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STRONG CONVERGENCE OF FUNCTIONS ON KÖTHE SPACES

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Abstract. Let Λ be a rearrangement invariant Köthe space over a nondiscrete group G with Haar measure μ. For a function $f \in \Lambda$ and relatively compact 0-neighborhood U in G the function

$$T_U f(x) = \frac{1}{\mu(U)} \cdot \int_{U+x} f \, d\mu$$

is continuous and also belongs to Λ. The convergence $T_U f \to f$ (as $U \to 0$) for the strong Köthe topology on Λ is involved in establishing compactness criteria for subsets of a Köthe space. The main result of this paper is a necessary and sufficient condition for convergence $T_U f \to f$ in the strong topology on Λ.

1. In [3], [4] and [5] Köthe studied pairs of subspaces of real sequences that were in weak duality. Dieudonné later generalized the theory to subspaces of locally integrable functions over a locally compact measure space E with Radon measure μ. If E is a σ-compact, locally compact Hausdorff space with regular Radon measure μ, we let Ω be the space of all functions which are integrable on each compact set in E. For a subset Γ of Ω, the Köthe space associated with Γ is $\Lambda = \Lambda(\Gamma) = \{f \in \Omega : \int_E |fg| \, d\mu < \infty$ for all $g \in \Gamma\}$ and the Köthe dual is $\Lambda^* = \Lambda^*(\Gamma) = \Lambda(\Lambda(\Gamma))$. The pair (Λ, Λ^*) is in weak duality; an example of such a pair is (L^p, L^q). The set Λ can be made into a complete locally convex topological vector space under the strong topology $S(\Lambda, \Lambda^*) = S$ defined by the seminorms

$$S_H(f) = \sup_{g \in H} \int |fg| \, d\mu$$

as H runs through the weakly bounded subsets of the Köthe dual Λ^*.

For $E = G$ an additive topological group, $f \in \Omega$, and U a relatively compact 0-neighborhood in G, we define the continuous function

$$T_U f(x) = \frac{1}{\mu(U)} \cdot \int_{U+x} f \, d\mu.$$

In [2] these functions were used in giving compactness criteria for subsets of Köthe spaces over G. The importance of the convergence of $T_U f$ to f (for the strong

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topology) as U runs through the relatively compact neighborhoods of 0 in G for each function f in a Köthe space will appear in papers by Welland and Goes which are as yet unpublished. In this paper a necessary and sufficient condition is established for the convergence of $T_U f$ to f for the strong topology on a rearrangement invariant Köthe space.

2. Two functions f and g are said to be equimeasurable or rearrangement invariant if

$$\mu(\{x : |f(x)| > r\}) = \mu(\{x : |g(x)| > r\})$$

for all nonnegative r. A Köthe space Λ is rearrangement invariant if $f' \in \Lambda$ whenever f' is equimeasurable with some $f \in \Lambda$. It is known that such a Λ is contained in the direct sum of $L^1(E, \mu)$ and $L^\infty(E, \mu)$, and that the Köthe dual Λ^* is also rearrangement invariant (proved in [7]).

A set of functions H is normal if $g \in H$ and $|h| \leqq |g|$ implies $h \in H$. In [6] it was proven that if E has no atoms (i.e. a set $S \subset E$ of positive measure such that $S_1 \subset S$ implies $\mu(S_1)=0$ or $\mu(S_1)=\mu(S)$) there is a fundamental system of normal, rearrangement invariant, weakly bounded subsets $\{H\}$ of Λ^* for which the seminorms S_H, which generate the strong topology of Λ, have the property that $S_H(f) = S_H(f')$ for f equimeasurable with f'.

In this paper G will be a σ-compact, locally compact, Hausdorff, nondiscrete topological group; μ will denote invariant Haar measure on G. The family of relatively compact neighborhoods of 0 in G will be denoted by $\mathscr{U}$. In addition, we will often write $\int_G f(x)\,dx$ to mean $\int_G f\,d\mu$, and f_y to be the function $f_y(x)=f(y+x)$ for $f \in \Omega$ and $y \in G$.

3. The following lemma will enable us to use the information we know about rearrangement invariant Köthe spaces over a nonatomic space G.

