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WEIGHTED NORM INEQUALITIES FOR THE HARDY MAXIMAL FUNCTION

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Abstract. The principal problem considered is the determination of all non-negative functions, $U(x)$, for which there is a constant, C, such that

$$\int_J [f^*(x)]^p U(x)\,dx \leqq C \int_J |f(x)|^p U(x)\,dx,$$

where $1<p<\infty$, J is a fixed interval, C is independent of f, and f^* is the Hardy maximal function,

$$f^*(x) = \sup_{y\neq x;\, y\in J} \frac{1}{y-x} \int_x^y |f(t)|\,dt.$$

The main result is that $U(x)$ is such a function if and only if

$$\left[\int_I U(x)\,dx\right]\left[\int_I [U(x)]^{-1/(p-1)}\,dx\right]^{p-1} \leqq K|I|^p$$

where I is any subinterval of J, $|I|$ denotes the length of I and K is a constant independent of I.

Various related problems are also considered. These include weak type results, the problem when there are different weight functions on the two sides of the inequality, the case when $p=1$ or $p=\infty$, a weighted definition of the maximal function, and the result in higher dimensions. Applications of the results to mean summability of Fourier and Gegenbauer series are also given.

1. **Introduction.** The original inequality of the type

$$\int_J [f^*(x)]^p U(x)\,dx \leqq C \int_J |f(x)|^p U(x)\,dx \tag{1.1}$$

was the well-known one of Hardy and Littlewood [3] showing that (1.1) is true if $U(x)=1$ and $1<p<\infty$. Stein in [10] showed that (1.1) is true for $J=(-\infty,\infty)$ if $1<p<\infty$, $U(x)=|x|^a$ and $-1/p<a<1-1/p$. Fefferman and Stein in [1] showed that (1.1) is true for $J=(-\infty,\infty)$ if $1<p<\infty$ and $U^*(x)\leqq CU(x)$ for almost every x.

Theorems of this sort are important in proving weighted mean convergence results for orthogonal series since the error terms can almost always be majorized by some version of $f^*(x)$; this was done, for example, in [6], [7] and [8]. They can also be used to prove mean summability results; several examples of this are given in this paper. It also turns out that the results here are needed to determine all the

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weight functions for which (1.1) is true if f^* is replaced by the conjugate function or the Hilbert transform. This problem was considered by Helson and Szegö in [4] and Forelli in [2] among others but has been solved completely only for the conjugate function when $p=2$. A sequel to this paper by this author and R. Wheeden will give the solution to this problem for $1<p<\infty$ for both the Hilbert transform and the conjugate function.

As a first step in obtaining the main result the "weak type" problem is considered in §2. Since it introduces no additional complications, a slightly more general result is proved. The problem considered is to find all pairs of nonnegative functions, $U(x)$, $V(x)$, for which there is a constant, B, such that

$$\int_{E_a} U(x)\,dx \leqq Ba^{-p}\int_J |f(x)|^p V(x)\,dx \tag{1.2}$$

where E_a is the subset of J where $f^*(x)>a$, $0<a<\infty$, and B is independent of f and a. A necessary and sufficient condition is given for this; it follows easily that

$$\left[\int_I U(x)\,dx\right]\left[\int_I [U(x)]^{-1/(p-1)}\,dx\right]^{p-1} \leqq K|I|^p \tag{1.3}$$

for every subinterval, I, of J is a necessary condition for (1.1) to hold.

Proving that (1.3) is a sufficient condition for (1.1) to hold is somewhat harder and requires several lemmas concerning functions that satisfy (1.3). These are proved in §3. Hölder's inequality shows that if U satisfies (1.3) for a given p, it satisfies (1.3) for all larger p. The major part of §3 is devoted to proving that a function satisfying (1.3) for a given p also satisfies it for a smaller p. Once this is known an interpolation argument in §4 completes the proof that (1.3) implies (1.1).

Several interesting corollaries follow from the results in §§2 and 3. If $U(x)$ satisfies (1.1) and $-\min[p-1, 1/(p-1)] \leqq a \leqq 1$, then $[U(x)]^a$ also satisfies (1.1). Furthermore, there is always an $a>1$ and an $a<-\min[p-1, 1/(p-1)]$ for which $[U(x)]^a$ satisfies (1.1). It also follows that the set of all p for which a function, $U(x)$, satisfies (1.1) is always an interval of the form (p_0, ∞). These corollaries and some results concerning the cases when $p=1$ and $p=\infty$ are given in §4.

Surprisingly, the problem of finding all U and V such that

$$\int_J [f^*(x)]^p U(x)\,dx \leqq C\int_J |f(x)|^p V(x)\,dx \tag{1.4}$$

is much harder. In this case the condition for (1.2) is still necessary since (1.4) implies (1.2), but this condition is not sufficient. In §5 several examples are given to illustrate the difficulties.

In dealing with mean convergence problems for various orthogonal series in [6], [7], and [8] it was natural to consider the weighted Hardy maximal function,

$$\sup_{y\neq x;\, y\in J} \frac{\int_x^y |f(t)|\,dm(t)}{\int_x^y dm(t)},$$

where $m(t)$ was a suitable measure. In §6 a necessary and sufficient condition is given for an inequality like (1.1) to hold for this maximal function. The result follows from the same reasoning as the theorem concerning $f^*(x)$.

The n dimensional case is sketched in §7. Except for one basic lemma the development is similar to the one dimensional case.

Finally, it is shown in §8 that weighted Cesaro and Abel mean summability results hold for Fourier series if and only if the weight function satisfies the condition (1.3). A similar result is proved for mean Abel summability of Gegenbauer series.

Throughout this paper $0\cdot\infty$ will be taken to be 0, C will denote a constant not necessarily the same at each occurrence, $|E|$ will denote the Lebesgue measure of a set E, and p' will be defined by $1/p+1/p'=1$. When $p=1$, $[\int_I |g(x)|^{-1/(p-1)}\,dx]^{p-1}$ will be taken to mean ess $\sup_{x\in I} |g(x)|^{-1}$.

2. **A weak type result.** The theorem to be proved in this section is the following.

THEOREM 1. *Let J be a fixed interval, $1\leqq p<\infty$, $0<a<\infty$, $U(x)$ and $V(x)$ be nonnegative functions on J, and given $f(x)$ on J let E_a be the subset of J where $f^*(x)>a$. Then there is a constant, B, independent of f and a such that*

$$\int_{E_a} U(x)\,dx \leqq Ba^{-p}\int_J |f(x)|^p V(x)\,dx \tag{2.1}$$

if and only if there is a constant, K, such that for every subinterval, I, of J

$$\left[\int_I U(x)\,dx\right]\left[\int_I [V(x)]^{-1/(p-1)}\,dx\right]^{p-1} \leqq K|I|^p. \tag{2.2}$$

Furthermore, if B and K are the least constants for which (2.1) *and* (2.2) *are true, $K\leqq B\leqq 2^pK$.*

For $p=1$ the convention stated at the end of §1 makes (2.2) take the form $\int_I U(x)\,dx \leqq K|I|$ ess $\inf_{x\in I} V(x)$. It is immediate then that (2.2) for all subintervals, I, of J with $p=1$ is equivalent to the statement $U^*(x)\leqq KV(x)$ for almost every x in J; this is the condition used by Stein and Fefferman in [1].

