_{\xi,\nu} = \Theta_{\xi^s, s(\nu-\rho)+\rho}$, by Theorem 3.1. Hence the result follows from Theorem 2.1.

We now apply the results to $G = \mathrm{SL}(n, \boldsymbol{R})$. In this case $K = \mathrm{SO}(n)$, A is the group of all diagonal matrices with positive entries and determinant 1. M is the group of all diagonal matrices with entries ± 1 and determinant one. If $m \in M$ then

$$m = \begin{bmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{bmatrix};$$

set $\varepsilon_i(m) = m_i$. Let $\varepsilon_0(m) = 1$ for all $m \in M$. Then every element of $\hat{M}$ is of the form $\varepsilon_0 \varepsilon_{i_1} \cdots \varepsilon_{i_r}$ with $0 < i_1 < \cdots < i_r \leqq n-1$. Let $\xi_r = \varepsilon_0 \varepsilon_1 \cdots \varepsilon_r$, $r = 0, \ldots, n-1$. Let $V^0 = \boldsymbol{C}$ with G acting trivially. Let $V^1 = \boldsymbol{C}^n$ with G acting in the usual (matrix) fashion on $\boldsymbol{C}^n$. Let $V^r = \Lambda^r V^1$, $r = 1, \ldots, n-1$. Then V^{n-r} is an extension of ξ_r. Let λ_r be the action of A on $(V^{n-r})^{\bar{N}}$ ($\bar{N}$ is the group of all lower triangular matrices with ones on the diagonal). Then

$$\lambda_r \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} = a_{r+1} \cdots a_n.$$

$\mathfrak{a}$ is the set of all diagonal $n \times n$ real matrices with trace 0. Let for h in $\mathfrak{a}$

$$h = \begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix},$$

$\phi_i(h) = h_i$. Then $\mathfrak{a}_C^* = \{\sum c_i \phi_i \mid \sum c_i = 0\}$. $\lambda_r(a) = \exp[\lambda_r(\log a)]$ where $\lambda_r = \sum_{j=1}^{n-r} \phi_{r+j}|_{\mathfrak{a}}$. $\rho = -\sum_{r=1}^{n-1} \lambda_r|_{\mathfrak{a}} = \sum_{j=1}^{n-1} (n-j)\phi_j|_{\mathfrak{a}}$.

THEOREM 5.3. *Let $G = \mathrm{SL}(n, \boldsymbol{R})$.*

(1) *Suppose $n = 2p+1$, p a positive integer. Let $\xi \in \hat{M}$, $\nu \in \mathfrak{a}_C^*$. If there is an $s \in W(A)$ ($W(A)$ acts by permuting entries in M and in A) so that $s \cdot \xi = \xi_r$ and if*

$$(s(\nu-\rho)+\lambda_{n-r})(h_\phi)/\phi(h_\phi)+\tfrac{1}{2}$$

and

$$(s(\rho-\bar{\nu})+\lambda_{n-r})(h_\phi)/\phi(h_\phi)+\tfrac{1}{2}$$

are not nonpositive integers then $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is irreducible.

(2) *Suppose that $n = 2p$, p a positive integer. If there is $s \in W(A)$ so that $s \cdot \xi = \xi_r$ and $r \neq p$, and if*

$$(s(\nu-\rho)+\lambda_{n-r})(h_\phi)/\phi(h_\phi)+\tfrac{1}{2}$$

and

$$(s(\rho-\bar{\nu})+\lambda_{n-r})(h_\phi)/\phi(h_\phi)+\tfrac{1}{2}$$

are not nonpositive integers then $(\pi_{\xi,\nu}, H^{\xi,\nu})$ is irreducible.

Proof. The multiplicity if ξ_r in V^{n-r} is 1 in each of the above cases (1) or (2). $n_\phi = 1$ and $m_\phi = \frac{1}{2}$ for all $\phi \in \Lambda^+$ $(= \Lambda_1^+)$. Thus Theorem 5.1 combined with Theorem 4.3 implies that $H^{\xi^s, s(\nu-\rho)+\rho}$ is irreducible. Since $\Theta_{\xi,\nu} = \Theta_{\xi^s, s(\nu-\rho)+\rho}$, Theorem 2.1 implies the result.

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