Lemma 1. *If G does not have the discrete topology, then Haar measure μ is nonatomic; that is G has no atoms.*

Proof. We first show that if G contains an atom S, then $\mu(S)$ must be finite. If not, and $\{K_n\}_{n=1}^{\infty}$ is the increasing sequence of compact sets whose union is G, then $\mu(K_n \cap S)$ is finite $(n=1, 2, \ldots)$ and strictly less than $\mu(S)=\infty$. It follows that $\mu(K_n \cap S)=0$ for each n, and $\mu(S)=0$; this is a contradiction. Therefore we must assume that $\mu(S)<\infty$. Now since $\mu(S)=\sup\{\mu(K) : K \subset S,\ K \text{ is compact}\}$ and S is an atom, there is a compact set K which is an atom satisfying $\mu(K)=\mu(S)$. By the nondiscreteness of G, there is a nonempty open set U containing 0 such that $\mu(U)<\mu(K)$. Since $\{U+x : x \in K\}$ is an open cover of the compact K, there must be a finite number of elements $\{x_1, x_2, \ldots, x_n\}$ in K such that $\bigcup_{i=1}^{n} U+x_i \supset K$.

But $\mu(U+x_i \cap K) \leqq \mu(U) < \mu(K)$ $(i=1, 2, \ldots, n)$ implies $\mu(U+x_i \cap K)=0$. It then follows that $\mu(S)=\mu(K)=0$.

LEMMA 2. *If Λ is a rearrangement invariant Köthe space over G, U is a compact neighborhood of* 0 *in G and f is a function in Λ, then*

(i) $T_U f \in \Lambda$;

(ii) *$T_U f$ is a uniformly continuous function on G;*

(iii) *$\rho(T_U f) \leqq \rho(f)$ as ρ runs through a certain family of seminorms that generate the strong topology of Λ.*

Proof. If $f \in \Lambda$ and U is compact in G, we show that $T_U f \cdot g$ is integrable for every $g \in \Lambda^*$; that is, $T_U f \in \Lambda$. As Λ^* is rearrangement invariant, there is a normal, rearrangement invariant, weakly bounded subset H of Λ^* which g belongs to, and satisfying $S_H(f') = S_H(f)$ whenever f' is equimeasurable with f. Since f_y is equimeasurable with f for each $y \in G$, we have

$$\begin{aligned}
\int_G |T_U f(x) \cdot g(x)|\, dx &= \int_G \frac{1}{\mu(U)} \cdot \left| \int_U f(x+y) \cdot g(x) \cdot dy \right| \cdot dx \\
&\leqq \frac{1}{\mu(U)} \cdot \int_G \int_U |f(x+y) \cdot g(x)| \cdot dy \cdot dx \\
&= \frac{1}{\mu(U)} \cdot \int_U \int_G |f(x+y) \cdot g(x)| \cdot dx \cdot dy \\
&\leqq \frac{1}{\mu(U)} \cdot \int_U \sup_{h \in H} \int |f(x+y) \cdot h(x)|\, dx \cdot dy \\
&= \frac{1}{\mu(U)} \int_U S_H(f_y)\, dy \\
&= \frac{1}{\mu(U)} \int_U S_H(f)\, d\mu = S_H(f) < \infty.
\end{aligned}$$

Thus $T_U f \in \Lambda$. Furthermore, it is clear that $\int_G T_U f \cdot g \cdot d\mu \leqq S_H(f)$ for all functions $g \in H$. Taking the supremum on all $g \in H$, we obtain $S_H(T_U f) \leqq S_H(f)$. Since the seminorms S_H generate the strong topology of Λ, (iii) is proved. In addition, this also shows that $T_U \colon \Lambda \to \Lambda$ is a strongly continuous linear function.

In order to show that $T_U f$ is uniformly continuous for U compact and $f \in \Lambda$, we observe first that $f = h + g$ where $h \in L^1(G, \mu)$ and $g \in L^\infty(G, \mu)$ (since Λ is rearrangement invariant); we then have $T_U f = T_U h + T_U g$. We must show that for any $\varepsilon > 0$ there is a 0-neighborhood V in G such that $x - y \in V$ implies $|T_U f(x) - T_U f(y)| < \varepsilon$.