To prove Theorem 1 it will first be shown that $B\leqq 2^pK$, and, consequently, that (2.2) implies (2.1). To do this fix a positive a and define

$$f_1^*(x) = \sup_{y<x;\,y\in J} \frac{1}{x-y}\int_y^x |f(t)|\,dt \quad\text{and}\quad f_2^*(x) = \sup_{y>x;\,y\in J} \frac{1}{y-x}\int_x^y |f(t)|\,dt.$$

As shown on p. 31, Vol. I of [12], the set where $f_1^*(x)>a$ is a disjoint union of open intervals, (a_i, b_i), and $\int_{a_i}^{b_i} |f(t)|\,dt\geqq a|(a_i, b_i)|$. The same is, of course, true for $f_2^*(x)$. Therefore, the set where $f^*(x)>a$, E_a, is open and can be written as the disjoint union of intervals, (c_i, d_i). Since either f_1^* or f_2^* is greater than a on at least half of each (c_i, d_i), it follows that

$$\int_{c_i}^{d_i} |f(t)|\,dt \geqq \tfrac{1}{2}a|(c_i, d_i)|. \tag{2.3}$$

Now assume that $f(x)=0$ at almost every point where $V(x)=\infty$; otherwise (2.1) is immediate for any positive B. Then

$$\int_{E_a} U(x)\,dx = \sum_i \int_{c_i}^{d_i} U(x)\,dx. \tag{2.4}$$

Using (2.3), the integral on the right side of (2.4) is bounded above by

$$\left[\frac{2}{a|(c_i, d_i)|}\int_{c_i}^{d_i} |f(t)|\,dt\right]^p \int_{c_i}^{d_i} U(x)\,dx. \tag{2.5}$$

Using Hölder's inequality, (2.5) is bounded by

$$\frac{2^p a^{-p}}{|(c_i, d_i)|^p}\int_{c_i}^{d_i} |f(t)|^p V(t)\,dt\left[\int_{c_i}^{d_i} [V(x)]^{-1/(p-1)}\,dx\right]^{p-1}\int_{c_i}^{d_i} U(x)\,dx \tag{2.6}$$

provided $0<V(x)<\infty$ almost everywhere in (c_i, d_i). If $V(x)=0$ on a subset of (c_i, d_i) of positive measure, $\int_{c_i}^{d_i} U(x)\,dx=0$ so (2.5) is bounded by (2.6) in this case. If $V(x)=\infty$ on a set of positive measure, the fact that $f(x)=0$ almost everywhere on the set again shows that (2.6) bounds (2.5). Then replacing the integrals on the right side of (2.4) with (2.6) and using (2.2) shows that

$$\int_{E_a} U(x)\,dx \leqq 2^p K a^{-p} \sum_i \int_{c_i}^{d_i} |f(t)|^p V(t)\,dt.$$

This immediately implies (2.1) with $B \leqq 2^p K$.

To show that $K \leqq B$ when $p>1$, fix a finite subinterval, I, of J and define $A=\int_I [V(x)]^{-1/(p-1)}\,dx$. If $A=0$, (2.2) is true for any K. If $0<A<\infty$, let $f(x)=[V(x)]^{-1/(p-1)}$ on I and 0 elsewhere. Then $f^*(x) \geqq A/|I|$ on all of I and (2.1) shows that

$$\int_I U(x)\,dx \leqq B|I|^p A^{-p}\int_I [V(x)]^{-p/(p-1)} V(x)\,dx. \tag{2.7}$$

Since $A<\infty$, $V(x)>0$ almost everywhere on I and the integral on the right side of (2.7) equals A. Multiplying both sides of (2.7) by A^{p-1} then proves (2.2) with $K \leqq B$.

If $A=\infty$, then $[V(x)]^{-1/p}$ is not in $L^{p'}$ on I. Then there is a function, $g(x)$, which is in L^p on I and 0 outside of I such that $\int_I g(x)[V(x)]^{-1/p}\,dx=\infty$. Define $f(x)=g(x)[V(x)]^{-1/p}$. Then since $[f(x)]^p V(x) \leqq [g(x)]^p$, $[f(x)]^p V(x)$ is integrable on I while $f^*(x)=\infty$ everywhere on I. Then (2.1) implies that $\int_I U(x)\,dx=0$ and this immediately implies that (2.2) is true in this case for any K. This completes the proof that $K \leqq B$ when $p>1$.

To prove that $K \leqq B$ when $p=1$, fix a finite subinterval, I, of J. If $\operatorname{ess\,inf}_{y\in I} V(y)=\infty$, (2.2) is true for all K. Otherwise, given $\varepsilon>0$, there is a subset, E, of I such that $|E|>0$ and $V(x)<\varepsilon+\operatorname{ess\,inf}_{y\in I} V(y)$ for all x in E. Define $f(x)=1$ on E and 0 elsewhere. Then $f^*(x) \geqq |E|/|I|$ on I and (2.1) implies that

$$\int_I U(x)\,dx \leqq \frac{B|I|}{|E|}\int_E V(x)\,dx \leqq B|I|\left[\varepsilon+\operatorname*{ess\,inf}_{x\in I} V(x)\right].$$

Since this is true for every $\varepsilon > 0$, (2.2) follows with $K \leqq B$. This completes the proof of Theorem 1.

3. **Facts concerning condition (1.3).** This section contains various lemmas that will be needed in §4 to show that (1.3) implies (1.1). The principal result is Lemma 5.

LEMMA 1. *Let $g(x)$ be nonnegative and integrable on a finite interval I, let a be a positive constant, and let Q be a collection of open subintervals of I such that, for every H in Q, $\int_H g(x)\,dx \leqq a|H|$. Then if $R = \bigcup_{H\in Q} H$, $\int_R g(x)\,dx \leqq 2a|R|$.*

Given $\varepsilon > 0$, choose $\delta > 0$ so that if $|E| < \delta$ and $E \subset I$, then $\int_E g(x)\,dx < \varepsilon$. Choose a finite collection $H_1, \ldots, H_n$ from Q so that $|\bigcup_{k=1}^n H_k| \geqq |R| - \delta$ and no point of I is contained in more than two H_k's. Then $\int_R g = \int_{R-\cup H_k} g + \int_{\cup H_k} g \leqq \varepsilon + \sum \int_{H_k} g \leqq \varepsilon + a \sum |H_k| \leqq \varepsilon + 2a|R|$. Since ε is arbitrary, this proves the lemma.

LEMMA 2. *Let I be a finite interval, B a subset of I and Q a collection of open subintervals of I such that, for every H in Q, $|H \cap B| \geqq 2|H|/3$. Then if $R = \bigcup_{H\in Q} H$, $|B \cap R| \geqq |R|/3$.*

Given $\varepsilon > 0$, choose $H_1, \ldots, H_n$ from Q so that $|\bigcup H_k| > |R| - \varepsilon$ and no point of I is contained in more than two H_k's. Then $|B \cap R| \geqq |B \cap \bigcup H_k| \geqq \frac{1}{2} \sum |B \cap H_k| \geqq \frac{1}{3} \sum |H_k| \geqq |R|/3 - \varepsilon$ and the lemma follows.

The next lemma, Lemma 3, is essentially a proof that if a function satisfies (1.3) for every subinterval, I, of a given interval, then the nonincreasing rearrangement satisfies the same condition. To avoid some minor technical difficulties and to simplify the proof of Lemma 5, however, Lemma 3 is stated in a slightly weaker form.

