ON THE DEGREES AND RATIONALITY OF CERTAIN CHARACTERS OF FINITE CHEVALLEY GROUPS

BY

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Abstract. Let $\mathscr{S}$ be a system of finite groups with (B, N)-pairs, with Coxeter system (W, R) and set of characteristic powers $\{q\}$ (see [4]). Let A be the generic algebra of the system, over the polynomial ring $\mathfrak{o} = Q[u]$. Let K be $Q(u)$, $\overline{K}$ an algebraic closure of K, and $\mathfrak{o}^*$ the integral closure of $\mathfrak{o}$ in $\overline{K}$. For the specialization $f: u \to q$ mapping $\mathfrak{o} \to Q$, let $f^*: \mathfrak{o}^* \to \overline{Q}$ be a fixed extension of f. For each irreducible character χ of the algebra $A^{\overline{K}}$, there exists an irreducible character ζ_{χ,f^*} of the group $G(q)$ in the system corresponding to q, such that $(\zeta_{\chi,f^*}, 1_{B(q)}^{G(q)}) > 0$, and $\chi \to \zeta_{\chi,f^*}$ is a bijective correspondence between the irreducible characters of $A^{\overline{K}}$ and the irreducible constituents of $1_{B(q)}^{G(q)}$. Assume almost all primes occur among the characteristic powers $\{q\}$. The first main result is that, for each χ, there exists a polynomial $d_\chi(t) \in Q[t]$ such that, for each specialization $f: u \to q$, the degree $\zeta_{\chi,f^*}(1)$ is given by $d_\chi(q)$. The second result is that, with two possible exceptions in type E_7, the characters ζ_{χ,f^*} are afforded by rational representations of $G(q)$.

1. **Introduction.** The results in this paper are concerned with the irreducible complex characters χ of a finite Chevalley group or a twisted type G, with the property that χ appears with positive multiplicity in the induced permutation representation 1_B^G from a Borel subgroup B of the group G. It is proved (with certain exceptions) that these characters are afforded by rational representations of G, and that their degrees are polynomials in q with rational coefficients, where q is the order of the finite field associated with G.

These results were proved by Steinberg [17] for the general linear groups over a finite field, and had been verified for some other special cases. Some of the methods in this paper were introduced by Curtis, Iwahori and Kilmoyer [4], and explicit formulas for the degrees of certain characters in 1_B^G for a general Chevalley group or twisted type were derived in [4, §§9, 10].

From the point of view of Harish-Chandra's approach to characters of finite Chevalley groups using cusp-forms, the results in this paper deal with a special case of problem (b), p. C-19 of Springer [16], that is, splitting up an induced representation $\Omega(c, \omega) = 1_B^G$, where c is the equivalence class of the cuspidal pair (B, H) where

H is a maximal torus in the Borel subgroup B, and ω is the trivial class in the discrete series of the torus H.

Here is a sketch of the contents of the paper. §2 contains precise statements of the main results, together with most of the definitions and notations used later in the paper. §3 contains a proof of the result about the degrees of the characters, which is independent of the classification of the Coxeter systems of the groups. §4 contains some general results about the generic rings of Coxeter systems, which, together with some results in §2, achieve a reduction of the proofs of the main theorems to verifying certain facts about finite Coxeter groups. These facts seem to be of independent interest, and are proved for the various types of Coxeter groups using a case-by-case analysis, in the subsequent sections.

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2. **Statements of results.**   In general, the notations in [5] will be used. In order to discuss the results from a uniform point of view, we axiomatize the situation as follows (see [4]).

(2.1) DEFINITIONS. Let (W, R) be a finite Coxeter system, with finite Coxeter group W and set of distinguished generators R. For $w \in W$ we denote by $l(w)$ the length of w, i.e. the minimal number of factors among all possible expressions of w as a word in the distinguished generators. A *system $\mathscr{S}$ of (B, N)-pairs of type (W, R)* consists of the Coxeter system (W, R), an infinite set $\mathscr{CP}$ of prime powers $\{q\}$, called *characteristic powers*, a set of positive integers $\{c_r\}_{r \in R}$, and for each characteristic power q, a finite group $G = G(q)$ with a (B, N)-pair, $(B(q), N(q))$, such that (W, R) is the Coxeter system of $G(q)$, and:

  (i) $c_r = c_s$, for $r, s \in R$, if r and s are conjugate in W;
  (ii) for each group $G = G(q) \in \mathscr{S}$, the *index parameters*

$$\text{ind } r = [B(q) : B(q) \cap B(q)^r],$$

for $r \in R$, are given by ind $r = q^{c_r}$;

  (iii) if (W, R) is of type G_2, we assume that $c_r + c_s$ is even, for r and s in R.

To such a system corresponds a *generic ring $A = A(u)$*, which is an algebra with a free basis $\{a_w\}_{w \in W}$, over the polynomial ring $\mathfrak{o} = Q[u]$, satisfying, for $w \in W, r \in R$,

$$\begin{aligned} a_w a_r &= a_{wr}, & l(wr) &> l(w), \\ a_w a_r &= u^{c_r} a_{wr} + (u^{c_r} - 1) a_w, & l(wr) &< l(w). \end{aligned}$$

For each homomorphism $f: \mathfrak{o} \to Q$, we can define the specialized algebra over Q,

$$A_f = Q \otimes_{\mathfrak{o}} A,$$

which has as a Q-basis the elements $\{a_{wf}\}$ given by $a_{wf} = 1 \otimes a_w$, $w \in W$. The map $f: \sum u_w a_w \to \sum f(u_w) a_{wf}$, $u_w \in \mathfrak{o}$, can be viewed as a homomorphism of algebras over $\mathfrak{o}$, if we view A_f as an $\mathfrak{o}$-algebra, with ra defined to be $f(r)a$, $r \in \mathfrak{o}$, $a \in A_f$.

Let f be the specialization $u \to q$ for a characteristic power q. Then

$$A(q) = A_f \cong H_Q(G, B),$$

for $G = G(q) \in \mathscr{S}$, where $H_Q(G, B)$ denotes the Hecke algebra (or centralizer ring) $eQGe$, for $e = |B|^{-1} \sum_{x \in B} x$, and is isomorphic to $\mathrm{Hom}_{QG}(1_B^G, 1_B^G)$.

For the specialization $u \to 1$, we have $A(1) \cong QW$, the group algebra of W over Q.

Let $K = Q(u)$, $\bar{K}$, $\bar{Q}$ algebraic closures of K and Q respectively, and $\mathfrak{o}^*$ the integral closure of $\mathfrak{o}$ in $\bar{K}$.

The first result to be stated is proved in [4, Proposition (7.1)].

(2.2) PROPOSITION. *Let χ be an irreducible character of the generic algebra $A^{\bar{K}}$. Then $\chi(a_w) \in \mathfrak{o}^*$, for all $w \in W$. Let $f: u \to q$ or $u \to 1$ define a specialization belonging to a characteristic power q or 1. Let $f^*: \mathfrak{o}^* \to \bar{Q}$ be an extension of f. Then the linear map $\chi_{f^*}: A_f^{\bar{Q}} \to \bar{Q}$ defined by*

$$\chi_{f^*}(a_{wf}) = f^*(\chi(a_w))$$

is an irreducible character of $A_f^{\bar{Q}}$. For a fixed extension f^* of f, the map $\chi \to \chi_{f^*}$ is a bijection between the irreducible characters of $A^{\bar{K}}$ and those of $A_f^{\bar{Q}}$.

From [3] we have

(2.3) PROPOSITION. *With the notations as in (2.2), each specialized character χ_{f^*}, viewed as a character of $H_{\bar{Q}}(G, B)$, is the restriction to $H_{\bar{Q}}(G, B)$ of a unique absolutely irreducible character ζ_{χ,f^*} of $G(q)$ (or W), such that $\zeta_{\chi,f^*} \in 1_B^{G(q)}$. Every irreducible constituent of 1_B^G is obtained in this way. The degree of ζ_{χ,f^*} is given by*

$$\deg \zeta_{\chi,f^*} = [G:B] \deg \chi \left\{ \sum_{w \in W} (\mathrm{ind}\, w)^{-1} \chi_{f^*}(\hat{a}_{wf}) \chi_{f^*}(a_{wf}) \right\}^{-1}$$

where $\hat{a}_{wf}$ is the basis element of A_f corresponding to w^{-1} and

$$\mathrm{ind}\, w = [B : B \cap B^w], \qquad w \in W.$$

For the specialization $f_0: u \to 1$, the formula becomes $\deg \zeta_{\chi,f_0^*} = \deg \chi$.