Let $\varepsilon > 0$ be given. Since h is integrable, there is a $\delta > 0$ such that A measurable and $\mu(A) < \delta$ implies $\int_A |h|\, d\mu < \varepsilon/4$. By the compactness of U in G and the regularity of μ, there is a symmetrical 0-neighborhood V such that $\mu(V+U \backslash U)$

$<\min\{\frac{1}{4}\cdot\varepsilon/\|g\|_\infty;\ \delta\}$. Then for $x-y\in V$, we will have

$$\begin{aligned}
&\mu(U)\cdot|T_Uf(x)-T_Uf(y)|\\
&\quad=\mu(U)\cdot|T_Uh(x)-T_Uh(y)+T_Ug(x)-T_Ug(y)|\\
&\quad\leqq\left|\int_{U+x}h(t)\,dt-\int_{U+y}h(t)\cdot dt\right|+\left|\int_{U+x}g(t)\,dt-\int_{U+y}g(t)\,dt\right|\\
&\quad\leqq\int_{x+U\backslash y+U}|h(t)|\,dt+\int_{y+U\backslash x+U}|h(t)|\,dt+\int_{x+U\backslash y+U}|g(t)|\,dt+\int_{y+U\backslash x+U}|g(t)|\,dt\\
&\quad=\int_{x-y+U\backslash U}|h(t+y)|\cdot dt+\int_{y-x+U\backslash U}|h(t+x)|\,dt+\int_{x-y+U\backslash U}|g(t+y)|\cdot dt\\
&\qquad+\int_{y-x+U\backslash U}|g(t+x)|\,dt\\
&\quad\leqq\int_{V+U\backslash U}|h(t+y)|\,dt+\int_{V+U\backslash U}|h(t+x)|\,dt+\int_{V+U\backslash U}|g(t+y)|\,dt\\
&\qquad+\int_{V+U\backslash U}|g(t+x)|\,dt\\
&\quad<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+2\cdot\|g\|_\infty\cdot\frac{\varepsilon}{\|g\|_\infty}\cdot\frac{1}{4}=\varepsilon.
\end{aligned}$$

Thus T_Uf is uniformly continuous.

Remark. On the real line such functions T_Uf are of the form

$$T^hf(x)=\frac{1}{h}\int_{x-h/2}^{x+h/2}f(t)\,dt\quad\text{for } h>0.$$

The T^h operation takes a function f and smoothes it out to a function that approximates the original function; in fact, $\lim_{h\to\infty}T^hf(x)=f(x)$ a.e. whenever f is locally integrable. As an example let us consider $f=\chi_{[a,b]}$ where a and b are real numbers with $a<b$. Then

$$\begin{aligned}
T^hf(x)&=(x-(a-h/2))/h &&\text{for } a-h/2\leqq x\leqq a+h/2;\\
&=1 &&\text{for } a+h/2\leqq x\leqq b-h/2;\\
&=(b+h/2-x)/h &&\text{for } b-h/2\leqq x\leqq b+h/2.
\end{aligned}$$

Obviously, $\lim_{h\to 0}T^hf(x)=1$ for $x\in(a,b)$.

We now give an example of two locally integrable functions f and g on R^1 such that $\int_{R^1}f(t)\cdot g(t)=0$, but $\int_{R^1}T^hf(t)\cdot g(t)\,dt=\infty$ for all $h>0$. This example will show that there is a Köthe space over R^1, $\Lambda=L_g^1=\{f\in\Omega:\int_{R^1}|fg|\,d\mu<\infty\}$, such that $f\in\Lambda$, but $T^hf\notin\Lambda$ for any h.

For each integer $n\geqq 5$ we choose numbers a_n, b_n, c_n and d_n such that $n<a_n<b_n<c_n<d_n$, $b_n-a_n=c_n-b_n=d_n-c_n=1/n$, and $n+1-d_n=a_n-n<\frac{1}{8}$. Set $A_n=(a_n,b_n)\cup(c_n,d_n)$, $B_n=(b_n,c_n)$ $(n\geqq 5)$, $f=\sum_{n=5}^\infty n\cdot\chi_{A_n}$ and $g=\sum_{n=5}^\infty n\cdot\chi_{B_n}$. Clearly,