LEMMA 3. *Assume that $1 \leqq q < \infty$, I is a fixed finite interval, $g(x)$ is nonnegative and for every subinterval H, of I*

$$\left[\int_H g(x)\,dx\right]\left[\int_H [g(x)]^{-1/(q-1)}\,dx\right]^{q-1} \leqq K|H|^q \tag{3.1}$$

where K is independent of H. Let $h(s)$ be the nonincreasing rearrangement of $g(x)$ on $[0, |I|]$; $h(s) = \sup_{|E|=s; E\subset I}[\inf_{x\in E} g(x)]$. Then if $0 < s \leqq |I|/20$,

$$\int_0^s h(t)\,dt \leqq 20Ks3^{q-1}h(s).$$

Fix s and let $a = (1/s)\int_0^s h(t)\,dt$. If $a \leqq 20h(s)$, there is nothing to prove since, by Hölder's inequality, $K \geqq 1$. If $a = \infty$, $\int_I g(x)\,dx = \infty$ and (3.1) with $H = I$ implies that $g(x) = \infty$ for almost every x in I and, consequently, that $h(s) = \infty$. The conclusion of the lemma is then immediate in this case also.

Therefore, assume that $20h(s) < a < \infty$, and let B be the set of x in I such that $g(x) > h(s)$. Then $\int_B g(x)\,dx \geqq \int_0^s h(t)\,dt - sh(s)$; the subtracted term compensates for the fact that $h(t)$ may equal $h(s)$ on part of $[0, s]$. The definition of a and the fact that $a > 20h(s)$ then imply that

$$\int_B g(x)\,dx > 19sa/20. \tag{3.2}$$

Next, let Q be the (possibly empty) set of all subintervals H of I such that $\int_H g(x)\,dx = a|H|/10$ and $|H \cap B| \geqq 2|H|/3$. Let $R = \bigcup_{H \in Q} H$. By Lemma 1

$$\int_R g(x)\,dx \leqq a|R|/5, \tag{3.3}$$

and by Lemma 2

$$|B \cap R| \geqq |R|/3. \tag{3.4}$$

By (3.4), $|R| \leqq 3|B \cap R| \leqq 3|B| \leqq 3s$ so (3.3) implies that

$$\int_{B \cap R} g(x)\,dx \leqq 3sa/5. \tag{3.5}$$

Then (3.2) shows that

$$\int_{B \cap R} g(x)\,dx < \frac{12}{19} \int_B g(x)\,dx. \tag{3.6}$$

Let $\tilde{R}$ denote the complement of R. If for every interval, H, containing an x in $B \cap \tilde{R}$, $\int_H g(x)\,dx < a|H|/10$, then $g(x) \leqq a/10$ almost everywhere in $B \cap \tilde{R}$ and

$$\int_{B \cap \tilde{R}} g(x)\,dx \leqq a|B \cap \tilde{R}|/10. \tag{3.7}$$

Again, by the definition of B and (3.2), $|B \cap \tilde{R}| \leqq |B| \leqq s \leqq (20/19a) \int_B g(x)\,dx$ so (3.7) implies that

$$\int_{B \cap \tilde{R}} g(x)\,dx \leqq \frac{2}{19} \int_B g(x)\,dx. \tag{3.8}$$

Since (3.6) and (3.8) are inconsistent, there must be an interval, G, containing an x in $B \cap \tilde{R}$ such that $G \subset I$ and $\int_G g(x)\,dx \geqq a|G|/10$.

Since $s \leqq |I|/20$ and $a > 20h(s)$, $\int_I g(x)\,dx = \int_0^{|I|} h(t)\,dt \leqq sa + (|I| - s)h(s) \leqq a|I|/10$. Therefore, G can be enlarged to make an interval, F, such that $F \subset I$ and

$$\int_F g(x)\,dx = a|F|/10. \tag{3.9}$$

The interval, F, is not in Q since it contains an x that is not in R. Since F satisfies one part of the definition of an H in Q, it must violate the other. Therefore,

$$|F \cap B| < 2|F|/3. \tag{3.10}$$

Now by (3.1),

$$K|F|^q \geqq \left[\int_F g(x)\,dx\right]\left[\int_F [g(x)]^{-1/(q-1)}\,dx\right]^{q-1}. \tag{3.11}$$

Also,

$$\left[\int_F [g(x)]^{-1/(q-1)}\,dx\right]^{q-1} \geqq \left[\int_{F\cap\tilde{B}} [g(x)]^{-1/(q-1)}\,dx\right]^{q-1} \geqq \frac{|F\cap\tilde{B}|^{q-1}}{h(s)}.$$

By (3.10), $|F\cap\tilde{B}|\geqq|F|/3$. Using these facts on the second part of the right side of (3.11) and (3.9) on the first part shows that

$$K|F|^q \geqq \frac{a3^{1-q}|F|^q}{10h(s)}. \tag{3.12}$$

Substituting the definition of a into (3.12) then gives the assertion of the lemma.

LEMMA 4. *Let $h(t)$ be nonnegative and monotone decreasing on $[0,j]$ and assume that there is a constant, D, such that, for $0\leqq s\leqq j/20$, $\int_0^s h(t)\,dt\leqq Dsh(s)$. Then if $1\leqq r<D/(D-1)$,*

$$\int_0^j [h(t)]^r\,dt \leqq \frac{(20)^r j^{1-r} D}{D-r(D-1)}\left[\int_0^j h(t)\,dt\right]^r.$$

By hypothesis

$$h(s)\Big/\int_0^s h(t)\,dt \geqq \frac{1}{Ds} \tag{3.13}$$

for $0\leqq s\leqq j/20$. Let $b=j/20$ and let u satisfy $0\leqq u\leqq b$. Integrating (3.13) from u to b gives

$$\log\left[\int_0^b h(t)\,dt\Big/\int_0^u h(t)\,dt\right] \geqq \frac{\log(b/u)}{D}.$$

Exponentiating gives

$$\int_0^u h(t)\,dt \leqq (u/b)^{1/D}\int_0^b h(t)\,dt.$$

Since h is decreasing, this implies that for $0<u\leqq b$

$$h(u) \leqq \frac{1}{u}\left(\frac{u}{b}\right)^{1/D}\int_0^b h(t)\,dt. \tag{3.14}$$

Since h is decreasing, it is immediate that

$$\int_0^j [h(u)]^r\,du \leqq 20\int_0^b [h(u)]^r\,du. \tag{3.15}$$

Using (3.14) in the right side of (3.15) then gives the assertion of the lemma.

The following definition will simplify the statement and proof of the next two lemmas.

DEFINITION. A function, $g(x)$, satisfies condition A_p on an interval, I, with constant, K, if $g(x)$ is nonnegative and, for every subinterval, H, of I, (3.1) holds.

LEMMA 5. *If* $1<p<\infty$ *and* $U(x)$ *satisfies condition* A_p *on the interval,* J, *with constant,* K, *then there exist constants,* r *and* L, *depending only on* p *and* K *such that* $1<r<p$ *and* $U(x)$ *satisfies condition* A_r *on* J *with constant,* L.

