

A REPRESENTATION THEOREM FOR FUNCTIONS HOLOMORPHIC OFF THE REAL AXIS

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Abstract. Let f be holomorphic in the union of the upper and lower half planes, and let $p \in [1, \infty)$. We prove that there exists an entire function φ and a sequence $\{f_n\}$ in $L^p(\mathbb{R})$ satisfying $\|f_n\|_p^{1/n} \rightarrow 0$ such that

$$f(z) = \varphi(z) + \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (t-z)^{-n-1} f_n(t) dt.$$

This complements an earlier result of the author's on representation of function holomorphic outside a compact subset of the Riemann sphere. A principal tool in both proofs is the Köthe duality between the spaces of functions holomorphic on and off a subset of the sphere. A corollary of the present result is that each hyperfunction of one variable can be represented by a sum of Cauchy integrals over the real axis.

1. In [1] we considered functions holomorphic in the (extended) complement of a compact subset K of the complex plane. We proved that, when K is locally connected, each such function f which vanishes at ∞ has a representation

$$(1) \quad f(z) = \sum_{n=0}^{\infty} \int_K (t-z)^{-n-1} d\mu_n(t) \quad (z \notin K)$$

where the μ_n are complex Borel measures on K satisfying $\lim_{n \rightarrow \infty} \|\mu_n\|^{1/n} = 0$. In other words, a function holomorphic in a neighborhood of ∞ can, under fairly general conditions, be represented by a sum of Cauchy integrals over its singular set.

In this paper we consider functions f holomorphic in $\mathbb{C} - \mathbb{R}$, where \mathbb{C} denotes the finite complex plane and \mathbb{R} denotes the real axis. Thus f is in fact a pair of functions f_+ and f_- , independently prescribed, one holomorphic in the upper half plane, the other in the lower half plane. The singular set is now $\mathbb{R} \cup \{\infty\}$. There are very few such f having a representation (1) with $\|\mu_n\|^{1/n} \rightarrow 0$ and \mathbb{R} in place of K , since those that do satisfy

$$|f(z)| \leq \sum_{n=0}^{\infty} |\operatorname{Im} z|^{-n-1} \|\mu_n\|.$$

Thus, they are bounded in every region $|\operatorname{Im} z| \geq \delta > 0$ and tend to zero as $|\operatorname{Im} z| \rightarrow \infty$.

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On the other hand, we are going to prove the following.

THEOREM 1. *Let f be holomorphic in $C-R$. Then for any p , $1 \leq p < \infty$, there exists a sequence $\{f_n\}$ in $L^p(R)$ satisfying $\|f_n\|_p^{1/n} \rightarrow 0$ and an entire function φ such that*

$$(2) \quad f(z) = \varphi(z) + \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (t-z)^{-n-1} f_n(t) dt \quad (z \in C-R).$$

Here $\|\cdot\|_p$ denotes the norm in $L^p(R)$. To see that the sum on the right converges, let q be such that $p^{-1} + q^{-1} = 1$. It is not hard to show that

$$\int_{-\infty}^{\infty} |t-z|^{-(n+1)q} dt \leq |\operatorname{Im} z|^{1-(n+1)q} \int_{-\infty}^{\infty} (1+t^2)^{-q/2} dt$$

for $1 < q < \infty$, $n \geq 0$. So, by Hölder's inequality, we have

$$(3) \quad \left| \int_{-\infty}^{\infty} f_n(t)(t-z)^{-n-1} dt \right| \leq C \|f_n\|_p |\operatorname{Im} z|^{q^{-1}-n-1},$$

where C depends on q only. When $p=1$, (3) obviously holds, with $C=1$. It follows from (3) that the sum in (2) converges uniformly, and is bounded, on the sets $|\operatorname{Im} z| \geq \delta > 0$.

If we start with a function f_+ holomorphic in the upper half plane Π , then it follows from Theorem 1 that f_+ has a representation (2) for $z \in \Pi$. Simply apply the theorem to the function f defined to be f_+ in Π and (say) zero in the lower half plane.

2. The theorem asserts that a function holomorphic in $C-R$ can be split into two parts, one of which (the sum of the Cauchy integrals) carries all the finite singularities and has a "weak singularity" at ∞ (it is bounded on the tangent half planes $|\operatorname{Im} z| \geq \delta > 0$). This splitting off leads, via conformal mapping, to the following result for functions in the unit disk.