$\int_{R^1} f(t)\cdot g(t)\,dt=0$. However, choosing $h\leqq\frac{1}{8}$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} T^h f(x)\cdot g(x)\,dx &= \int_{-\infty}^{\infty}\frac{1}{h}\cdot\int_{-h/2}^{h/2} f(x+t)\cdot g(x)\,dt\,dx\\ &= \frac{1}{h}\cdot\int_{-h/2}^{h/2}\int_{-\infty}^{\infty} f(x+t)\cdot g(x)\,dx\,dt\\ &= \frac{1}{h}\cdot\int_{-h/2}^{h/2}\sum_{n=5}^{\infty}\int_{n}^{n+1} n^2\cdot\chi_{A_n}(x+t)\cdot\chi_{B_n}(x)\,dx\,dt\\ &= \sum_{n=5}^{\infty}\frac{1}{h}\cdot\int_{-h/2}^{h/2} n^2\cdot\mu(A_n-t\cap B_n)\,dt\\ &\geqq \frac{1}{h}\sum_{n\geqq 2/h} n^2\cdot\int_{0}^{1/n}\mu(A_n+t\cap B_n)\,dt\\ &= \frac{1}{h}\sum_{n\geqq 2/h} n^2\cdot\int_{0}^{1/n} t\cdot dt = \frac{1}{h}\cdot\sum_{n\geqq 2/h} n^2\cdot\frac{1}{n^2}\cdot\frac{1}{2} = \infty.\end{aligned}$$

In the following theorem $C_0=C_0(G)$ will denote the continuous functions of compact support on G and $S(\Lambda,\Lambda^*)=S$ will denote the strong Köthe topology on Λ generated by the rearrangement invariant seminorms

$$S_B(f) = \sup_{g\in B}\int_G fg\,d\mu \quad (\text{for } f\in\Lambda)$$

where B runs through the normal, rearrangement invariant, weakly bounded subsets of Λ^*. Recall that $\lim_{U\in\mathscr{U}} T_U f=f$ for the topology S means that for each weakly bounded $B\subset\Lambda^*$ there is a relatively compact 0-neighborhood U_0 in G such that $S_B(T_U f-f)\leqq 1$ for all 0-neighborhoods $U\subset U_0$.

THEOREM. *If Λ is a rearrangement invariant Köthe space over G with the strong topology S, then the following are equivalent*:

(1) *C_0 is dense in Λ;*

(2) *$\lim_{U\in\mathscr{U}} T_U f=f$ for each $f\in\Lambda$; $\lim_{n\to\infty}\varphi\cdot\chi_{K_n}=\varphi$ for each nonnegative uniformly continuous φ in Λ.*

Proof. The implication (1) implies (2) will be proven first. We begin by showing that $\lim_{U\in\mathscr{U}} T_U\varphi=\varphi$ for each $\varphi\in C_0$. Let $\varphi\in C_0$ have its support on K compact, and let B be a normal, weakly bounded subset of Λ^*. If U is a compact and symmetric 0-neighborhood in G, then $U+K=\{u+k : u\in U, k\in K\}$ is compact, $d=S_B(\chi_{U+K})$ is a finite number, and the support of $T_U\varphi$ is contained in $U+K$. In order to see this last statement we observe that for $x\in E\backslash U+K$ we will have $U+x\cap K=\varnothing$; for if there was a $u\in U$ and $k\in K$ satisfying $u+x=k$, we would have $x=k-u\in K+U$ (as U is symmetrical), a contradiction. Thus for $x\in E\backslash U+K$, we obtain

$$T_U\varphi(x) = \frac{1}{\mu(U)}\cdot\int_{U+x}\varphi(y)\,dy = \frac{1}{\mu(U)}\cdot\int_{U+x\cap K}\varphi(y)\,dy = 0.$$

If $d=0$ the above argument shows that $S_B(T_V\varphi-\varphi)=0$ for all 0-neighborhoods $V\subset U$. We can then assume that $d>0$. Now since φ is uniformly continuous, there is a symmetric 0-neighborhood V with $V+V\subset U$ such that $x-y\in V$ implies $|\varphi(x)-\varphi(y)|<1/d$. The 0-neighborhood V is relatively compact, and for any 0-neighborhood $V'\subset V$ we have

$$\begin{aligned} S_B(T_{V'}\varphi-\varphi) &= S_B(\chi_{K+U}\cdot(T_{V'}\varphi-\varphi)) \\ &= \sup_{g\in B}\int_G |\chi_{K+U}(x)\cdot(T_{V'}\varphi(x)-\varphi(x))\cdot g(x)|\,dx \\ &\leqq \sup_{g\in B}\int_G \chi_{K+U}(x)\cdot\left\{\frac{1}{\mu(V')}\cdot\int_{V'}|\varphi(x)-\varphi(x+y)|\,dy\right\}\cdot g(x)\,dx \\ &< \sup_{g\in B}\int_G \chi_{K+U}\cdot\frac{1}{d}\cdot g\cdot d\mu = \frac{1}{d}\cdot S_B(\chi_{K+U}) = 1. \end{aligned}$$