Let I be a finite subinterval of J. Then $[U(x)]^{-1/(p-1)}$ satisfies condition $A_{p'}$ on I with constant $K^{p'-1}$. Next apply Lemmas 3 and 4 with $g(x)=[U(x)]^{-1/(p-1)}$ and $q=p'$. This shows that if $u=(D-\frac{1}{2})/(D-1)$ where $D=20(3K)^{p'-1}$, there is a constant, C, depending only on p and K such that

$$\int_0^{|I|}[V(t)]^u\,dt \leqq C|I|^{1-u}\left[\int_0^{|I|}V(t)\,dt\right]^u \tag{3.16}$$

where $V(t)$ is the decreasing rearrangement of $[U(x)]^{-1/(p-1)}$ on I. Now (3.16) immediately implies that

$$\int_I [U(x)]^{-u/(p-1)}\,dx \leqq C|I|^{1-u}\left[\int_I [U(x)]^{-1/(p-1)}\,dx\right]^u. \tag{3.17}$$

Combining (3.17) with the fact that $U(x)$ satisfies condition A_p on J with constant, K, shows that

$$\left[\int_I U(x)\,dx\right]\left[\int_I [U(x)]^{-u/(p-1)}\,dx\right]^{(p-1)/u} \leqq KC^{(p-1)/u}|I|^{1+(p-1)/u}. \tag{3.18}$$

Since C depends only on p and K, this shows that $U(x)$ satisfies condition A_r on J with constant, L, where $r=1+(p-1)/u$ and $L=KC^{(p-1)/u}$. It is immediate that r and L depend only on K and p and that $1<r<p$.

LEMMA 6. *If* $1\leqq p<\infty$ *and* $U(x)$ *satisfies condition* A_p *with constant,* K, *on an interval,* J, *then there are constants,* r *and* L, *depending only on* p *and* K, *such that* $r>1$ *and* $[U(x)]^r$ *satisfies condition* A_p *on* J *with constant,* L.

Let I be a finite subinterval of J. If $p>1$, the first paragraph of the proof of Lemma 5 shows that (3.17) is true with C and u depending only on K and p and $u>1$. The same reasoning applied to $U(x)$ instead of $[U(x)]^{-1/(p-1)}$ shows that there are constants, v and C, depending only on p and K such that $v>1$ and

$$\int_I [U(x)]^v\,dx \leqq C|I|^{1-v}\left[\int_I U(x)\,dx\right]^v. \tag{3.19}$$

Using (3.17), (3.19) and the hypothesis then shows that there is a constant, C, depending only on p and K, such that

$$\left[\frac{1}{|I|}\int_I [U(x)]^v\,dx\right]^{1/v}\left[\frac{1}{|I|}\int_I [U(x)]^{-u/(p-1)}\,dx\right]^{(p-1)/u} \leqq C. \tag{3.20}$$

Now let $r=\min(u, v)$. Hölder's inequality and (3.20) then show that

$$\left[\frac{1}{|I|}\int_I [U(x)]^r\,dx\right]^{1/r}\left[\frac{1}{|I|}\int_I [U(x)]^{-r/(p-1)}\,dx\right]^{(p-1)/r} \leqq C. \tag{3.21}$$

Taking the rth power of (3.21) then completes the proof of Lemma 6 for $p>1$.

If $p=1$, the same reasoning still proves (3.19). Since by hypothesis

$$\frac{1}{|I|}\int_I U(x)\,dx \leqq K \operatorname*{ess\,inf}_{x\in I} U(x),$$

(3.19) shows that

$$\frac{1}{|I|}\int_I [U(x)]^v\,dx \leqq CK^v \operatorname*{ess\,inf}_{x\in I} [U(x)]^v$$

as desired.

4. **Strong type results.** The solution of the main problem of this paper is given here as Theorem 2. Some other corollaries of the preceding section are given and the cases $p=1$ and $p=\infty$ are considered.

THEOREM 2. *If $U(x)$ is nonnegative on an interval, J, and $1<p<\infty$, then there is a constant, C, independent of f, for which* (1.1) *holds if and only if there is a constant, K, independent of I for which* (1.3) *holds for all subintervals, I, of J. If C and K are the least constants for which* (1.1) *and* (1.3) *are true, then $K\leqq C\leqq g(K,p)$ where g is a function depending only on the indicated arguments.*

If there is a C for which (1.1) holds, it is immediate that (2.1) is true for all $a>0$ with $B=C$ and $V(x)=U(x)$. Theorem 1 then proves that (1.3) holds with $K=C$.

If (1.3) holds for a given K, then by Lemma 5 there are constants, r and L, depending only on K and p, such that $1<r<p$ and

$$\left[\int_I U(x)\,dx\right]\left[\int_I [U(x)]^{-1/(r-1)}\,dx\right]^{r-1} \leqq L|I|^r. \tag{4.1}$$

By Hölder's inequality it is also true that

$$\left[\int_I U(x)\,dx\right]\left[\int_I [U(x)]^{-1/(2p-1)}\,dx\right]^{2p-1} \leqq K|I|^{2p}. \tag{4.2}$$

Applying Theorem 1 to (4.1) and (4.2) shows that for all functions, f,

$$\int_{E_a} U(x)\,dx \leqq 2^r La^{-r}\int_J |f(x)|^r U(x)\,dx \tag{4.3}$$

and

$$\int_{E_a} U(x)\,dx \leqq 2^{2p} Ka^{-2p}\int_J |f(x)|^{2p} U(x)\,dx \tag{4.4}$$

where E_a is the subset of J where $f^*(x)>a$. The Marcinkiewicz interpolation theorem, p. 112, Vol. II of [12], then proves (1.1) with constant, C, depending only on K and p.

COROLLARY 1. *If $U(x)$ is nonnegative on an interval, J, and $1<p<\infty$, then there is a C for which* (1.1) *is true if and only if there is a B for which* (1.2) *is true with $V(x)=U(x)$.*

This immediate consequence of Theorems 1 and 2 is interesting since it says that the weight functions for strong type and weak type are the same.

COROLLARY 2. *If $U(x)$ is nonnegative on an interval, J, $1<p<\infty$, and there is a C for which* (1.1) *is true, then the set of all a for which*

$$\int_J [f^*(x)]^p[U(x)]^a\,dx \leqq C\int_J |f(x)|^p[U(x)]^a\,dx \tag{4.5}$$

where C is independent of f (but not of a) is an open interval containing $[b, 1]$ *where* $b=\max[-(p-1), -1/(p-1)]$.

The fact that $U(x)$ satisfies (1.3) and Hölder's inequality imply that $[U(x)]^b$ also satisfies (1.3). If $[U(x)]^a$ satisfies (1.3), Hölder's inequality implies that (1.3) is true for $[U(x)]^d$ for any d between 0 and a. This shows that the set of all a for which (4.5) is true is an interval containing $[b, 1]$. The fact that it is an open interval follows immediately from Lemma 6.

COROLLARY 3. *Given a nonnegative $U(x)$ on an interval, J, the set of all p for which there is a C that makes* (1.1) *true is an interval of the form* (p_0, ∞).

Since (1.1) cannot be true for $p=1$ except for trivial cases as shown below in Theorem 5, Corollary 3 follows immediately from Theorem 1, Lemma 5 and Hölder's inequality.

COROLLARY 4. *Assume that $1<p<\infty$, $U(x)$ is nonnegative and has period b and let $Mf(x)=\sup_{y\neq x}(1/(y-x))\int_x^y |f(t)|\,dt$. Then there is a constant, C, independent of f such that*

$$\int_0^b [Mf(x)]^pU(x)\,dx \leqq C\int_0^b |f(x)|^pU(x)\,dx \tag{4.6}$$

for every f of period b if and only if there is a constant, K, independent of I for which (1.3) *holds for all intervals, I, satisfying $|I|\leqq b$.*

If (1.3) is true for all I satisfying $|I|\leqq b$, there is a constant for which it is true for all $I\subset[-b, 2b]$. Given f with period b, let $g(x)=f(x)$ on $[-b, 2b]$ and 0 elsewhere and let $g^*(x)$ be the maximal function of g on $[-b, 2b]$. By Theorem 2 there is a C such that

$$\int_{-b}^{2b} [g^*(x)]^pU(x)\,dx \leqq C\int_{-b}^{2b} |g(x)|^pU(x)\,dx.$$

Since $g^*(x)=Mf(x)$ on $[0, b]$, this implies (4.6).