COROLLARY 1. *Let $f(w)$ be holomorphic in $|w| < 1$. Then there exists a function ψ , holomorphic on the sphere except at $w=1$, such that $f(w) - \psi(w) \rightarrow 0$ as $w \rightarrow 1$ inside any Stolz angle with vertex at $w=1$, and $f - \psi$ is bounded in each tangent circle $|w-r| < 1-r$, $0 < r < 1$.*

Proof. Apply Theorem 1, with $p=1$, to f^* , where

$$\begin{aligned} f^*(z) &= f((z+i)/(z-i)) & (\operatorname{Im} z > 0), \\ &= 0 & (\operatorname{Im} z < 0). \end{aligned}$$

Let h be the sum of the Cauchy integrals. We have already observed that h is bounded on the tangent half planes $|\operatorname{Im} z| \geq \delta > 0$. Moreover, from (3), with $q=\infty$, we see that $h(z) = O(|\operatorname{Im} z|^{-1})$ as $|\operatorname{Im} z| \rightarrow \infty$. The corollary follows.

M. Sato [4] introduced the class of hyperfunctions. These generalize Schwartz distributions. Loosely speaking, a hyperfunction (of one variable) is supposed to be like a "boundary jump" $f(x+i0) - f(x-i0)$ of some function holomorphic in $C-R$. Precisely, the vector space of hyperfunctions is defined to be the quotient of the space of all functions holomorphic in $C-R$ modulo the space of all entire functions. From Theorem 1 we obtain the following result concerning representation of hyperfunctions by integrals over the real axis.

COROLLARY 2. *Every hyperfunction on R has a representing holomorphic function of the form*

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f_n(t)(t-z)^{-n-1} dt \quad (\text{Im } z \neq 0)$$

with $\|f_n\|_p^{1/n} \rightarrow 0$ as $n \rightarrow \infty$, $1 \leq p < \infty$.

3. The main idea in the proof of Theorem 1 is the duality between the spaces of functions holomorphic on and off a subset of the Riemann sphere, which was developed by Köthe [2]. We will also use some facts from the theory of locally convex spaces [5]. In particular, we will use the following criterion for a linear map between Fréchet spaces to be onto.

PROPOSITION. *Let E and F be Fréchet spaces and let T be a linear map from E into F . If $(T')^{-1}(B)$ is equicontinuous in E' whenever B is an equicontinuous set in F' , then $T(E)=F$.*

Here E' and F' denote the topological dual spaces and T' denotes the adjoint map. Note that "equicontinuous" can be interchanged with "weakly bounded," since Fréchet spaces are barreled.

Proof. According to [6, p. 198], T is onto if (and only if) the closure of $T(U)$ is a zero-neighborhood in F whenever U is a convex circled zero-neighborhood in E . Since

$$T(U)^0 = (T')^{-1}(U^0)$$

where 0 denotes the polar with respect to one of the dualities $\langle E, E' \rangle$ or $\langle F, F' \rangle$, and U^0 is equicontinuous, it follows from the hypothesis that $T(U)^0$ is equicontinuous, hence $T(U)^{00}$ is a zero-neighborhood in F . For convex sets in E the closure in the original topology coincides with the closure in the weak topology. Since $T(U)^{00}$ is the weak closure of $T(U)$, we are done.

4. Proof of the theorem. We introduce three Fréchet spaces, E_1 , E_2 , and F . E_1 is the space of all entire functions and F is the space of all functions holomorphic in $C-R$. These two spaces are given the usual topology of uniform convergence on compact subsets (of C and $C-R$, respectively). E_2 is the space of all sequences $\{f_n\}_0^\infty$ in $L^p(R)$ such that $\|f_n\|_p^{1/n} \rightarrow 0$ as $n \rightarrow \infty$. It is topologized by the sequence of

norms $\| \cdot \|_k$ defined by

$$\| \{f_n\} \|_k = \sup_n k^n \|f_n\|_p \quad (k = 1, 2, \dots).$$

Then E_2 is a locally convex metrizable space, and, as the reader may easily check, it is complete. Spaces like E_2 were studied in [3].