Thus $\lim_{U\in\mathscr{U}} T_U\varphi=\varphi$ for each $\varphi\in C_0$. Fix $f\in\Lambda$. We now show $T_Uf\to f$ in Λ. Let B be a normal, rearrangement invariant, weakly bounded subset of Λ^* whose associated seminorm S_B is rearrangement invariant. Since C_0 is strongly dense in Λ, there is a $\varphi\in C_0$ such that $S_B(\varphi-f)<\frac{1}{3}$.

The convergence of $T_U\varphi$ to φ implies there is a relatively compact 0-neighborhood U_0 such that $S_B(T_U\varphi-\varphi)<\frac{1}{3}$ for all 0-neighborhoods $U\subset U_0$. By Lemma 2 we have $S_B(T_Uf-f)\leqq S_B(T_Uf-T_U\varphi)+S_B(T_U\varphi-\varphi)+S_B(\varphi-f)\leqq 2\cdot S_B(f-\varphi)+S_B(T_U\varphi-\varphi)<1$ for $U\subset U_0$. Thus $\lim_{U\in\mathscr{U}} T_Uf=f$ for each $f\in\Lambda$.

For the second part of (1) implies (2) we suppose that $\psi\in\Lambda$ is a nonnegative and uniformly continuous function. If $B\subset\Lambda^*$ is weakly bounded and normal, there is a function $\varphi\in C_0$ such that $S_B(\psi-\varphi)<1$. Suppose the support of φ is contained in K_m for some integer m. If $n\geqq m$ we have

$$0 = |\psi(x)-\chi_{K_n}\cdot\psi(x)| \leqq |\varphi(x)-\psi(x)| \quad \text{for any } x\in K_n;$$

$$|\psi(x)-\chi_{K_n}\cdot\psi(x)| = |\psi(x)| = |\psi(x)-\varphi(x)| \quad \text{for } x\in E\backslash K_n.$$

Therefore, for $n\geqq m$ we have

$$S_B(\psi-\chi_{K_n}\cdot\psi) \leqq S_B(\psi-\varphi) < 1,$$

which was to be shown.

In order to prove (2) implies (1) we must first prove that if B is a normal, rearrangement invariant, weakly bounded subset of Λ^* whose associated seminorm S_B is rearrangement invariant, and $\varepsilon>0$ is arbitrary, then there is a $\delta>0$ (dependent upon ε) such that $S_B(\chi_A)<\varepsilon$ for any measurable set A satisfying $\mu(A)<\delta$. Since G is nondiscrete, there is a compact set K with nonempty interior K° for which the boundary of K, $\partial K=K\mid K^\circ$, contains a point x_0 with the property that every open set about x_0 has a nonempty intersection with the open sets K° and $E\backslash K$. Since we have assumed $\lim_{U\in\mathscr{U}} T_U\chi_K=\chi_K$ for the strong topology of Λ, there is a compact, symmetric 0-neighborhood U_0 in G such that $S_B(T_U\chi_K-\chi_K)<\varepsilon/2$ for any 0-neighborhood $U\subset U_0$. Now we show that there is a 0-neighborhood $V_0\subset U_0$ such that $T_{V_0}\chi_K(x) > \frac{1}{2}$ for all x in some nonempty open set contained in $E\backslash K$. As U_0+x_0 is an open neighborhood of x_0 meeting K° in a nonempty open set and μ is a regular measure, there is a symmetric 0-neighborhood $U' \subset U_0$ such that $0 < \mu(U') < \mu(U_0+x_0 \cap K)$. The 0-neighborhood $V_0 = (U'+x_0 \cap E\backslash K) \cup (U_0+x_0 \cap K)-x_0 \supset (U'+x_0 \cap E\backslash K) \cup (U'+x_0 \cap K)-x_0 = U'$ is contained in U_0.