(2.4) DEFINITION. With the notation as above let

$$d_\chi = \left[\sum_{w \in W} \nu(a_w) \right] \deg \chi \left\{ \sum_{w \in W} \nu(a_w)^{-1} \chi(\hat{a}_w) \chi(a_w) \right\}^{-1}$$

where $\nu: A \to \mathfrak{o}$ is the homomorphism such that $\nu(a_r) = u^{c_r}$, $r \in R$. d_χ is called the *generic degree* associated with χ.

(2.5) PROPOSITION. *For each χ and f^*, as above, d_χ belongs to the specialization ring of f^*, and $f^*(d_\chi) = \deg(\zeta_{\chi,f^*})$.*

Proof. From [4, Lemmas (5.2) and (5.9)], we have $f(\sum_{w \in W} \nu(a_w)) = [G:B]$, and $f(\nu(a_w)) = \mathrm{ind}\, w$. The rest of the proof is clear, taking into account the formula for $\deg \zeta_{\chi,f^*}$ given in Proposition (2.3), and Proposition (2.2).

(2.6) Theorem. *Suppose the set of characteristic powers of $\mathscr{S}$ contains almost all primes. Then for each irreducible character χ of $A^{\bar{K}}$, the generic degree $d_\chi \in Q[u]$.*

Note that this theorem does not apply to twisted G_2, B_2 or F_4, but the conclusion is true for twisted G_2 or B_2 since in these cases, the representation of G on the cosets of B is doubly transitive. In the case of twisted F_4, the generic degrees do not belong to $\mathfrak{o}$ (see [4, §9]).

Definition. A character χ of $A^{\bar{K}}$ is said to be *rational* in case $\chi(a_w) \in \mathfrak{o}$ for all $w \in W$.

Note that if χ is rational, then the specialized characters χ_{f^*} defined in Proposition (2.2) have the property that χ_{f^*} is independent of the extension f^* of f, and $\chi_{f^*}(a_{wf})$ $\in Q$ for all $w \in W$.

(2.7) Theorem. *Suppose χ is an irreducible character of $A^{\bar{K}}$ and $J \subset R$. Set $e_J = \sum_{w \in W_J} a_w$, $E_J = \nu(e_J)$ and let $m_J(\chi)$ be the nonnegative integer such that $\chi(e_J) = m_J(\chi)E_J$. Suppose for each irreducible character $\chi' \neq \chi$, there exists a $J \subset R$ such that $m_J(\chi) \neq m_J(\chi')$. Then χ is a rational character of $A_{\bar{R}}$, i.e. $\chi(a_w) \in \mathfrak{o}$, $w \in W$.*

Proof. For each specialization $f: u \to q$ or 1, $m_J(\chi) = (1_{G_J}^G, \zeta_{\chi, f^*})$ where G_J is the parabolic subgroup BW_JB (see [4, §7]). Suppose χ is not rational; then $\chi \neq \chi^\sigma$ for some field automorphism σ of $\bar{K}$ over K. But $m_J(\chi) = m_J(\chi^\sigma)$ for all subsets $J \subset R$, contradicting the hypothesis. This completes the proof.

Using Theorem (2.7), the following theorem can be proved.

(2.8) Theorem. *Suppose (W, R) is of type A_n $(n \geq 1)$, B_n (or C_n) $(n \geq 2)$, D_n $(n \geq 4)$, G_2, F_4, E_6, E_7, E_8. Then every irreducible character of the generic algebra $A^{\bar{K}}$ is rational, with the possible exception of the two irreducible characters of degree 512, in case (W, R) is of type E_7.*

In case W is of type E_7, it is the direct product of a cyclic group C of order 2 and the Chevalley group $B_3(2)$. If 1_C and ε_C are the two irreducible characters of C and ζ is the Steinberg character of $B_3(2)$, the characters $1_C \otimes \zeta$ and $\varepsilon_C \otimes \zeta$ of W occur with the same multiplicity in 1_P^W for every parabolic subgroup P of W. The two characters of $A^{\bar{K}}$ corresponding to $1_C \otimes \zeta$ and $\varepsilon_C \otimes \zeta$ fail to satisfy the hypothesis of Theorem (2.7) and account for the exception in Theorem (2.8).

We remark that Theorem (2.8), combined with the formula for generic degrees (2.4), gives an independent proof of Theorem (2.6) (on the rationality of generic degrees) for generic rings of the type listed in Theorem (2.8), with the possible exception of the two characters of degree 512 in the case of E_7.

Before stating the next theorem, some new concepts have to be introduced. An irreducible character χ of a finite dimensional algebra A over a field K is said to be afforded by a primitive idempotent $e \in A$ in case χ is the character of the irreducible representation of A on the minimal left ideal Ae in A. Now suppose K is the quotient field of a principal ideal domain R, and that A has a K-basis $\{a_1, \ldots, a_n\}$ for which

the structure constants all belong to R. For each $a \in A$, we let $\delta(a)$ denote a genera-tor of the ideal in R consisting of all elements $x \in R$ such that $xa \in \sum Ra_i$; $\delta(a)$ will be called the *denominator* of a.

(2.9) Theorem. *Let $\mathscr{S}$ be a system of groups with (B, N)-pairs of any one of the types A_n ($n \geqq 1$), B_n (or C_n) ($n \geqq 2$), D_n ($n \geqq 4$), G_2, F_4, E_6, E_7, or E_8. Then every ir-reducible character χ of the generic algebra A^K is afforded by a primitive idempotent $a_\chi \in A^K$, with the possible exception of the two characters of degree 512 in case $\mathscr{S}$ is of type E_7. Thus, except for the case of E_7, K is a splitting field for A^K. Let a_χ be any one of these rational idempotents, and let f be a specialization associated with a characteristic power q or 1. Then $f(\delta(a_\chi)) \neq 0$, and $f(\delta(a_\chi))^{-1}f[\delta(a_\chi)a_\chi] = a_{\chi f}$ is a primitive idempotent in the group algebra $QG(q)$, affording the irreducible complex character $\zeta_{\chi, f}$ of G. Thus every irreducible character $\zeta \in 1_B^G$, for $G \in \mathscr{S}$, is afforded by a rational representation of G, with the possible exception of two characters of G associated with the characters of degree 512 of the generic algebra, in case $\mathscr{S}$ is of type E_7.*

The methods to be given below provide explicit formulas for the idempotents a_χ, in terms of the characters of A^K and its subalgebras $A_J^K = \langle a_r; r \in J \rangle^K, J \subset R$.

As a corollary to the proof of Theorem (2.9), the following results can be stated.

(2.10) Corollary. *Let (W, R) be a Coxeter system of type A_n ($n \geqq 1$), B_n ($n \geqq 2$), D_n ($n \geqq 4$), or G_2, F_4, E_6, E_7, E_8. Let χ be an irreducible complex character of W. There exist subsets $R = J_1 \supset J_2 \supset \cdots \supset J_s$ of R, for $s = 2$ or 3, and irreducible characters $\chi_1 = \chi, \chi_2, \ldots, \chi_s$ of $W_{J_1}, \ldots, W_{J_s}$ respectively, such that $(\chi_i, \chi_{i+1}^{W_{J_i}}) = 1$, for $1 \leqq i \leqq s - 1$, with χ_s a linear character of W_{J_s}. Let e_χ be the idempotent $e(\chi_1)e(\chi_2) \cdots e(\chi_s)$, where*

$$e(\chi_i) = \chi_i(1)|W_{J_i}|^{-1} \sum_{x \in W_{J_i}} \chi_i(x^{-1})x.$$

Then e_χ is primitive, belongs to QW, and affords χ. Thus each irreducible character of W is afforded by a rational representation.