Define the linear map $T: E_1 \oplus E_2 \rightarrow F$ by

$$T(f \oplus \{f_n\})(z) = f(z) + \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (t-z)^{-n-1} f_n(t) dt \quad (z \in C-R).$$

T is continuous, because if K is a compact subset of $C-R$, then, if

$$k^{-1} \leq \frac{1}{2} \inf \{ |\operatorname{Im} z| : z \in K \},$$

we deduce from (3) that

$$\|T(f \oplus \{f_n\})\|_K \leq \|f\|_K + C' \|\{f_n\}\|_k$$

where C' depends only on K and p , and $\| \cdot \|_K$ denotes sup over K .

The conclusion of Theorem 1 is that T is onto. We want to deduce this from the proposition, so we look for concrete representations of the dual spaces.

First of all, for any pair of locally convex spaces E_1 and E_2 , we can identify $(E_1 \oplus E_2)'$ and $E_1' \oplus E_2'$ by the rule

$$\langle x_1 \oplus x_2, y \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$$

where $x_i \in E_i$, $y \in (E_1 \oplus E_2)'$, and $y_i \in E_i'$ is the restriction of y to $E_i \oplus \{0\}$.

Next, if G is an open subset of the Riemann sphere S with $\infty \notin G$, and E is the space of all functions holomorphic in G , with the topology of uniform convergence on compact subsets, then, by Köthe's theory [2], E' can be identified with the spaces of all (equivalence classes of) functions g holomorphic in some (variable) neighborhood of $S-G$ which vanish at ∞ . The duality is given by

$$\langle f, g \rangle = \frac{1}{2\pi i} \int_{\Gamma} f(z) g(z) dz \quad (f \in E)$$

where Γ is the union of finitely many simple closed rectifiable curves in $G \cap V$, V being a neighborhood of $S-G$ in which g is holomorphic, and Γ is such that

$$\begin{aligned} \int_{\Gamma} (t-z)^{-1} dt &= 2\pi i & (z \in G-V), \\ &= 0 & (z \in S-G). \end{aligned}$$

In our case, E_1' can be identified with the space of functions holomorphic in some neighborhood of ∞ , and F' can be identified with the space of functions holomorphic outside some compact subset of $C-R$. In both cases, the functions vanish at ∞ . Note that $F' \subset E_1'$.

The dual of E_2 can be identified with the set of all sequences $\{g_n\}$ in L^q such that $\|g_n\|_q = O(k^n)$ ($n \rightarrow \infty$) for some positive integer k . The duality is given by

$$\langle \{f_n\}, \{g_n\} \rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f_n(t) g_n(t) dt.$$

These statements are easy to prove and are left to the reader. (Or, the proof of Proposition 3 of [3, p. 45] can be adapted.)

Define the linear map $T_0: F' \rightarrow E'_1 \oplus E'_2$ by

$$(5) \quad T_0 g = g \oplus \{(-1)^n (n!)^{-1} D^n g\}_0^{\infty},$$

where $D^n g$ denotes the n th derivative of g , restricted to R . To see that the sequence on the right satisfies the growth condition for E'_2 , select a pair of simple closed rectifiable curves, one in the upper half plane, the other in the lower half plane, such that g is holomorphic outside a compact subset of the union of the interiors. Orient both curves counterclockwise, and call their union Γ . Then we have (since g vanishes at ∞)

$$(5) \quad D^n g(t) = -\frac{n!}{2\pi i} \int_{\Gamma} g(z) (z-t)^{-n-1} dz \quad (t \in R).$$

Take $h \in L^p$ with $\|h\|_p \leq 1$. Using Fubini's theorem and the estimate used in (3) for the L^q norm of $(t-z)^{-n-1}$, we find

$$\int_{-\infty}^{\infty} |h(t)| |D^n g(t)| dt \leq Cn! (2\pi)^{-1} \left(\int_{\Gamma} |g(z)| |dz| \right) k^{n+1-q^{-1}}$$

where C is the constant of (3) and $k^{-1} \leq \inf \{|\operatorname{Im} z| : z \in \Gamma\}$. It follows from this that $\{(-1)^n (n!)^{-1} D^n g\} \in E'_2$.

