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A CHARACTERIZATION OF COMPACT MULTIPLIERS

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Abstract. Let G be a compact abelian group and φ a complex-valued function defined on the dual Γ. The main result of this paper is that φ is a compact multiplier of type (p, q), $1 \leqq p < \infty$ and $1 \leqq q \leqq \infty$, if and only if it satisfies the following condition: Given $\varepsilon > 0$ there corresponds a finite set $K \subset \Gamma$ such that $|\sum a_\gamma b_\gamma \varphi(\gamma)| < \varepsilon$ whenever $P = \sum a_\gamma \gamma$ and $Q = \sum b_\gamma \gamma$ are trigonometric polynomials satisfying $\|P\|_p \leqq 1$, $\|Q\|_{q'} \leqq 1$ (q' the conjugate index of q) and $b_\gamma = 0$ for $\gamma \in K$. Using the above characterization we obtain the following necessary and sufficient condition for φ to be the Fourier transform of a continuous complex-valued function on G: Given $\varepsilon > 0$ there corresponds a finite set $K \subset \Gamma$ such that $|\sum b_\gamma \varphi(\gamma)| < \varepsilon$ whenever $Q = \sum b_\gamma \gamma$ is a trigonometric polynomial satisfying $\|Q\|_1 \leqq 1$ and $b_\gamma = 0$ for $\gamma \in K$.

Throughout the paper G is a compact abelian group, φ a complex-valued function defined on the dual Γ and $L^p(G)$ $(1 \leqq p \leqq \infty)$ the usual Lebesgue space of index p formed with respect to Haar measure on G. Let $M(G)$ denote the convolution algebra of complex-valued regular measures which are bounded on G, and $C(G)$ the class of all continuous complex-valued functions defined on G.

The Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)\, dx \qquad (\gamma \in \Gamma)$$

and the Fourier-Stieltjes transform $\hat{\mu}$ of a measure $\mu \in M(G)$ by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma)\, d\mu(x) \qquad (\gamma \in \Gamma).$$

The function φ is said to be a multiplier of type (p, q) if given $f \in L^p(G)$ there corresponds a $g \in L^q(G)$ such that $\varphi\hat{f} = \hat{g}$.

Now any (p, q) multiplier induces a bounded linear operator from $L^p(G)$ into $L^q(G)$, T_φ, where $(T_\varphi f)^\wedge = \varphi\hat{f}$ and T_φ commutes with translation. Conversely for $p < \infty$, there corresponds to any such bounded linear operator T mapping $L^p(G)$ into $L^q(G)$, a unique multiplier φ of type (p, q) such that $T = T_\varphi$ (see [5, pp. 249–250]). We say that φ is a compact multiplier if T_φ is a compact operator. Let $M_p^q(\Gamma)$ denote the set of all multipliers of type (p, q) and $m_p^q(\Gamma)$ the set of all $\varphi \in M_p^q(\Gamma)$ which are compact. Then $M_p^q(\Gamma)$ is a Banach space where the norm $\|\cdot\|_{(p,q)}$ of the multiplier φ is defined to be the norm of the multiplier operator T_φ. Let $\mathfrak{J}(G)$

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denote the set of all trigonometric polynomials on G and $\mathfrak{J}(G)^\wedge$ the set of all functions on Γ which are Fourier transforms of functions in $\mathfrak{J}(G)$. The following lemma is important in the sequel.

LEMMA. *Let $1 \leqq p < \infty$ and $1 \leqq q \leqq \infty$. Then the closure of $\mathfrak{J}(G)^\wedge$ in $M_p^q(\Gamma)$ is precisely $m_p^q(\Gamma)$.*

Proof. If $P \in \mathfrak{J}(G)$ then the convolution product $P * L^p(G)$ is finite dimensional. Therefore convolution by P is an operator of finite rank. Thus the closure of $\mathfrak{J}(G)^\wedge$ is contained in $m_p^q(\Gamma)$.

On the other hand, if $\varphi \in m_p^q(\Gamma)$, let $\varphi_\alpha = \varphi \hat{e}_\alpha$, where e_α is a bounded approximate identity in $L^1(G)$ consisting of trigonometric polynomials. Then $\varphi_\alpha \in \mathfrak{J}(G)^\wedge$ and $\|\varphi_\alpha - \varphi\|_{(p,q)} \to 0$. (See Gaudry [8] or Bachelis and Gilbert [1] for details.)