The statements about the existence of the characters $\{\chi_i\}$ and their multiplicities are proved in subsequent sections. The fact that e_χ is a primitive idempotent affording χ follows from a result of Janusz [11]. The third statement, that $e_\chi \in QW$, is a consequence of the known result that all the characters of W are rational (without exception).

We remark that the proof of Theorem (2.8) given below for (W, R) of type A_n, B_n and D_n gives a new proof of the rationality of the characters of Coxeter groups of these types. The last statement of Corollary (2.10), on the rationality of the representations of the Coxeter groups of the types listed, is not a new result. The last open problems on the subject were settled by Benard [1] for types E_6, E_7 and E_8, by an analysis of the elementary subgroups of these Coxeter groups and using some results about Schur indices. The result was already known for the other types (see [1] for references).

(2.11) COROLLARY. *Let the hypothesis of Theorem* (2.9) *be satisfied. Then* $H_Q(G(q), B(q)) \cong QW$, *except possibly for W of type* E_7.

To prove this corollary, one first observes that by the Deformation Theorem (see Steinberg [18, p. 249]), both $H_Q(G(q), B(q))$ and QW have the same numerical invariants. It then follows immediately from Theorem (2.9) and Corollary (2.10) that, with the possible exception of E_7, the field of rational numbers Q is a splitting field for $H_Q(G(q), B(q))$ and QW, and the corollary follows.

We conclude this section by calling attention to two of the combinatorial methods and results used in later sections to discuss the various types of Coxeter groups. The first is the systematic use of Clifford's theorem in the form given by Gallagher [10, Theorem 2] on characters of semidirect products, for computation of characters of B_n in §§6 and 7. The second is the following result about characters of the symmetric group (Theorem (6.7) below).

(2.12) PROPOSITION. *Suppose* ω *and* ω' *are characters of the symmetric group* S_n, $n \geq 3$, *with the property that* $\omega(p)/\omega(1) = \omega'(p)/\omega'(1)$, *for all cycles p of lengths* $2, 3, \ldots, n-1$. *Then* $\omega = \omega'$.

3. **Rationality of generic degrees.** This section contains a proof of Theorem (2.6). Let χ be an irreducible character of A^K, and let

$$c_\chi = \sum_{w \in W} \nu(a_w)^{-1} \chi(\hat{a}_w) \chi(a_w).$$

We assert first that it is sufficient to prove that $c_\chi \in K = Q(u)$. For if that is the case, then $d_\chi \in K$, and is a rational function in $Q(u)$ which becomes an integer upon substitution of infinitely many integer values of u. An elementary argument (which is omitted) then shows that $d_\chi \in Q[u]$.

From Proposition (2.2), we have $\chi(a_w) \in \mathfrak{o}^*$, $w \in W$, where $\mathfrak{o}^*$ is the integral closure of $\mathfrak{o}$ in $\bar{K}$. Therefore, letting $c_\chi' = [\text{lcm}_{w \in W} \nu(a_w)]c_\chi$, we have $c_\chi' \in \mathfrak{o}^*$, and it is sufficient to prove that $c_\chi' \in K$. Since $c_\chi' \in \mathfrak{o}^*$, we have Irr $(c_\chi', K, t) \in \mathfrak{o}[t]$. Letting

$$F(u, t) = \text{Irr}(c_\chi', K, t) = t^m + \varphi_1(u)t^{m-1} + \cdots + \varphi_m(u), \qquad \varphi_i(u) \in \mathfrak{o},$$

we see that $F(u, t)$ is an irreducible polynomial in $Q[u, t]$.

Now let $f: u \to q$ be a specialization, for some $q \in \mathscr{CP}$, and let $f^*: \mathfrak{o}^* \to \bar{Q}$ be an extension of f. Then $f^*(c_\chi') \in Q$, since $f^*(d_\chi) \in Z$, by Proposition (2.5).

Suppose the theorem is false, and that $c_\chi' \notin K$. Then $m = \deg \text{Irr}(c_\chi', K, t) > 1$. Let $f: u \to q$ be a specialization, for $q \in \mathscr{CP}$, and f^* an extension of f to $\mathfrak{o}^*$. Then the rational number $f^*(c_\chi')$ is a zero of the polynomial $F(q, t) \in Q[t]$, where $F(q, t)$ is monic, and of degree $m > 1$. On the other hand, the Hilbert Irreducibility Theorem (Lang [13, p. 148]) implies that for infinitely many primes p, the specialized polynomial $F(p, t)$ is irreducible in $Q[t]$. Some such p must belong to $\mathscr{CP}$, by the assumption that $\mathscr{CP}$ contains almost all primes, and we have arrived at a contradiction. This completes the proof of Theorem (2.6).

4. Primitive idempotents and multiplicities in generic algebras.   In this section, some general results about idempotents in generic algebras will be proved, which will provide the means of establishing Theorem (2.9) for each of the individual types of systems of groups with (B, N)-pairs. We begin with some general (and presumably well-known) lemmas. Let K be the quotient field of a principal ideal domain $\mathfrak{o}$. Let A be a finite dimensional algebra over K, with a basis $\{a_1, \ldots, a_n\}$ whose constants of structure lie in $\mathfrak{o}$. Let B be the subring $\sum \mathfrak{o}a_i$ of A. As in §2, for $a \in A$ we let $\delta(a)$ be a generator of the ideal consisting of all $x \in R$ such that $xa \in B$. An element $b \in B$ is said to have *unit content* if $b = \sum r_i a_i$, $r_i \in \mathfrak{o}$, and $\gcd \{r_i\} = 1$ $(1 \leq i \leq n)$.

(4.1) LEMMA. *Let e be an idempotent $(\neq 0)$ in A. Then $e' = \delta(e)e \in B$, and has unit content.*

Proof We have $(e')^2 = \delta(e)e'$. Let d be the g.c.d. of the coefficients of e'. Then $(d, \delta(e)) = 1$, otherwise $[\delta(e)/(d, \delta(e))]e \in B$, contrary to the definition of $\delta(e)$. We have $e' = de''$, with e'' of unit content, and $(e')^2 = \delta(e)e' = d^2(e'')^2$, which implies that $\delta(e)e'' = d(e'')^2$. Then d is a unit, since e'' has unit content.

(4.2) LEMMA. *Let b be an element of unit content in B. Then $Kb \cap B = \mathfrak{o}b$.*

Proof. Let $s^{-1}rb = b' \in B$, where $r, s \in \mathfrak{o}$ and $(r, s) = 1$. Then $rb = sb'$, and since b has unit content, $s \mid r$. Therefore s is a unit, and $s^{-1}r \in \mathfrak{o}$.

We turn now to a system of groups with (B, N)-pairs of type (W, R) in the sense of Definition (2.1), and let A be the generic ring associated with the system. We shall apply Lemmas (4.1) and (4.2) to the pair (A^K, A) with A playing the role of B, and $K = Q(u)$, the quotient field of $\mathfrak{o} = Q[u]$.

(4.3) LEMMA. *Let $J \subset R$. Then*

(i) *$A_J = \langle a_r, r \in J \rangle$ is a generic ring of a system of type (W_J, J), with the same set of characteristic powers, where for each $q \in \mathscr{CP}$, the corresponding group with a (B, N)-pair is the parabolic subgroup $G_J(q) = B(q)W_JB(q)$ of $G(q)$.*

(ii) *For each specialization $f: u \to q$ or 1, A_{Jf} is isomorphic to, and will be identified with, $H_Q(G_J(q), B(q))$ or W_J respectively.*

The proof of this lemma is entirely routine and will be omitted.