Since $\mu(U'+x_0 \cap E\backslash K) \leqq \mu(U'+x_0) = \mu(U') < \mu(U_0+x_0 \cap K)$, we have

$$\begin{aligned} T_{V_0}\chi_K(x_0) &= \frac{1}{\mu(V_0)} \cdot \int_{V_0+x_0} \chi_K \, d\mu = \frac{\mu(V_0+x_0 \cap K)}{\mu(V_0+x_0)} \\ &= \frac{\mu(V_0+x_0 \cap K)}{\mu(U'+x_0 \cap E\backslash K)+\mu(U_0+x_0 \cap K)} \\ &= \frac{\mu(U_0+x_0 \cap K)}{\mu(U'+x_0 \cap E\backslash K)+\mu(U_0+x_0 \cap K)} \\ &> \frac{\mu(U_0+x_0 \cap K)}{\mu(U_0+x_0 \cap K)+\mu(U_0+x_0 \cap K)} = \frac{1}{2}. \end{aligned}$$

The fact that $T_{V_0}\chi_K$ is continuous and $T_{V_0}\chi_K(x_0) > \frac{1}{2}$ implies that there is an open set $V(x_0)$ about x_0 such that $T_{V_0}\chi_K(x) > \frac{1}{2}$ for all $x \in V(x_0)$. By the choice of x_0, $V = V(x_0) \cap E\backslash K$ is a nonempty open set contained in $E\backslash K$ for which $x \in V$ implies $T_{V_0}\chi_K(x) > \frac{1}{2}$. It follows that $\frac{1}{2} \cdot \chi_V \leqq T_{V_0}\chi_K \cdot \chi_{E\backslash K}$. Since $V_0 \subset U_0$, for any measurable set $A \subset V$ we have

$$\begin{aligned} \tfrac{1}{2} \cdot S_B(\chi_A) &\leqq S_B(\tfrac{1}{2}\chi_V) \leqq S_B(T_{V_0}\chi_K \cdot \chi_{E\backslash K}) \\ &= S_B(\chi_{E\backslash K} \cdot (T_{V_0}\chi_K - \chi_K)) \leqq S_B(T_{V_0}\chi_K - \chi_K) < \varepsilon/2. \end{aligned}$$

Set $\delta = \frac{1}{2}\mu(V) > 0$, and let A be a measurable set with $\mu(A) < \delta$. Since G has no atoms, there is a measurable set A' contained in V satisfying $\mu(A') = \mu(A)$. The equimeasurability of $\chi_{A'}$ with χ_A implies

$$S_B(\chi_A) = S_B(\chi_{A'}) \leqq S_B(\chi_V) < \varepsilon.$$

We now show that C_0 is dense in Λ. Let $f \in \Lambda$; we assume without loss of generality that $f \geqq 0$. Given $B \subset \Lambda^*$ normal, rearrangement invariant and weakly bounded, there is a compact 0-neighborhood U_0 such that $S_B(T_{U_0}f - f) < \frac{1}{3}$. The function $\varphi = T_{U_0}f$ is nonnegative, uniformly continuous and contained in Λ. By the hypothesis of (2), there is an integer n for which $S_B(\varphi - \varphi\chi_{K_n}) < \frac{1}{3}$. Setting $d = \sup_{x \in K_{n+1}} \varphi(x)$, we can find a $\delta > 0$ such that $\mu(A) < \delta$ implies $S_B(\chi_A) < 1/3d$. Let U be an open set such that $K_n \subset U \subset K_{n+1}$ and $\mu(U|K_n) < \delta$, and let g be a continuous Urysohn function with its support contained in U and having the properties that $g \equiv 1$ on K_n and $g(x) \leqq 1$ for all $x \in G$. The function $g \cdot \varphi = \psi$ is a continuous function whose compact support is contained in U satisfying

$$\begin{aligned} S_B(\varphi \cdot \chi_{K_n} - \psi) &= S_B(\varphi \cdot \chi_{K_n} - \varphi \cdot g) = S_B(\varphi \cdot (g - \chi_{K_n})) = S_B(\varphi \cdot \chi_U \cdot (g - \chi_{K_n})) \\ &= S_B(g \cdot \varphi\chi_{U|K_n}) \leqq d \cdot 1 \cdot S_B(\chi_{U|K_n}) < \frac{d}{3d} = \frac{1}{3}. \end{aligned}$$

We finally have

$$S_B(f-\psi) \leqq S_B(f-\varphi)+S_B(\varphi-\varphi\chi_{K_n})+S_B(\varphi\chi_{K_n}-\psi) < 1.$$

Thus C_0 is dense in Λ for the strong topology. The Theorem is proved.