If (4.6) is true, let I be an interval with $|I| \leqq b$. Given f on I, extend its definition so that it has period b and is 0 where not otherwise defined, and let f^* be the maximal function of f on I. Then $Mf(x) \geqq f^*(x)$ on I and by (4.6)

$$\int_I |f^*(x)|^p U(x)\, dx \leqq C \int_I |f(x)|^p U(x)\, dx.$$

Theorem 2 then implies that (1.3) is true for this I with $K = C$. This completes the proof of Corollary 4.

Weighted norm inequalities for $p = \infty$ are much simpler and will be done with two weight functions. The result depends upon whether the version of (1.1) to be considered is

$$\left[\int_J |f^*(x)|^p U(x)\, dx\right]^{1/p} \leqq C\left[\int_J |f(x)|^p V(x)\, dx\right]^{1/p} \tag{4.7}$$

or

$$\left[\int_J |f^*(x)U(x)|^p\, dx\right]^{1/p} \leqq C\left[\int_J |f(x)V(x)|^p\, dx\right]^{1/p}. \tag{4.8}$$

For $p < \infty$, these are equivalent problems, and it was more convenient to use the form (4.7). For $p = \infty$, however, they give the following rather different results.

THEOREM 3. *Let $U(x)$ and $V(x)$ be nonnegative functions on an interval, J. Then there is a constant, C, independent of f, for which* (4.7) *holds for $p = \infty$ if and only if $V(x) > 0$ for almost every x in J or $U(x) = 0$ for almost every x in J.*

THEOREM 4. *Let $U(x)$ and $V(x)$ be nonnegative functions on an interval, J. Then there is a constant, C, independent of f, for which* (4.8) *holds for $p = \infty$ if and only if $[1/V(x)]^* \leqq C/U(x)$ for almost every x in J.*

Theorem 3 is trivial; if $V(x)$ were 0 on a set of positive measure, E, let $f(x)$ be the characteristic function of E. This produces an immediate contradiction unless $U(x) = 0$ almost everywhere. The opposite implication follows immediately from the fact that $\|f^*(x)\|_\infty \leqq \|f(x)\|_\infty$.

Theorem 4 is almost as simple. For a given value, A, of $\|f(x)V(x)\|_\infty$ the left side of (4.8) is maximized if $f(x) = A/V(x)$. Therefore, if $[1/V(x)]^* \leqq C/U(x)$, (4.8) holds. The converse is immediate by taking $f(x) = 1/V(x)$.

For $p = 1$ the following settles the problem.

THEOREM 5. *Let $U(x)$ be a nonnegative function on an interval J. Then there is a C for which* (1.1) *holds with $p = 1$ if and only if $U(x) = 0$ almost everywhere on J or $U(x) = \infty$ almost everywhere on J.*

If $U(x)$ satisfies either condition, then (1.1) is certainly true. Conversely, if (1.1) is true, suppose that $0 < U(x) < \infty$ on a set of positive measure. Then there is a positive number, A, such that the set where $A < U(x) \leqq 2A$ has positive measure;

call this set E. Let b be a point of density of E and not the right end of J and let $f(x)$ equal $|x-b|^{-1}|\log|x-b||^{-2}$ on E and 0 elsewhere. Then the right side of (1.1) is finite. Since b is a point of density of E, for all sufficiently large n, $[a+2^{-n}, a+2^{-n+1}] \cap E$ has measure greater than 2^{-n-1}. From this it follows that, for $x>b$ and near b, $f^*(x) \geqq \frac{1}{4}|x-b|^{-1}|\log|x-b||^{-1}$, and that the left side of (1.1) is infinite.

5. **The two weight function problem.** For $1 \leqq p < \infty$, the problem of determining all pairs of nonnegative functions, $U(x)$ and $V(x)$, such that

$$\int_J [f^*(x)]^p U(x)\,dx \leqq C\int_J |f(x)|^p V(x)\,dx \tag{5.1}$$

is quite different from the case when $U(x)=V(x)$. Determining all such pairs is of interest because in mean convergence and certain mean summability problems for Hermite and Laguerre series many of the estimates involve Hardy maximal functions and the weight functions must be different on the two sides of the inequality.

Two general statements can be made about the problem. If (5.1) holds, then by Theorem 1 there is a K for which

$$\left[\int_I U(x)\,dx\right]\left[\int_I [V(x)]^{-1/(q-1)}\,dx\right]^{q-1} \leqq K|I|^q \tag{5.2}$$

for every subinterval, I, of J with $q=p$. The proof of Theorem 2 also shows that if (5.2) is true for some $q<p$, then (5.1) follows. Examples will now be given to show that the obvious generalizations of most of the results in §§3 and 4 do not apply to this problem.

First, observe that if $J=[0,\frac{1}{2}]$ and $q=1$, then $U(x)=x^{-1}(\log x)^{-2}$, $V(x)=(-x\log x)^{-1}$ satisfy (5.2); therefore, they also satisfy (5.1) for $p=2$ and (5.2) for $q=2$. It is immediate that $U(x)=-x\log x$, $V(x)=x(\log x)^2$ satisfy (5.2) for $q=2$. They do not, however, satisfy (5.1) for $p=2$ as the example $f(x)=x^{-1}(\log x)^{-2}$ shows. Therefore, (5.2) for $q=p$ does not imply (5.1) and the analogue of Theorem 2 is false. This pair is also an example of the fact that the analogue of Lemma 5 is not true.

The analogue of Lemma 6 is not true for either U or V in this problem even with the stronger hypothesis that they satisfy (5.1). The first example in which $U(x)=x^{-1}(\log x)^{-2}$ is an example of this; no power of $U(x)$ greater than the first can satisfy (5.2) for $I=J=[0,\frac{1}{2}]$ unless $V(x)=\infty$ almost everywhere. To obtain a similar example for $V(x)$ it will be shown that

$$\int_0^{1/2} \frac{[f^*(x)]^2 x\,dx}{(\log x)^2} \leqq C\int_0^{1/2} |f(x)|^2 x(\log x)^2\,dx. \tag{5.3}$$

This follows by observing that

$$f^*(x) \leqq \sup_{|y-x|\leqq x/2;\, y\in[0,1/2]} \frac{1}{y-x}\int_x^y |f(t)|\,dt + \frac{2}{x}\int_0^{1/2} |f(t)|\,dt.$$

For the first term (5.3) follows by writing the integral on the left as the sum of integrals over $[2^{-n-1}, 2^{-n}]$, $n=1, 2, \ldots$, and using the unweighted Hardy maximal function inequality. For the second term (5.3) follows by use of Hölder's inequality.

No power of $x(\log x)^2$ can be the V in (5.2) with $q=2$ and $I=[0, \frac{1}{2}]$ unless $U(x)=0$ almost everywhere on $[0, \frac{1}{2}]$. This shows that Lemma 6 does not apply to $V(x)$ either.

The two weight function version of Corollary 1 is also false since, as shown above, (5.2) with $q=p$ is equivalent to (2.1) but is not equivalent to (5.1). Corollary 3 is also false; the functions in (5.3) satisfy (5.1) for $p \geqq 2$ but not for $p<2$.