(4.4) PROPOSITION. *Let A, A_J be as above. Let χ and φ be irreducible characters of A^K and A_J^K respectively, and suppose φ is afforded by a primitive idempotent e belonging to A_J^K. Let $e' = \delta(e)e$. Then*

$$\chi(e') = m(\chi, \varphi)\delta(e),$$

where $m(\chi, \varphi)$ is a nonnegative integer. For every specialization $f: u \to q$ or 1, let f^* denote a fixed extension of f to the integral closure $\mathfrak{o}^*$ of $\mathfrak{o}$ in $\bar{K}$. Then from §2, there are corresponding irreducible characters $\zeta_{\chi, f^*}, \eta_{\varphi, f^*}$ of $G(q)$ and $G_J(q)$ respectively. For every specialization f such that $f(\delta(e)) \neq 0$, we have

$$m(\chi, \varphi) = (\zeta_{\chi, f^*}, \eta_{\varphi, f^*}^{G(q)}).$$

Proof. For $a \in A$ or A_J, let a_f denote the image in the specialized algebra A_f or A_{Jf} of a, under the homomorphism of $\mathfrak{o}$-algebras defined by f. We prove first that in case $f(\delta(e)) \neq 0$, $e'_f / f(\delta(e))$ is a primitive idempotent in $A_{Jf} = H_Q(G_J(q), B(q))$ affording the character η_{φ, f^*}.

We have $e'A_J e' = Ke' \cap A_J = \mathfrak{o} e'$ by Lemma (4.2). Moreover $(e')^2 = \delta(e)e'$. Applying the homomorphism f to these equations, and using the fact that $e'_f \neq 0$ since e' has unit content (by Lemma (4.1)), it follows that $e'_f / f(\delta(e))$ is a primitive idempotent in A_{Jf}.

Finally, since $\varphi(e) = 1$, we have $\varphi(e') = \delta(e)$, and hence

$$\eta_{\varphi, f^*}(e'_f) = \varphi_{f^*}(e'_f) = f(\delta(e)) \neq 0.$$

This completes the proof that $e'_f / f(\delta(e))$ is primitive and affords η_{φ, f^*}.

We next have from $\chi(e') = m(\chi, \varphi)\delta(e)$ the result that

$$\zeta_{\chi, f^*}(e'_f) = \chi_{f^*}(e'_f) = m(\chi, \varphi)f(\delta(e)).$$

From this formula we conclude, in case $f(\delta(e)) \neq 0$, that $m(\chi, \varphi) = (\zeta_{\chi, f^*}, \eta_{\varphi, f^*}^{G(q)})$, and the proposition is proved.

For an irreducible character χ of A^K, we shall denote by $e(\chi)$ the central primitive idempotent corresponding to χ.

(4.5) PROPOSITION. *Let A, A_J, χ, φ, e be as above, and suppose that the irreducible character χ of A^K is rational, in the sense that $\chi(a_w) \in \mathfrak{o}$ for all $w \in W$. Finally suppose the multiplicity $m(\chi, \varphi) = 1$. Then $e(\chi)e$ is a primitive idempotent in A^K affording χ. Let $f : u \to q$ or 1 be a specialization such that $f(\delta(e)) \neq 0$. Then $f(\delta(e(\chi)e)) \neq 0$, and, letting $\tilde{e} = \delta(e(\chi)e)e(\chi)e$, $\tilde{e}_f / f(\delta(e(\chi)e))$ is a primitive idempotent in $A_f = H_Q(G(q), B(q))$ affording ζ_{χ, f^*}.*

Proof. From Corollary (5.12) of [4], with $J = \varnothing$, we know that A^K is a split semisimple symmetric algebra, with dual bases $\{a_w\}$ and $\{\nu(a_w)^{-1}\hat{a}_w\}$ with respect to the bilinear form defined in [4]. From Fossum [6], the block idempotent $e(\chi)$ is given by

$$e(\chi) = \chi(1)c_\chi^{-1} \sum_{w \in W} \nu(a_w)^{-1}\chi(\hat{a}_w)a_w,$$

where $c_\chi = \sum_{w \in W} \nu(a_w)^{-1}\chi(\hat{a}_w)\chi(a_w) \in K$. These remarks show that $e(\chi) \in A^K$ if χ is rational. Let $m = \text{lcm}_{w \in W} \{\nu(a_w)\}$. Then $f(m) \neq 0$, $mc_\chi \in \mathfrak{o}$, and $f(mc_\chi) \neq 0$ because of Proposition (2.5), and the formula for the generic degree d_χ. It follows that

$$mc_\chi \cdot \delta(e)ee(\chi) \in A,$$

and hence $\delta(ee(\chi)) \neq 0$ since $f(mc_\chi\delta(e)) \neq 0$.

Next, $e \in A_f^K$ affords φ and $\chi(e) = 1$, by our hypothesis. This means that if E is a primitive idempotent in A^K affording χ, then $\dim eA^K E = 1$, and by [5, (54.16)], the number of composition factors of $A^K E$ isomorphic to $A_J^K e$ is one.

We can now state the following lemma, which is an immediate consequence of Theorems 6, 7, and 8 of Nakayama [15], and the facts that $A_f^{\bar{K}}$ and $A^{\bar{K}}$ are semisimple algebras over an algebraically closed field $\bar{K}$.

(4.6) LEMMA. *Let $A_f^{\bar{K}}$, $A^{\bar{K}}$, e and E be as above. Then the number of composition factors of $A^{\bar{K}}E$ which are isomorphic to $A_f^{\bar{K}}e$ is equal to the number of composition factors of $eA^{\bar{K}}$ isomorphic to $EA^{\bar{K}}$.*

Applying the lemma, we conclude $e(\chi)eA^{\bar{K}}$ is a minimal right ideal in $A^{\bar{K}}$, and hence $A^{\bar{K}}e(\chi)e$ is a minimal left ideal affording the character χ. Now let f be a specialization such that $f(\delta(e)) \neq 0$. From what has been proved, and from the proof of Proposition (4.4), it follows that $\tilde{e}_f/f(\delta(e(\chi)e))$ is a primitive idempotent in A_f. It remains to check that it affords the character ζ_{χ,f^*}. Since $\tilde{e}/\delta(e(\chi)e)$ affords χ, we have $\chi(\tilde{e}) = \delta(e(\chi)e)$. Therefore

$$\chi_{f^*}(\tilde{e}_f) = f(\delta(e(\chi)e))$$

and hence, for a specialization f satisfying our hypotheses,

$$\zeta_{\chi,f^*}(\tilde{e}_f) = \chi_{f^*}(\tilde{e}_f) = f(\delta(e(\chi)e)) \neq 0.$$

This completes the proof of the proposition.

In the applications of these results in subsequent sections, the construction of idempotents via Proposition (4.5) will always begin with some subset $J \subset R$ for which the character φ of A_J is either $\nu: A_J \to \mathfrak{o}$ or $\varepsilon: A_J \to \mathfrak{o}$, given by $\nu(a_r) = u^{c_r}$, $r \in R$, and $\varepsilon(a_r) = -1, r \in R$, respectively. The idempotents e affording these characters are

$$\left[\sum_{w \in W_J} \nu(a_w) \right]^{-1} \sum_{w \in W_J} a_w$$

and

$$\left[\sum_{w \in W_J} \nu(a_w)^{-1} \right]^{-1} \sum_{w \in W_J} \nu(a_w)^{-1}(-1)^{l(w)} a_w$$

respectively, and thus $f(\delta(e)) \neq 0$ for all specializations $f: u \to q$ or 1. This means that the hypothesis about f needed to apply Proposition (4.5) will automatically be satisfied in the applications given in later sections.

The rationality of an irreducible character of a generic algebra $A^{\bar{K}}$, when it is needed to apply Proposition (4.5), will follow from previously checked cases of Theorem (2.8), and the following result.

(4.7) PROPOSITION. *Let A be a generic ring of type (W, R), and suppose that $R = J_1 \cup \cdots \cup J_k$, with $\{J_s\}$ indecomposable, $(J_s, J_{s'}) = 1$, and $J_s \cap J_{s'} = \varnothing$ for $s \neq s'$. Then $A \cong A_{J_1} \otimes \cdots \otimes A_{J_k}$. If all the irreducible characters of $A_{J_s}, 1 \leq s \leq k$, are rational, the same is true of $A^{\bar{K}}$.*

The proof is omitted.