We claim that T_0 can be identified with T' . Take $f \in E_1$, $\{f_n\} \in E_2$, and $g \in F'$. Then, writing $\langle \rangle_E$ to denote $\langle E, E' \rangle$ duality, we have

$$(6) \quad \begin{aligned} \langle f \oplus \{f_n\}, T_0 g \rangle_{E_1 \oplus E_2} &= \langle f, g \rangle_{E_1} + \langle \{f_n\}, \{(-1)^n (n!)^{-1} D^n g\} \rangle_{E_2} \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(z) g(z) dz + \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (-1)^n (n!)^{-1} D^n g(t) f_n(t) dt \end{aligned}$$

where Γ , which depends on g , is the same as in (5). (This path is a suitable one for effecting the Köthe duality in both $\langle E_1, E'_1 \rangle$ and $\langle F, F' \rangle$.)

On the other hand, writing

$$F_n(z) = \int_{-\infty}^{\infty} (t-z)^{-n-1} f_n(t) dt,$$

we have

$$(7) \quad \begin{aligned} \langle T(f \oplus \{f_n\}), g \rangle_{F'} &= \langle f + \sum F_n, g \rangle_{F'} = \langle f, g \rangle_{F'} + \sum \langle F_n, g \rangle_{F'} \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(z) g(z) dz + \frac{1}{2\pi i} \sum \int_{\Gamma} F_n(z) g(z) dz. \end{aligned}$$

From Fubini's theorem and (5) it follows that

$$\frac{1}{2\pi i} \int_{\Gamma} F_n(z) g(z) dz = (-1)^n (n!)^{-1} \int_{-\infty}^{\infty} f_n(t) D^n g(t) dt,$$

hence the right-hand sides of (6) and (7) are the same. Thus T_0 and T' can be identified, as asserted.

To finish the proof, it suffices, by the proposition, to show that $T_0^{-1}(B)$ is equicontinuous in F' whenever B is equicontinuous in $E'_1 \oplus E'_2$. We can assume $B = B_1 \oplus B_2$, where B_1 is the set of all functions holomorphic in $|z| > \rho$, continuous and ≤ 1 in $|z| \geq \rho$, and vanishing at ∞ , for some fixed ρ [2, Satz 12, p. 39], while B_2 is any superset of the polar of $\{\{f_n\} \in E_2 : \|\{f_n\}\|_k \leq 1\}$ for some fixed positive integer k . We choose

$$B_2 = \{\{g_n\} : \|g_n\|_q \leq k^n, n = 0, 1, 2, \dots\}.$$

A set in F' is equicontinuous if and only if it is contained in one of the form

$$(8) \quad \{h : h \text{ is holomorphic in } S-K, \\ \text{continuous on the closure of } S-K, \text{ and } \|h\|_{S-K} \leq M\}$$

for some fixed compact subset K of $C-R$ and some fixed constant M .

If $g \in F'$ then $T_0 g \in B_1 \oplus B_2$ iff $g \in B_1$ and

$$(9) \quad \|D^n g\|_q \leq n! k^n \quad (n = 0, 1, 2, \dots).$$

From

$$D^n g(t) = - \int_t^{2\rho} D^{n+1} g(\tau) d\tau + D^n g(2\rho)$$

(the ρ used in defining B_1) we deduce by Hölder's inequality and (9) that

$$|D^n g(t)| \leq (n+1)! k^{n+1} (4\rho)^{1/p} + |D^n g(2\rho)|$$

holds when $-2\rho < t < 2\rho$. Since $g \in B_1$, Cauchy's formula shows that $|D^n g(2\rho)| \leq n! \rho^{-n}$. Thus the series

$$\sum_{n=0}^{\infty} (n!)^{-1} D^n g(t) (z-t)^n \quad (-2\rho < t < 2\rho)$$

converges whenever $|z-t| < \min(k^{-1}, \rho) = \sigma$, its sum is independent of t , and so provides a single valued analytic continuation of g to the rectangle

$$Q = \{z : |\operatorname{Im} z| < \frac{1}{2}\sigma, |\operatorname{Re} z| < 2\rho\}.$$

Moreover, $|g| \leq M'$ in Q , where M' depends only on k, p , and ρ , and g is continuous on the closure of Q . Thus we have proved that $T_0^{-1}(B_1 \oplus B_2)$ is contained in the set (8), with $K = \{z : |z| \leq \rho, z \notin Q\}$, $M = \max(1, M')$, and so $T_0^{-1}(B_1 \oplus B_2)$ is equicontinuous.

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