Theorem 5 also fails since there are nontrivial pairs of functions, $U(x)$, $V(x)$, for for which (5.1) holds for $p=1$. A simple example would be to take $J=[-1, 1]$, $U(x)=x$ on $[0, 1]$ and 0 elsewhere and $V(x)=1$ on $[-1, 0]$ and ∞ elsewhere. The proof of Theorem 5 shows only that $U(x)[V(x)]^{-1}=0$ for almost every x.

6. **A weighted maximal function.** Given a measure, m, on an interval, J, define

$$f_m^*(x) = \sup_{y \in J} \frac{\int_x^y |f(t)|\, dm(t)}{\int_x^y dm(t)} \tag{6.1}$$

where the quotient is to be taken as 0 if the numerator and denominator are both 0 or both ∞. The following generalizations of Theorems 1 and 2 are true for this maximal function; they are also a generalization of Lemma 1, p. 232 of [5]. An application of Theorem 7 is given in §8.

THEOREM 6. *Let m be a Borel measure on an interval, J, which is* 0 *on sets consisting of single points. Let $U(x)$ and $V(x)$ be nonnegative functions on J, assume that $1 \leqq p < \infty$ and $0 < a < \infty$, and given $f(x)$ on J let E_a be the subset of J where $f_m^*(x) > a$. Then there is a constant, B, independent of f and a such that*

$$\int_{E_a} U(x)\, dm(x) \leqq Ba^{-p} \int_J |f(x)|^p V(x)\, dm(x) \tag{6.2}$$

if and only if there is a constant, K, such that for every subinterval, I, of J

$$\left[\int_I U(x)\, dm(x)\right]\left[\int_I [V(x)]^{-1/(p-1)}\, dm(x)\right]^{p-1} \leqq K[m(I)]^p. \tag{6.3}$$

If B and K are the least constants for which (6.2) *and* (6.3) *are true, then $K \leqq B \leqq 2^p K$.*

THEOREM 7. *Let m be a Borel measure on an interval, J, which is* 0 *on sets consisting of single points. Let $U(x)$ be a nonnegative function on J and assume that $1 < p < \infty$. Then there is a constant, C, independent of f, for which*

$$\int_J [f_m^*(x)]^p U(x)\, dm(x) \leqq C \int_J |f(x)|^p U(x)\, dm(x) \tag{6.4}$$

if and only if there is a constant, K, such that for every subinterval, I, of J

$$\left[\int_I U(x)\, dm(x)\right]\left[\int_I [U(x)]^{-1/(p-1)}\, dm(x)\right]^{p-1} \leqq K[m(I)]^p. \tag{6.5}$$

If C and K are the least constants for which (6.4) *and* (6.5) *are true, then* $K \leqq C \leqq g(K, p)$ *where g is a function depending only on the indicated arguments.*

With more assumptions on m these could be proved easily from Theorems 1 and 2 by a change of variables. In this form they are proved by repeating the proofs of Theorems 1 and 2 with $|E|$ replaced by $m(E)$ and dx replaced by $dm(x)$ throughout. In Lemma 3, $h(s)$ should be defined as $\sup \inf_{x \in E} g(x)$ where the sup is taken over all sets, E, such that $E \subset J$ and $m(E)=s$. The dt used with rearranged functions in Lemmas 3–5 should, of course, not be replaced by $dm(t)$.

7. **Higher dimensional case.** Given a real valued function on R^n the maximal function to be considered is

$$f^*(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(t)|\, dt \tag{7.1}$$

where the sup is taken over all cubes, Q, with center x and $|Q|$ denotes the measure of Q. Analogues of Theorems 1 and 2 will be proved in this section; the proofs are almost the same. The following modification of the Calderón-Zygmund lemma, Theorem 4, p. 17 of [9], will be needed.

Lemma 7. *Let $f(x)$ be a nonnegative integrable function on R^n and let a be a positive constant. Then there is a sequence of sets, S_k, and a sequence of cubes Q_k such that*

1. *The intersection of different S_k's has measure* 0.
2. $Q_k \subset S_k \subset 3Q_k$ *where $3Q_k$ is the cube with the same center as Q_k and with sides parallel to Q_k's and three times as long.*
3. $4^{-n}a|Q_k| \leqq \int_{S_k} f(t)\, dt \leqq a|Q_k|$.
4. *If $f^*(x) > a$, x is in $\bigcup S_k$.*

By the Calderón-Zygmund lemma there is a sequence of cubes, Q_k, whose interiors are disjoint,

$$4^{-n}a|Q_k| < \int_{Q_k} f(t)\, dt \leqq 2^{-n}a\, Q_k, \tag{7.2}$$

and $f(x) \leqq 4^{-n}a$ for almost every x not in $\bigcup Q_k$. Furthermore, it is shown in the proof that the Q_k's can be chosen so that all are members of a collection, C, of cubes consisting of a fixed mesh of equal cubes with disjoint interiors that fill R^n and cubes obtained from these by successive divisions of these cubes into 2^n equal smaller cubes by bisecting the sides. It also appears in the proof that if Q is in C and is not a subset of a Q_k, then

$$\int_Q |f(t)|\, dt \leqq 4^{-n}a|Q|. \tag{7.3}$$

Now define the S_k's successively to be the set of all x in $3Q_k$ that are not in the interior of any Q_j for $j \neq k$ nor in the interior of any S_j for $j < k$. Then statements 1 and 2 are immediate.

The first part of 3 follows from (7.2) and the fact that $Q_k \subset S_k$. For the second part observe that

$$\int_{S_k} f(t)\,dt = \int_{Q_k} f(t)\,dt + \int_{S_k - Q_k} f(t)\,dt. \tag{7.4}$$

Now $S_k - Q_k$ has at most a set of measure 0 in common with $\bigcup Q_k$ so $f(x) \leqq 4^{-n}a$ for almost every x in $S_k - Q_k$ and

$$\int_{S_k - Q_k} f(t)\,dt \leqq |S_k - Q_k| 4^{-n}a \leqq (3^n - 1)|Q_k| 4^{-n}a.$$

Using this and the second part of (7.2) on (7.4) then proves the second part of 3 since $2^{-n} + (3^n - 1)4^{-n} \leqq 1$.

To prove 4 let x be a point such that $f^*(x) > a$. Then there is a cube, P, with center at x such that $\int_P f(t)\,dt > a|P|$. There is a cube, Q, in the collection, C, with $|P| \leqq |Q| < 2^n|P|$, such that

$$\int_{Q \cap P} f(t)\,dt \geqq 2^{-n}a|P| \tag{7.5}$$

since no more than 2^n of the cubes in C of this size can intersect P in a set of positive measure. Then from (7.5) and the inequality on the size of $|Q|$ it follows that $\int_Q f(t)\,dt > 4^{-n}a|Q|$. Since Q does not satisfy (7.3), it must be a subset of some Q_k. Then $P \subset 3Q_k$ and x is in $3Q_k$. By the definition of the S_j's, x is in some S_j, and, therefore, in $\bigcup S_k$.