5. **Proof of Theorems (2.8) and (2.9) for** A_n. We first remark that since $m_J(\chi) = (1_{G_J}^G, \zeta_{\chi,f^*})$ (see the proof of Theorem (2.7)) for all specializations $f: u \to q$ or $u \to 1$, the proof of Theorem (2.8) is reduced, using Theorem (2.7), to showing that an irreducible character ω of W is determined by the multiplicities $m_J(\omega) = (\omega, 1_{W_J}^W)$. Theorem (2.8) for W of type A_n thus follows from the following proposition.

(5.1) PROPOSITION. *Let* (W, R) *be of type* A_n *and let* $R = \{w_1, \ldots, w_n\}$, *indexed so that* $(w_i w_{i+1})^3 = 1$, $i = 1, \ldots, n-1$; $(w_i w_j)^2 = 1$, $|j - i| > 1$. *Then, for irreducible characters* ω, ω' *of* W, $m_J(\omega) = m_J(\omega')$ *for all* $J \subset R$ *implies* $\omega(w_J) = \omega'(w_J)$ *where* w_J *is the product in any order of the elements in* J. *Moreover* $\omega = \omega'$.

Proof. In this case W is the symmetric group S_{n+1}, and the w_i are the transpositions $(i, i+1)$. For a given subset $J \subset R$, the possible choices of w_J are all conjugate, and every element of W_J which is not conjugate to w_J is conjugate to a $w_{J'}$, for $J' \subset J, J' \neq J$. The proposition follows by induction on $|J|$, using the fact that

$$m_J(\omega) = \omega(e_J), \qquad e_J = |W_J|^{-1} \sum_{w \in W_J} w.$$

We now prove Theorem (2.9) for the case A_n.

(5.2) DEFINITION. The irreducible character ζ_{χ,f^*} is said to be of *parabolic type* if $(\zeta_{\chi,f^*}, 1_{G_J}^G) = 1$ for some $J \subset R$.

Using Propositions (4.4), (4.5) and (4.7), Theorem (2.9) for the case $W = A_n$ follows from the following proposition.

(5.3) PROPOSITION. *Every irreducible character* ω *of* (W, R) *of type* A_n *is of parabolic type.*

Proof. To ω we associate a standard Young tableau D, [7]. Then the group $R(D)$ of row permutations of D coincides with W_J for some J. Moreover $1_{W_J}^W$ is a Frobenius compound character [7], and $(1_{W_J}^W, \omega) = 1$ [7] (or [11]).

6. **Rationality of characters of a generic algebra of type** B_n. If (W, R) is of type B_n, we may consider W as the group of signed permutations of n letters [2], and $W = NS_n$, the semidirect product of an elementary abelian 2-group N by the symmetric group on n letters. Moreover, N has a basis over Z_2 consisting of the negative 1-cycles $\{-(1), -(2), \ldots, -(n)\}$ and we may take the set R to be $\{w_1, \ldots, w_n\}$, $w_i = (i, i+1)$, $i = 1, \ldots, n-1$, and $w_n = -(n)$. From [2], two elements are conjugate in W if and only if they have the same signed cycle type.

By one of Clifford's theorems in the form given by Gallagher [10, Theorem 2], the irreducible characters of W are constructed as follows. Each irreducible character appears with positive multiplicity in a unique induced character ψ^W, where $\{\psi\}$ are representatives of the different equivalence classes, under conjugacy by W, of linear characters of N. For the representatives $\{\psi\}$, we may select characters $\{\psi_{kl}\}$ for $k + l = n$, such that $\psi_{kl}(-(i)) = 1$, $1 \leq i \leq k$, and $\psi_{kl}(-(i)) = -1$, $k + 1 \leq i \leq n$. For

such a character ψ_{kl}, the centralizer T_{kl} in W is the semidirect product NS, where $S \subset S_n$, and $S \cong S_k \times S_l$, the direct product of symmetric groups on $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$. We shall write $S = S_k \times S_l$. Every irreducible constituent of ψ_{kl}^W has the form

$$\eta = (\tilde{\psi}_{kl} \omega)^W = (\tilde{\psi}_{kl} \omega_k \omega_l)^W,$$

where $\tilde{\psi}_{kl}$ is the extension of ψ_{kl} to T_{kl} such that $\tilde{\psi}_{kl} | S = 1$, and ω is an irreducible character of $S = S_k \times S_l$, and can be expressed in the form $\omega_k \omega_l$, where ω_k and ω_l are uniquely determined irreducible characters of the symmetric groups S_k and S_l respectively. We ask the reader to live with the possibility that $k = l$ does not imply that the characters ω_k and ω_l are the same.

The subgroup T_{kl} is the direct product of the groups of signed permutations on $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$ respectively, and we shall use the notation $T_{kl} = B_k \times B_l$. The meaning of the notations $S_k \times B_l$, $S_k \times S_l$, etc., for subgroups of T_{kl} will be clear.

For complementary subsets K and L of $\{1, \ldots, n\}$, define $\psi_{KL}: N \to \{\pm 1\}$ to be the character such that $\psi_{KL}(-(i)) = 1$, $i \in K$, and $\psi_{KL}(-(i)) = -1$, $i \in L$. Then $\psi_{KL} \cong \psi_{|K|, |L|}$, and $\psi_{KL}^W = \psi_{|K|, |L|}^W$.

As in §5, to prove Theorem (2.8) for the case $W = B_n$, it is sufficient to show that the irreducible characters η of W are determined by their multiplicities $(1_{W_J}^W, \eta)$. Thus assume, for Lemmas (6.1) and (6.4), that $\eta = (\tilde{\psi}_{kl} \omega_k \omega_l)^W$ and $\eta' = (\tilde{\psi}_{k'l'} \omega_{k'}' \omega_{l'}')^W$ and that $(1_P^W, \eta) = (1_P^W, \eta')$ for all parabolic subgroups P of W.

(6.1) LEMMA. $k = k'$, $l = l'$ and $\omega_k(1)\omega_l(1) = \omega_{k'}'(1)\omega_{l'}'(1)$.

Proof. If $P = \{1\}$, we have

$$(6.2) \qquad \eta(1) = [W:T_{kl}]\omega_k(1)\omega_l(1) = \eta'(1) = [W:T_{k'l'}]\omega_{k'}'(1)\omega_{l'}'(1).$$

Letting $P = \{1, w_n\}$ we have $\eta(w_n) = \eta'(w_n)$. But

$$(6.3) \qquad \begin{aligned} \eta(w_n) &= \frac{\omega_k(1)\omega_l(1)}{|T_{kl}|} \sum_{x \in W} \tilde{\psi}_{kl}(x^{-1} w_n x) \\ &= (|C_W(w_n)|/|T_{kl}|)\omega_k(1)\omega_l(1)(k-l) \end{aligned}$$

where $C_W(w_n)$ is the centralizer in W of w_n. Setting $\eta(w_n) = \eta'(w_n)$ and using (6.2) yields $k - l = k' - l'$ which implies $k = k'$, $l = l'$ since $l = n - k$, $l' = n - k'$. Then $\omega_k(1)\omega_l(1) = \omega_{k'}'(1)\omega_{l'}'(1)$ from (6.2).

We write p for a cycle of length p in either S_k or S_l and set $\bar{\omega}_k(p) = \omega_k(p)/\omega_l(1)$, $\bar{\omega}_l(p) = \omega_l(p)/\omega_l(1)$, etc.

(6.4) LEMMA. $\bar{\omega}_k(p) = \bar{\omega}_k'(p)$ and $\bar{\omega}_l(p) = \bar{\omega}_l'(p)$ if the cycle length of p is less than n.