Our main result is the following characterization of $m_p^q(\Gamma)$:

THEOREM. *Let φ be a complex-valued function defined on Γ, $1 \leqq p < \infty$ and $1 \leqq q \leqq \infty$. The following statements are equivalent:*

(i) $\varphi \in m_p^q(\Gamma)$;

(ii) *Given $\varepsilon > 0$, there corresponds a finite subset $K \subset \Gamma$ such that $|\sum a_\gamma b_\gamma \varphi(\gamma)| < \varepsilon$ whenever $P = \sum a_\gamma \gamma$ and $Q = \sum b_\gamma \gamma$ are trigonometric polynomials satisfying $\|P\|_p \leqq 1$, $\|Q\|_{q'} \leqq 1$ (q' the conjugate index of q) and $b_\gamma = 0$ for $\gamma \in K$.*

If $p = 1$, then both (i) *and* (ii) *are equivalent to*

(iii) *Given $\varepsilon > 0$, there corresponds a finite subset $K \subset \Gamma$ such that $|\sum b_\gamma \varphi(\gamma)| < \varepsilon$ whenever $Q = \sum b_\gamma \gamma$ is a trigonometric polynomial with $\|Q\|_{q'} \leqq 1$ and $b_\gamma = 0$ for $\gamma \in K$.*

Proof. (i) $\Rightarrow$ (ii). Let $\varepsilon > 0$ be given. There corresponds by the preceding lemma a trigonometric polynomial L such that $\|L - T_\varphi\|_{(p,q)} < \varepsilon$.

Let K be the finite support of $\hat{L}$ and let P and Q be as in (ii). Then $Q * L = 0$, so

$$\begin{aligned} \left|\sum \hat{P}(\gamma)\hat{Q}(\gamma)\varphi(\gamma)\right| &= |T_\varphi(P) * Q(0)| = |T_\varphi(P) * Q(0) - L * P * Q(0)| \\ &\leqq \|T_\varphi(P) - L * P\|_q \|Q\|_{q'} \leqq \|T_\varphi - L\|_{(p,q)} \|P\|_p \|Q\|_{q'} \\ &< \varepsilon. \end{aligned}$$

(ii) $\Rightarrow$ (i). The function φ induces a linear mapping of $\mathfrak{J}(G)$ into $\mathfrak{J}(G)$ as follows:

$$T(P) = \sum \hat{P}(\gamma)\varphi(\gamma)\gamma \qquad (P \in \mathfrak{J}(G)).$$

Let $\varepsilon > 0$. We claim it is enough to show that there exists a trigonometric polynomial N such that

$$(*) \qquad \|T(P) - N * P\|_q < \varepsilon$$

for all trigonometric polynomials P such that $\|P\|_p \leqq 1$. Since $p < \infty$, this implies that T has a continuous extension $\tilde{T}$ to $L^p(G)$ which necessarily is compact and commutes with translation. Hence, $\tilde{T} = T_\psi$ for some ψ in $m_p^q(\Gamma)$. Since $\tilde{T}(\gamma)^\wedge(\gamma) = \varphi(\gamma)$ we may conclude $\psi = \varphi$.

So let K be as in (ii) corresponding to $\varepsilon/3$. Now choose a trigonometric polynomial R such that $\|R\|_1 \leqq 2$ and $\hat{R}|K=1$; see [10, p. 53]. Put $N=\sum \hat{R}(\gamma)\varphi(\gamma)\gamma$. To show (*) it suffices to prove that

$$(**) \qquad |(T(P)-N*P)*Q(0)| < \varepsilon$$

for all trigonometric polynomials Q such that $\|Q\|_{q'} \leqq 1$. Given such a Q let $Q_1=\frac{1}{3}(Q-Q*R)$. Then

$$\|Q_1\|_{q'} \leqq \tfrac{1}{3}(\|Q\|_{q'}+\|Q\|_{q'}\|R\|_1) \leqq 1 \quad \text{and} \quad \hat{Q}_1|K = 0.$$

Thus by the choice of K, $|\sum \hat{P}(\gamma)\hat{Q}_1(\gamma)\varphi(\gamma)| < \varepsilon/3$. But

$$\begin{aligned}\sum \hat{P}(\gamma)\hat{Q}_1(\gamma)\varphi(\gamma) &= \tfrac{1}{3}[\sum \hat{P}(\gamma)\hat{Q}(\gamma)\varphi(\gamma) - \sum \hat{P}(\gamma)\hat{Q}(\gamma)\hat{R}(\gamma)\varphi(\gamma)] \\ &= \tfrac{1}{3}[T(P)*Q(0)-N*P*Q(0)]\end{aligned}$$

which proves (**).