THEOREM 8. *Let J be a subset of R^n, $1 < p < \infty$, $0 < a < \infty$, $U(x)$ and $V(x)$ be non-negative functions on J and given $f(x)$ let E_a be the subset of J where $f^*(x) > a$. Then there is a constant, B, independent of f and a such that*

$$\int_{E_a} U(x)\,dx \leqq Ba^{-p} \int_J |f(x)|^p V(x)\,dx \tag{7.6}$$

where f is any real valued function vanishing outside J if and only if there is a constant, K, independent of Q, such that

$$\left[\int_{Q \cap J} U(x)\,dx\right]\left[\int_{Q \cap J} [V(x)]^{-1/(p-1)}\,dx\right]^{p-1} \leqq K|Q|^p \tag{7.7}$$

for any cube, Q. Furthermore, if B and K are the minimum constants for which (7.6) *and* (7.7) *are true, then $K \leqq B \leqq (12)^{np}K$.*

First assume that (7.7) is true. Given a, apply Lemma 7 to $|f(x)|$ to obtain sequences S_k and Q_k with the stated properties. Then

$$\int_{E_a} U(x)\,dx \leqq \sum_k \int_{S_k \cap J} U(x)\,dx. \tag{7.8}$$

Part 3 of Lemma 7 shows that the right side of (7.8) is bounded by

$$\sum_k \left[\frac{4^n}{a|Q_k|}\int_{S_k\cap J}|f(t)|\,dt\right]^p \int_{S_k\cap J} U(x)\,dx.$$

The proof then proceeds as the proof of Theorem 1 does after (2.5); the sets on which $U(x)$ and $[V(x)]^{-1/(p-1)}$ are integrated in the analogue of (2.6) must be changed from $S_k \cap J$ to $3Q_k \cap J$ in order to apply (7.7). This shows that $B \leqq (12)^{np}K$. The opposite implication and the fact that $K \leqq B$ follow from the same reasoning as used to prove that part of Theorem 1.

THEOREM 9. *Let J be a cube in R^n or all of R^n, $1<p<\infty$, and let $U(x)$ be a non-negative function on J. Then there is a constant, C, independent of $f(x)$ such that*

$$\int_J [f^*(x)]^p U(x)\,dx \leqq C\int_J |f(x)|^p U(x)\,dx \tag{7.9}$$

where f is any real valued function that vanishes outside J, if and only if there is a constant, K, independent of Q such that

$$\left[\int_Q U(x)\,dx\right]\left[\int_Q [U(x)]^{-1/(p-1)}\,dx\right]^{p-1} \leqq K|Q|^p \tag{7.10}$$

for any cube, Q, satisfying $Q\subset J$. If C and K are the least constants for which (7.9) *and* (7.10) *are true, then $K\leqq C\leqq g(K,p)$ where g is a function depending only on the indicated arguments.*

The set, J, could be somewhat more general in Theorem 9 and some such results follow easily as corollaries of this theorem. Examples would be rectangular solids in which the ratio of the longest side to the shortest is bounded, a half space, the set where certain components are positive, etc.

The proof of Theorem 9 involves showing that analogues of Lemmas 1–5 are true for n dimensional space with different constants. An outline of the principal differences from the one dimensional case are given below.

The obvious version of Lemma 1 with "interval" replaced by "cube" throughout is false even if the constant in the conclusion is changed. An additional hypothesis is needed; it is that there is a constant, D, such that

$$\int_{5H\cap J} g(t)\,dt \leqq D\int_H g(t)\,dt \tag{7.11}$$

for every subcube, H, of J. The proof then uses the lemma on p. 9 of [9] with spheres replaced by cubes. The proof of that lemma shows that there is a sequence of disjoint cubes, H_i, such that $H_i \in Q$ and $R\subset 5H_i \cap J$. Then

$$\int_R g \leqq \sum\int_{5H_i\cap J} g \leqq D\sum\int_{H_i} g \leqq Da\sum|H_i| \leqq Da|R|.$$

This completes the proof of the n dimensional Lemma 1 with the 2 in the conclusion replaced by D.

The obvious analogue of Lemma 2 is true if the $\frac{1}{2}$ in the conclusion is replaced by $(\frac{2}{3})5^{-n}$. This is proved by using the lemma on p. 9 of [9] for cubes to choose a disjoint sequence, H_i, of cubes in Q such that $|B| \leqq 5^n \sum |H_i|$. The proof is completed by observing that $\sum |H_i| \leqq (3/2) \sum |H_i \cap B| \leqq (3/2)|R \cap B|$.

The obvious n dimensional version of Lemma 3 is also true; the constant in the conclusion is different but depends only on K and q. The proof is like the one dimensional version with different constants; the only essential difference is that (7.11) must be proved in order to apply the n dimensional version of Lemma 1. To do this let H be a subcube of J and let G be the smallest subcube of J that contains $5H \cap J$. Then applying (3.1) to G shows that

$$\int_{5H \cap J} g(x)\, dx \leqq K|G|^p \left[\int_G [g(x)]^{-1/(p-1)}\, dx\right]^{1-p}. \tag{7.12}$$

Since $G \supset H$, the right side of (7.12) is bounded by

$$K|G|^p \left[\int_H [g(x)]^{-1/(p-1)}\, dx\right]^{1-p}.$$

Hölder's inequality shows that this is bounded by $K|G|^p|H|^{-p} \int_H U(x)\, dx$. Since $|G|/|H| \leqq 5^n$, this proves (7.11) with $D = 5^{np}K$.

Lemma 4 is used unchanged. The n dimensional version of Lemma 5 follows in the same way as the one dimensional version. The proof of Theorem 9 is then exactly like the proof of Theorem 2.

Lemma 6 and the other theorems and corollaries in §§4 and 6 can also be carried over easily to the n dimensional case.

8. **Application to mean summability theorems.** To illustrate some uses of Theorems 2 and 7 the following will be proved. For further information on the definitions used in Theorems 10 and 11 see [12] and for the definitions in Theorem 12 see [8].

THEOREM 10. *Assume that* $1 \leqq p < \infty$, $f(\theta)$ *is integrable on* $[0, 2\pi]$, $U(\theta) \geqq 0$, f *and* U *have period* 2π *and* $f(r, \theta)$ *is the Poisson integral of* $f(\theta)$. *Then the following are equivalent*:

(i) $\lim_{r \to 1^-} \int_0^{2\pi} |f(r, \theta) - f(\theta)|^p U(\theta)\, d\theta = 0$ *for every function* f, *satisfying* $\int_0^{2\pi} |f(\theta)|^p U(\theta)\, d\theta < \infty$.

(ii) *There is a constant*, C, *independent of* f *such that* $\int_0^{2\pi} |f(r, \theta)|^p U(\theta)\, d\theta \leqq C \int_0^{2\pi} |f(\theta)|^p U(\theta)\, d\theta$.

(iii) *For every interval*, I, *with* $|I| \leqq 2\pi$,

$$\left[\int_I U(\theta)\, d\theta\right]\left[\int_I [U(x)]^{-1/(p-1)}\, dx\right]^{p-1} \leqq K|I|^p$$

where K *is independent of* I.

THEOREM 11. *Theorem* 10 *remains true if* $f(r,\theta)$ *is replaced by* $\sigma_n(f,\theta)$, *the sequence of the first arithmetic means of* f*'s Fourier series, and* $\lim_{r\to 1^-}$ *is replaced by* $\lim_{n\to\infty}$.

THEOREM 12. *Assume that* $1\leqq p<\infty$, $\lambda>0$, $f(\theta)\sin^{2\lambda}\theta$ *is integrable on* $[0,\pi]$, $\sum a_n P_n^\lambda(\cos\theta)$ *is the Gegenbauer expansion of* $f(\theta)$, *and for* $0\leqq r<1$, $f(r,\theta)=\sum_{n=0}^\infty a_n r^n P_n^\lambda(\cos\theta)$. *Then the following are equivalent*:

(i) $\lim_{r\to 1^-}\int_0^\pi |f(r,\theta)-f(\theta)|^p U(\theta)\,d\theta=0$ *for every function*, f, *satisfying* $\int_0^\pi |f(\theta)|^p U(\theta)\,d\theta<\infty$.