Proof. Taking $P = \langle w_1, \ldots, w_{p-1}, w_n \rangle$ we obtain, by an argument similar to Proposition (5.1), that for $p = (1, \ldots, p) = w_1 \cdots w_{p-1}$, $p \leq n-1$, $\eta(p) = \eta'(p)$ and $\eta(pw_n) = \eta'(pw_n)$. Computing $\eta(p)$ and $\eta(pw_n)$ we obtain

$$(6.5) \qquad \eta(p) = 2^{p-1}(p-1)! \frac{|C_W(p)|}{|T_{kl}|} \omega_k(1)\omega_l(1)\left[\binom{k}{p}\bar{\omega}_k(p) + \binom{l}{p}\bar{\omega}_l(p)\right].$$

By a similar calculation,

$$\eta(pw_n) = 2^{p-1}(p-1)! \frac{|C_W(pw_n)|}{|T_{kl}|} \omega_k(1)\omega_l(1)$$

(6.6)

$$\cdot \left[\binom{k}{p}(k-p-l)\bar{\omega}_k(p) - \binom{l}{p}(l-p-k)\bar{\omega}_l(p) \right],$$

where we set $\binom{k}{p}$ or $\binom{l}{p} = 0$ if k or $l < p$. Setting $\eta(p) = \eta'(p)$ and $\eta(pw_n) = \eta'(pw_n)$ the lemma follows since

$$\begin{vmatrix} 1 & 1 \\ k-p-l & -(l-p-k) \end{vmatrix} = 2p \neq 0.$$

(In the case where $\binom{k}{p}$ or $\binom{l}{p} = 0$ the lemma is immediate.)

By Lemma (6.4), in order to complete the proof of Theorem (2.8) for B_n, $n \geq 3$, it is sufficient to prove the following results about characters of the symmetric group S_m. We shall follow the notations in Murnaghan [14]. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of m, with $\lambda_1 \geq \lambda_2 \geq \cdots$, $\sum \lambda_i = m$, and ω_λ the character of S_m associated with λ. Let s be the length of the principal diagonal in the corresponding Young tableau, and let $(b_1, \ldots, b_s)$, $(a_1, \ldots, a_s)$ be defined as in Murnaghan [14, p. 137]. For another character $\omega_{\lambda'}$, we let λ'_i, b'_i, a'_i, etc., be defined in a corresponding way.

(6.7) THEOREM. *Let λ, λ' be partitions of $m \geq 3$, and ω_λ and $\omega_{\lambda'}$ the corresponding irreducible characters of S_m. Suppose $\bar{\omega}_\lambda(p) = \bar{\omega}_{\lambda'}(p)$, for $2 \leq p \leq m-1$. Then $\omega_\lambda = \omega_{\lambda'}$.*

Proof. From [14], $\bar{\omega}_\lambda(p)$ is the coefficient of y^{-1} in

$$y(y-1)\cdots(y-p+1)F(y-p)/F(y),$$

divided by $-pm(m-1)\cdots(m-p+1)$. The function $F(y-p)/F(y)$ has the expansion

$$1 - p[F'(y)/F(y)] + p^2[F''(y)/2!\ F(y)] + \cdots$$

where $F'(y)/F(y) = m/y^2 + c_3/y^3 + c_4/y^4 + \cdots$ and

$$c_{i+1} = \sum_{j=1}^{s} [b_j^i + (-1)^{i-1}(a_j+1)^i], \qquad i = 2, \ldots.$$

It is easily seen that

$$F(y-p)/F(y) = A_{0p} + A_{1p}/y + A_{2p}/y^2 + \cdots$$

where $A_{ip} = \gamma_{ip}c_i + f_i(m, p, c_3, \ldots, c_{i-1})$, $i \geq 3$, with $\gamma_i \neq 0$ and f_i independent of λ. Thus, setting $\bar{\omega}_\lambda(p) = \bar{\omega}_{\lambda'}(p)$, for $p = 2, \ldots, m-1$, we have $c_{i+1} = c'_{i+1}$, $i = 2, \ldots, m-1$, where $c'_{i+1} = \sum_{j=1}^{s} [(b'_j)^i + (-1)^{i-1}(a'_j+1)^i]$. Letting $\bar{a}_i = -(a_i+1)$, we have $c_{i+1} = \sum_{j=1}^{s} (b_j^i - \bar{a}_j^i)$, $i = 2, \ldots$. Now let

$$S_r(t) = \alpha_{r1}t + \alpha_{r2}t^2 + \cdots$$

be the polynomial in t such that $S_r(u) = 1^r + 2^r + \cdots + u^r$, for positive integers u. Then we have, for $1 \leqq r \leqq m-1$, $S_r(0) = 0$, and

$$\alpha_{r1} m + \alpha_{r2} c_3 + \cdots = \sum_{j=1}^{s} [S_r(b_j) + (-1)^r S_r(a_j)],$$

using the identity $S_r(-u) = (-1)^{r-1} S_r(u-1)$ for positive integers u and the fact that $\deg S_r(t) = r+1$. Now, setting $c_{i+1} = c'_{i+1}, 2 \leqq i \leqq m-1$, we have for all r, $1 \leqq r \leqq m-1$,

$$(6.8) \qquad \sum_{j=1}^{s} [S_r(b_j) + (-1)^r S_r(a_j)] = \sum_{j=1}^{s'} [S_r(b'_j) + (-1)^r S_r(a'_j)].$$

Now introduce $m-s$ new variables

$$u_1 = 1, \quad u_2 = 2, \quad \ldots, \quad u_{b_1} = b_1, \quad u_{b_1+1} = 1, \quad \ldots,$$
$$u_{b_1+b_2} = b_2, \quad \ldots, \quad u_{b_1+\cdots+b_s} = b_s,$$
$$u_{b_1+\cdots+b_s+1} = -1, \quad \ldots, \quad u_{b_1+b_2+\cdots+b_s+a_1+\cdots+a_s} = -a_s$$

and similarly introduce $\{u'_j\}_{1 \leqq j \leqq m-s}$. Then (6.8) implies that

$$\sum_{1}^{m-s} u_i^r = \sum_{1}^{m-s'} (u'_i)^r, \qquad 1 \leqq r \leqq m-1.$$

Applying Newton's identities we have first, since none of the variables are zero, that $s = s'$, and the u_i's and u'_i's are the same except for a permutation. Now $b_1 = \max\{u_i\}$, $b'_1 = \max\{u'_i\}$, and the maximums are unique. So $b_1 = b'_1$. Similarly $a_1 = a'_1$. Cancelling $\{1, 2, \ldots, b_1, -1, -2, \ldots, -a_1\}$ from both sides, we can repeat the argument to obtain $a_i = a'_i, b_i = b'_i, 1 \leqq i \leqq s$. This completes the proof of Theorems (6.7) and (2.8) for the case $B_n, n \geqq 3$. The proof of Theorem (2.8) for type B_2 is left as an exercise.

7. **Proof of Theorem (2.9) for B_n.** From Propositions (4.4), (4.5) and (4.7), and the results of §6, the proof of Theorem (2.9) for B_n is reduced to the proofs of Propositions (7.1) and (7.3) below.

(7.1) PROPOSITION. *Let $\eta = (\tilde{\psi}_{kl} \omega_k \omega_l)^W$ be an irreducible character of the Coxeter group W of type B_n, $n > 2$, where, as in §6, $\tilde{\psi}_{kl} \omega_k \omega_l$ is an irreducible character of $T_{kl} = B_k \times B_l$. Then $P = S_k \times B_l$ is a parabolic subgroup of W, $\chi = \omega_k(\tilde{\psi}_{0l} \omega_l)$ is an irreducible character of P, and $(\eta, \chi^W) = 1$.*

Proof. We have $P = \langle w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n \rangle$, and is a parabolic subgroup. We are using the convention that $P = \langle w_1, \ldots, w_n \rangle$ or $\langle w_2, \ldots, w_n \rangle$ if $k = 0$ or 1 respectively, and in these cases ω_k does not appear in χ.

The key to the result is repeated application of Clifford's Theorem [10, Theorem 2]. We first observe that by Frobenius reciprocity every irreducible component of $\chi^{T_{kl}}$ is a component of $\psi_{0l}^{T_{kl}}$, where ψ_{0l} is defined on $N_l = \langle -(k+1), \ldots, -(n) \rangle$, and sends all the generators of N_l to -1. We have, by an easy computation, ψ_{i0}^N

$= \sum_{J \subset K} \psi_{Jl}$, where ψ_{Jl} is the linear character on N which agrees with ψ_{0l} on N_l, and for $J \subset K = \{1, 2, \ldots, k\}$, $\psi_{Jl}(-(i)) = -1$ for $i \in J$ and $\psi_{Jl}(-(i)) = 1$ for $i \notin J$. Then

$$\psi_{0l}^{T_{kl}} = \psi_{kl}^{T_{kl}} + \sum_{J \subset K; J \neq \varnothing} \psi_{Jl}^{T_{kl}}.$$

Therefore, by Clifford's Theorem [10], $\chi^{T_{kl}} = \sum_{s \in S} \lambda_s + \sum_{t \in T} \mu_t$, where the $\{\lambda_s\}_{s \in S}$ are irreducible characters of T_{kl} which are components of $\psi_{kl}^{T_{kl}}$, and the $\{\mu_t\}_{t \in T}$ are irreducible characters of T_{kl} which are components of $\psi_{Jl}^{T_{kl}}$, for $J \neq \varnothing$.