COROLLARY. *If Λ is a rearrangement invariant Köthe space over G which contains all the constant functions, then $T_U f \to f\ (U \in \mathscr{U})$ strongly for each $f \in \Lambda$ if and only if the uniformly continuous functions are strongly dense in Λ.*

Proof. To show sufficiency we observe that $S_B(\chi_G)<\infty$ for $B \subset \Lambda^*$ weakly bounded. From this we can show, as in the Theorem, that $T_U\varphi \to \varphi$ for any uniformly continuous $\varphi \in \Lambda$. We again use the denseness of the uniformly continuous functions and the fact that $S_B(T_U f) \leqq S_B(f)$ for seminorms S_B generating the topology on Λ to show $T_U f \to f\ (U \in \mathscr{U})$ strongly. The necessity part of the Corollary is obvious since $T_U f$ is uniformly continuous for U compact.

REMARK. The L^p spaces for $1 \leqq p < +\infty$ are spaces in which $T_U f \to f\ (U \in \mathscr{U})$. L^∞ is a space which does not have this property; the continuous functions are not dense for the strong (norm) topology. We now give an example of a Köthe space that is not rearrangement invariant, in which the continuous functions of compact support are dense, and for which $f \in \Lambda$ implies $T_U f \in \Lambda$, but $T_U f$ does not converge to f for the strong topology.

Let $G=R^1$ and let μ be Lebesgue measure on R^1. Construct sequences of positive numbers $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, and $\{c_n\}_{n=1}^\infty$ such that the following is true:

$a_n < b_n < c_n < a_{n+1} \dots$;

$c_n - b_n = b_n - a_n = 1/n^2$ for each n;

if $A_n = (a_n, b_n)$ and $B_n = (b_n, c_n)$ $(n=1,2,\dots)$, then $\bigcup_{n=1}^\infty A_n \cup \bigcup_{n=1}^\infty B_n$ is contained in a compact interval.

Set $f=\sum_{n=1}^\infty n^{3/4}\cdot\chi_{A_n}$ and $g=\sum_{n=1}^\infty n^{3/4}\cdot\chi_{B_n}$. Both f and g are integrable, of compact support, and $f\cdot g \equiv 0$. Thus $f \in L_g^1$ which is Köthe space in which the continuous functions of compact support are dense, and in which $T^h f' \in L^1 g$ whenever $f' \in L^1 g$ (as $L^1 g$ contains all the continuous functions). We show $\lim_{h\to 0}\int_{-\infty}^\infty T^h f(x)\cdot g(x)\,dx = \infty$. Choosing h sufficiently small, we will have

$$\begin{aligned}
\int_{-\infty}^\infty T^h f(x)\cdot g(x)\,dx &= \sum_{n=1}^\infty \int_{B_n} \frac{1}{h}\cdot\int_{-h/2}^{h/2} f(x+t)\cdot g(x)\,dt\,dx \\
&\geqq \sum_{n=1}^\infty \frac{1}{h}\cdot\int_{-h/2}^{0}\int_{B_n} n^{3/2}\cdot\chi_{A_n}(x+t)\,dx\,dt \\
&\geqq \sum_{n\leqq n(h)} \frac{1}{h}\int_{-h/2}^{0}\int_{b_n}^{b_n+t} \chi_{A_n-t}(x)\,dx\,dt \quad (\text{where } n(h) = [\sqrt{(2/h)}]+1) \\
&= \sum_{n\leqq n(h)} n^{3/2}\cdot\frac{1}{h}\int_{-h/2}^{0} t\cdot dt = \frac{h}{8}\sum_{n\leqq n(h)} n^{3/2} \geqq \frac{1}{10}\frac{1}{h^{1/4}} \to \infty \quad \text{as } h \to 0.
\end{aligned}$$

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