(ii) *There is a constant*, C, *independent of* f *such that* $\int_0^\pi |f(r,\theta)|^p U(\theta)\,d\theta \leqq C\int_0^\pi |f(\theta)|^p U(\theta)\,d\theta$.

(iii) *For every subinterval*, I, *of* $[0,\pi]$,

$$\left[\int_I U(\theta)\,d\theta\right]\left[\int_I [U(\theta)]^{-1/(p-1)}[\sin\theta]^{2\lambda p'}\,d\theta\right]^{p-1} \leqq K\left[\int_I \sin^{2\lambda}\theta\,d\theta\right]^p$$

where K *is independent of* I.

Most of the proof of Theorem 10 is standard. It will be proved by showing that (i) implies (ii), (ii) and (iii) imply (i), (iii) implies (ii) and, finally, that (ii) implies (iii).

Part (i) implies part (ii) by the Banach-Steinhaus theorem. Part (iii) implies that either $U(x)=\infty$ almost everywhere in which case (i) is immediate or that $\int_0^{2\pi} U(x)\,dx<\infty$. In this latter case it follows that given $\varepsilon>0$ and f satisfying $\int_0^{2\pi}|f(\theta)|^p U(\theta)\,d\theta<\infty$, there is a function, $g(\theta)$, of the form

$$\sum_{n=0}^{k}[c_n\cos n\theta+d_n\sin n\theta]$$

such that $\int_0^{2\pi}|f(\theta)-g(\theta)|^p U(\theta)\,d\theta<\varepsilon$. Then writing $f(r,\theta)-f(\theta)$ as the sum of $f(r,\theta)-g(r,\theta)$, $g(r,\theta)-g(\theta)$ and $g(\theta)-f(\theta)$, integrating and using Minkowski's inequality and (ii) shows that

$$\limsup_{r\to 1}\int_0^{2\pi}|f(r,\theta)-f(\theta)|^p U(\theta)\,d\theta \leqq C\varepsilon.$$

This, of course, implies the limit statement in (i).

To prove that (iii) implies (ii) for $p>1$, use the well-known fact that $|f(r,\theta)| \leqq f^*(\theta)$; this follows, for example, immediately from Theorem 1, p. 232 of [5]. An application of Corollary 4 completes the proof of (ii). If $p=1$ use Fubini's theorem on the left side of the inequality in (ii) to show that it is bounded by

$$\int_0^{2\pi}|f(\phi)|\left(\frac{1}{\pi}\int_0^{2\pi}P(r,\theta,\phi)U(\theta)\,d\theta\right)d\phi \tag{8.1}$$

where

$$P(r,\theta,\phi) = \tfrac{1}{2}(1-r^2)/(1-2r\cos(\theta-\phi)+r^2) \tag{8.2}$$

is the Poisson kernel. Now the inner expression in (8.1) is just $U(r, \phi)$ and, therefore, is bounded by $U^*(\phi)$. Condition (iii) for $p=1$ is just the statement that $U^*(\phi) \leqq KU(\phi)$ almost everywhere. This shows that (8.1) is bounded by

$$K \int_0^{2\pi} |f(\phi)| U(\phi)\, d\phi$$

as desired.

This leaves the proof that (ii) implies (iii). To do this fix an interval, I, with $|I| \leqq 2\pi$ and let $r = \max(0, 1-|I|)$. Then if θ and ϕ are both in I, it is apparent from (8.2) that

$$P(r, \theta, \phi) \geqq (4|I|)^{-1}. \tag{8.3}$$

Now let $f(\theta)$ be any nonnegative function with period 2π that vanishes outside the set consisting of I and the images of I under translation by an integral multiple of 2π. Then if θ is in I

$$|f(r, \theta)| \geqq (13|I|)^{-1} \int_I |f(\theta)|\, d\theta. \tag{8.4}$$

Using (8.4) in condition (ii) and restricting the integration on the left to I shows that

$$\left[\int_I U(\theta)\, d\theta\right] \left[(13|I|)^{-1} \int_I f(\theta)\, d\theta\right]^p \leqq C \int_I |f(\theta)|^p U(\theta)\, d\theta. \tag{8.5}$$

Now the proof in Theorem 1 that (2.1) implies (2.2) can be repeated since all that was used in that proof was (8.5).

The proof of Theorem 11 is practically the same. To obtain the analogue of (8.3), n should be chosen as the larger of 0 and the greatest integer less than $-1+\pi/|I|$.

The proof of Theorem 12 also follows the same scheme as the proof of Theorem 10. The first two parts are the same. To show that (iii) implies (ii) the fact that $|f(r, \theta)| \leqq Cf_m^*(\theta)$ is needed where $dm = \sin^{2\lambda}\theta$ and f_m^* is the function defined in (6.1) with $J=[0, \pi]$. This inequality is proved in Lemma 3, p. 28 of [8]. Theorem 7 then completes the proof that (iii) implies (ii).

The only complications in the proof of Theorem 12 arise in showing that (ii) implies (iii). To do this fix an interval, $I=[a, b]$, and let $r = 1-(b-a)/6$. As shown in [11] the Poisson kernel in this case is

$$P(r, \theta, \phi) = \frac{\lambda}{\pi}(1-r^2) \int_0^\pi \frac{(\sin t)^{2\lambda-1}\, dt}{[1-2r(\cos\theta\cos\phi + \sin\theta\sin\phi\cos t) + r^2]^{\lambda+1}}.$$

It will be shown that there is a positive constant, C, independent of I, θ and ϕ such that if θ and ϕ are in I

$$P(r, \theta, \phi) \geqq C\left[\int_I \sin^{2\lambda} t\, dt\right]^{-1}. \tag{8.6}$$

To prove (8.6) assume first that $I \subset [0, 3\pi/4]$. Now the denominator of the integral in the definition of P can be written as

$$[(1-r)^2 + 2r(1-\cos\theta-\phi) + 2r\sin\theta\sin\phi\,(1-\cos t)]^{\lambda+1}.$$

If $a \geqq b/2$ and θ and ϕ are in I, $\frac{1}{2}\theta \leqq \phi \leqq 2\theta$ and

$$P(r, \theta, \phi) \geqq C(1-r)\int_{(1-r)/\theta}^{1} \frac{t^{2\lambda-1}\,dt}{(\theta^2 t^2)^{\lambda+1}} \geqq \frac{C\theta^{-2\lambda}}{1-r}$$

where the C's are positive and independent of I, θ and ϕ. If $a < b/2$ and θ and ϕ are in I, then

$$P(r, \theta, \phi) \geqq C(1-r)\int_0^1 \frac{t^{2\lambda-1}\,dt}{(1-r)^{2\lambda+2}} \geqq C(1-r)^{-2\lambda-1}$$

where the C's are positive and independent of I, θ and ϕ. In both cases (8.6) follows immediately.

By symmetry (8.6) is true if $I \subset [\frac{1}{4}\pi, \pi]$ and θ and ϕ are in I. If I intersects both $[0, \frac{1}{4}\pi)$ and $(3\pi/4, \pi]$ and θ and ϕ are in I, then $|I| \geqq \frac{1}{2}\pi$, $r \leqq 1-\pi/12$, $P(r, \theta, \phi)$ has a positive lower bound independent of I, θ and ϕ and (8.6) is immediate. Therefore, (8.6) is true if θ and ϕ are in I for any $I \subset [0, \pi]$. Then (8.6) can be used to show that (ii) implies (iii) in the same way that (8.3) was used to prove the same part of Theorem 10. This completes the proof of Theorem 12.

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