We shall first prove that there is only one character of the form λ_s, and that this character is $\check{\psi}_{kl}\omega_k\omega_l$. An arbitrary character in $\psi_{kl}^{T_{kl}}$ has the form $\check{\psi}_{kl}\omega'_k\omega'_l$, where ω'_k and ω'_l are characters of S_k and S_l respectively. We have $(\check{\psi}_{kl}\omega'_k\omega'_l, \chi^{T_{kl}})_{T_{kl}}$ $= (\check{\psi}_{kl}\omega'_k\omega'_l|_P, \chi)_P$, by Frobenius reciprocity. But $\check{\psi}_{kl}\omega'_k\omega'_l|_P = \omega'_k(\check{\psi}_{0l}\omega'_l)$, so the above multiplicity is zero unless $\omega'_k = \omega_k$, $\omega'_l = \omega_l$, and in that case it is one. Thus

(7.2) $$\chi^{T_{kl}} = \check{\psi}_{kl}\omega_k\omega_l + \sum_{t \in T} \mu_t.$$

We now have, since $(\check{\psi}_{kl}\omega_k\omega_l)^W = \eta$, $\chi^W = \eta + \sum_{t \in T} \mu_t^W$. But it is easily seen that the irreducible components of the μ_t^W are all components of characters $\psi_{k'l'}^W$, with $l' > l$, and cannot contribute another η to the formula for χ^W. This completes the proof.

(7.3) PROPOSITION. *Let P and χ be as in Proposition (7.1). Then there exists a parabolic subgroup P_1 of P such that $(\chi, 1_{P_1}^P) = 1$.*

Proof. From Proposition (5.3), there exist parabolic subgroups R_k and R_l of S_k and S_l respectively, such that $(1_{R_k}^{S_k}, \omega_k) = (1_{R_l}^{S_l}, \omega_l) = 1$. Using the fact that $(\check{\psi}_{0l}\omega_l)|_{R_l} = \omega_l|_{R_l}$ and Frobenius reciprocity, we have $(1_{R_l}^{B_l}, \check{\psi}_{0l}\omega_l) = 1$. The conclusion of Proposition (7.3) follows by taking $P_1 = R_k \times R_l$.

In applying Proposition (4.5), it is essential to have the fact that the characters of a generic algebra of type $A_k \times B_l$ are rational, and this follows from Proposition (4.7) and the results in §§5 and 6. This completes the proof of Theorem (2.9) for type B_n, $n > 2$. The case of B_2 is easily handled since every character is of parabolic type.

8. Rationality of characters of a generic algebra of type D_n.

Let (W, R) be of type D_n, $n \geq 4$. We shall view W as a subgroup of index two in the Coxeter group $W(B_n)$ of type B_n; and may take the set R to be $\{w_1, \ldots, w_n\}$, with $w_i = (i, i+1)$, $1 \leq i \leq n-1$, and $w_n = (n, -(n-1))$. From Carter [2, Proposition 25], two elements of W are conjugate (in W) if and only if they have the same signed cycle type, with the exception that there are two classes having the same signed cycle type if all the cycles are even and positive. As in §§5 and 6, we shall prove that an irreducible character ζ of W is determined by its multiplicities $(\zeta, 1_{W_J}^W)$, for subsets $J \subset R$. We shall make constant use of the corresponding results for $W(B_n)$.

(8.1) PROPOSITION. *Let ζ, ζ' be irreducible characters of W such that $(\zeta-\zeta', 1_{W_J}^W)$ $=0$ for all $J \subset R$. Then $\zeta(x) = \zeta'(x)$ for all elements x belonging to the parabolic subgroups $\langle w_1, \ldots, w_{n-1} \rangle$, and $\langle w_1, \ldots, w_{n-3}, w_{n-1}, w_n \rangle$.*

The proof is similar to that of Proposition (5.1) and is omitted.

We shall use the notation ψ^* for $\psi^{W(B_n)}$ for characters ψ of subgroups of $W(B_n)$. Then $\zeta^* | W = \zeta + \zeta^g$, for $g \in W(B_n)$, $g \notin W$. It follows that if $\zeta = \zeta^g$, then $\zeta^* = \eta + \varepsilon\eta$, for some irreducible character η of B_n, and a linear character $\varepsilon \neq 1$ of $W(B_n)$ such that $\varepsilon | W = 1$. If $\zeta \neq \zeta^g$, then ζ^* is irreducible.

We shall first prove Theorem (2.8) for W of type D_n, with $n \geq 4$ and n odd. In this case, two elements in W which are conjugate in $W(B_n)$ are conjugate in W, since not all cycles can be even in this case. It follows that if ζ and ζ' are as in Proposition (8.1), then $\zeta = \zeta^g$ for all $g \in W(B_n)$, and hence

$$\zeta^* = \eta + \varepsilon\eta, \qquad (\zeta')^* = \eta' + \varepsilon\eta',$$

for irreducible characters η and η' of $W(B_n)$. Moreover, by Proposition (8.1), $\eta(x) = \eta'(x)$ for x in one of the parabolic subgroups of W listed in Proposition (8.1).

(8.2) PROPOSITION. *Let the characters η and η' defined above have the forms $\eta = (\tilde{\psi}_{kl}\omega_k\omega_l)^*$, $\eta' = (\tilde{\psi}_{k'l'}\omega_{k'}'\omega_{l'}')^*$. Then $k = k'$ and $l = l'$ or $k = l'$ and $l = k'$.*

Proof. From Proposition (8.1) we have $\eta(1) = \eta'(1)$ and $\eta(w_{n-1}w_n) = \eta'(w_{n-1}w_n)$. We have

$$\eta(1) = [W(B_n):T_{kl}]\omega_k(1)\omega_l(1),$$

and

$$\eta(w_{n-1}w_n) = \frac{|C_{W(B_n)}(w_{n-1}w_n)|\omega_k(1)\omega_l(1)}{|T_{kl}|}\left[\binom{k}{2}+\binom{l}{2}-kl\right].$$

From these formulas we obtain

$$k(k-1)+l(l-1)-2kl = k'(k'-1)+l'(l'-1)-2k'l',$$

and the proposition follows.

The linear character ε has the property that $\varepsilon(-(i)) = -1$, $1 \leq i \leq n$. Therefore

$$\varepsilon\eta = \varepsilon(\tilde{\psi}_{kl}\omega_k\omega_l)^* = [(\varepsilon|T_{kl})(\tilde{\psi}_{kl}\omega_k\omega_l)]^*.$$

Since $(\varepsilon|T_{kl})\tilde{\psi}_{kl}$ is conjugate by $W(B_n)$ to $\tilde{\psi}_{lk}$ we have

$$(8.3) \qquad \varepsilon\eta = \varepsilon(\tilde{\psi}_{kl}\omega_k\omega_l)^* = (\tilde{\psi}_{lk}\omega_l\omega_k)^*.$$

Therefore, by Proposition (8.2), we may assume that $k = k'$ and $l = l'$.