Suppose now that $p=1$. We will show that (ii) $\Leftrightarrow$ (iii). If (ii) holds and $\varepsilon>0$, let K be as given by (ii) corresponding to $\varepsilon/2$.

If Q is a trigonometric polynomial with $\|Q\|_{q'} \leqq 1$ and $\hat{Q}|K=0$, choose a trigonometric polynomial P such that $\|P\|_1 \leqq 3/2$ and $P*Q=Q$.

Then

$$\left|\sum \hat{Q}(\gamma)\varphi(\gamma)\right| = \left|\sum \hat{P}(\gamma)\hat{Q}(\gamma)\varphi(\gamma)\right| < (\varepsilon/2)\|P\|_1 < \varepsilon.$$

Therefore (ii) $\Rightarrow$ (iii).

Suppose now that (iii) holds. Given $\varepsilon>0$ let K be as given by (iii). If P and Q are trigonometric polynomials with $\|P\|_1 \leqq 1$, $\|Q\|_{q'} \leqq 1$, and $\hat{Q}|K=0$, then

$$\|P*Q\|_{q'} \leqq \|P\|_1\|Q\|_{q'} \leqq 1 \quad \text{and} \quad (P*Q)^\wedge|K = 0.$$

Thus

$$\left|\sum \hat{P}(\gamma)\hat{Q}(\gamma)\varphi(\gamma)\right| = \left|\sum (P*Q)^\wedge(\gamma)\varphi(\gamma)\right| < \varepsilon$$

and this concludes the proof.

Applying the above characterization in the special cases $m_1^1(\Gamma)$ and $m_1^\infty(\Gamma)$ we obtain the following corollary:

COROLLARY. *Let φ be a complex-valued function defined on Γ.*

(a) *The function $\varphi \in L^1(G)^\wedge$ if and only if it satisfies the following condition: Given $\varepsilon>0$ there corresponds a finite subset $K \subset \Gamma$ such that $|\sum b_\gamma\varphi(\gamma)| < \varepsilon$ whenever $Q=\sum b_\gamma\gamma$ is a trigonometric polynomial satisfying $\|Q\|_\infty \leqq 1$ and $b_\gamma=0$ for $\gamma \in K$.*

(b) *The function $\varphi \in C(G)^\wedge$ if and only if it satisfies the following condition: Given $\varepsilon>0$ there corresponds a finite subset $K \subset \Gamma$ such that $|\sum b_\gamma\varphi(\gamma)| < \varepsilon$ whenever $Q=\sum b_\gamma\gamma$ is a trigonometric polynomial satisfying $\|Q\|_1 \leqq 1$ and $b_\gamma=0$ for $\gamma \in K$.*

Proof. Clearly it is enough to show that

(1) $$m_1^1(\Gamma) = L^1(G)^\wedge$$

and

(2) $$m_1^\infty(\Gamma) = C(G)^\wedge.$$

Now $M_1^1(\Gamma) = M(G)^\wedge$ and $M_1^\infty(\Gamma) = L^\infty(G)^\wedge$ (see [7, p. 368] and [9]), thus (1) and (2) follow from the lemma since the closure of $\mathfrak{J}(G)$ in $M(G)$ ($L^\infty(G)$) is $L^1(G)$ ($C(G)$).

REMARKS. For compact abelian groups, the above characterization of transforms of absolutely continuous measures is Theorem 2 of Doss [3, pp. 361–362]. Theorem 2 of [3] in the noncompact case may be obtained by simple modifications of the above proofs, which we omit. For an interesting reformulation of Theorem 2 of [3] the reader is referred to [2, p. 114]. For different characterizations of transforms of $L^1(G)$ and $C(G)$ functions, see Theorems 3 and 4 of [6, pp. 245–246]. In this connection see also Theorem 2 of [4, p. 78].

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