(8.4) PROPOSITION. *Let η and η' be as in Proposition (8.2), with $k = k'$, $l = l'$. Then $\bar{\omega}_k(p) = \bar{\omega}_l(p)$, $\bar{\omega}_l(p) = \bar{\omega}_{l'}(p)$, for cycles p in S_k or S_l respectively of lengths $1, 2, \ldots, n-2$.*

Proof. Recall that $\bar{\omega}_k(p)$ means $\omega_k(p)/\omega_k(1)$, etc. From Proposition (8.1) and the remark before Proposition (8.2) we have

$$(8.5) \qquad \eta(p) = \eta'(p), \qquad \eta(pw_{n-1}w_n) = \eta'(pw_{n-1}w_n).$$

By a calculation similar to the one used to derive (6.6) we have

$$\eta(pw_{n-1}w_n) = \frac{|C_{W(B_n)}(pw_{n-1}w_n)|2^{p-1}(p-1)!\,\omega_k(1)\omega_l(1)}{|T_{kl}|}$$

$$(8.6) \qquad \cdot \left\{ \binom{k}{p}\left[\binom{k-p}{2}+\binom{l}{2}\right] - (k-p)l \right]\bar{\omega}_k(p)$$

$$+ \binom{l}{p}\left[\binom{k}{2}+\binom{l-p}{2}\right] - (l-p)k \right]\bar{\omega}_l(p) \right\}.$$

The proposition follows from (6.5) and (8.6) since either $k<p$ or $l<p$, or the coefficient determinant for the resulting system of equations for $\bar{\omega}_k(p) - \bar{\omega}'_k(p)$ and $\bar{\omega}_l(p) - \bar{\omega}'_l(p)$ is $c(k-l)$, for some positive constant c, and since n is odd, $k-l \neq 0$.

In the proof of Theorem (6.7), the hypothesis actually used was that $\bar{\omega}_\lambda(p) = \bar{\omega}_{\lambda'}(p)$, for $p=2,\ldots,\max\{m-s, m-s'\}$, where s and s' are the lengths of the principal diagonals in the Young tableaus for the partitions λ and λ'. Therefore, from Proposition (8.4), we can apply Theorem (6.7) to conclude that $\omega_k=\omega'_k$ and $\omega_l=\omega'_l$, and hence $\zeta=\zeta'$, or else k or l is equal to n. In the latter case, we have $\eta(p)=\eta'(p)$ for all p, and hence by Theorem (6.7) again, we obtain $\zeta=\zeta'$. This completes the proof of Theorem (2.8) for type D_n, with n odd.

We assume now that n is even, and as before, that ζ and ζ' are irreducible characters of W such that $(\zeta-\zeta', 1_{W_J}^W)=0$ for all $J \subset R$. Let $g=-(n)$; then $W(B_n) = W \cup gW$. We shall consider three cases.

Case 1. $\zeta^g=\zeta$, $(\zeta')^g=\zeta'$. We proceed as in the case of n odd, using Propositions (8.2) and (8.4). In case $k \neq l$, we obtain $\zeta=\zeta'$ as before. Now suppose $k=l$. From (6.5) we have (recalling that ω_k and ω_l are not necessarily the same),

$$(8.7) \qquad \bar{\omega}_k(p) + \bar{\omega}_l(p) = \bar{\omega}'_k(p) + \bar{\omega}'_l(p),$$

for $p=2,\ldots,n-2$. Now let p and q be disjoint cycles in $\langle w_1,\ldots,w_{n-3}\rangle$ of lengths p and q respectively. Then by Proposition (8.1), we have

$$(8.8) \qquad \eta(pq) = \eta'(pq), \qquad \eta(pqw_{n-1}w_n) = \eta'(pqw_{n-1}w_n).$$

In case p and q have different lengths, we have

$$(8.9) \qquad \eta(pq) = |T_{kl}|^{-1}2^{p+q-2}(p-1)!\,(q-1)!\,\omega_k(1)\omega_l(1)|C_{W(B_n)}(pq)|F_{pq}$$

where

$$F_{pq} = \left[\binom{k}{p}\binom{l}{q}\bar{\omega}_k(p)\bar{\omega}_l(q) + \binom{k}{q}\binom{l}{p}\bar{\omega}_k(q)\bar{\omega}_l(p)\right.$$

$$\left. + \binom{k}{p}\binom{k-p}{q}\bar{\omega}_k(pq) + \binom{l}{p}\binom{l-p}{q}\bar{\omega}_l(pq)\right].$$

If p and q have equal lengths then the formula for $\eta(pq)$ has to be multiplied by $\frac{1}{2}$. Similarly, for p and q of unequal length, we have

$$(8.10) \quad \begin{aligned} &\eta(pqw_{n-1}w_n) \\ &= |T_{kl}|^{-1}2^{p+q-2}(p-1)!\,(q-1)!\,\omega_k(1)\omega_l(1)|C_{W(B_n)}(pqw_{n-1}w_n)|G_{pq}, \end{aligned}$$

where

$$\begin{aligned} G_{pq} = \Bigg\{ &\binom{k}{p}\binom{l}{q}\left[\binom{k-p}{2}+\binom{l-q}{2}-(k-p)(l-q)\right]\bar{\omega}_k(p)\bar{\omega}_l(q) \\ &+\binom{k}{q}\binom{l}{p}\left[\binom{k-q}{2}+\binom{l-p}{2}+(k-q)(l-p)\right]\bar{\omega}_k(q)\bar{\omega}_l(p) \\ &+\binom{k}{p}\binom{k-p}{q}\left[\binom{k-p-q}{2}+\binom{l}{2}-(k-p-q)l\right]\bar{\omega}_k(pq) \\ &+\binom{l}{p}\binom{l-p}{q}\left[\binom{l-p-q}{2}+\binom{k}{2}-(l-p-q)k\right]\bar{\omega}_l(pq)\Bigg\}. \end{aligned}$$

Again, if p and q have the same length, the expression is multiplied by $\frac{1}{2}$.

Since $k=l$, (8.8), (8.9) and (8.10) give a system of homogeneous equations in

$$(8.11) \quad A_{pq} = (\bar{\omega}_k(p)\bar{\omega}_l(q)+\bar{\omega}_k(q)\bar{\omega}_l(p))-(\bar{\omega}'_k(p)\bar{\omega}'_l(q)+\bar{\omega}'_k(q)\bar{\omega}'_l(p))$$

and

$$B_{pq} = (\bar{\omega}_k(pq)+\bar{\omega}_l(pq))-(\bar{\omega}'_k(pq)+\bar{\omega}'_l(pq)).$$

The coefficient determinant is a nonzero multiple of pq, and we obtain that $A_{pq}=0$ in (8.11) for all disjoint cycles p and q in $\langle w_1,\ldots,w_{n-3}\rangle$. In particular, if p and q have the same length, then, viewing ω_k, ω'_k, ω_l, ω'_l as characters of the symmetric group S_k, and p as a p-cycle in S_k, we have, for $p=2,\ldots,k-1$,

$$(8.12) \quad \bar{\omega}_k(p)\bar{\omega}_l(p) = \bar{\omega}'_k(p)\bar{\omega}'_l(p).$$

Combining (8.12) with (8.7) we have

$$(8.13) \quad \{\bar{\omega}_k(p),\bar{\omega}_l(p)\} = \{\bar{\omega}'_k(p),\bar{\omega}'_l(p)\},$$

for $2 \leqq p \leqq k-1$. If either $\bar{\omega}_k(p)=\bar{\omega}'_k(p)$, $2 \leqq p \leqq k-1$, or $\bar{\omega}_k(p)=\bar{\omega}'_l(p)$, $2 \leqq p \leqq k-1$, we have, by Theorem (6.7) and (8.13), that $\omega_k=\omega'_k$ and $\omega_l=\omega'_l$ or $\omega_k=\omega'_l$ and $\omega_l=\omega'_k$, respectively. Otherwise there exist cycles p and q, of lengths less than k, such that

$$(8.14) \quad \begin{aligned} \bar{\omega}_k(p) &= \bar{\omega}'_l(p), & \bar{\omega}_k(p) &\neq \bar{\omega}'_k(p) \\ \bar{\omega}_k(q) &= \bar{\omega}'_k(q), & \bar{\omega}_k(q) &\neq \bar{\omega}'_l(q). \end{aligned}$$

From (8.13) and (8.14), we have

$$(8.15) \quad \bar{\omega}_l(p) = \bar{\omega}'_k(p), \qquad \bar{\omega}_l(q) = \bar{\omega}'_l(q).$$

Using the fact that $A_{pq}=0$ in (8.11), together with (8.14) and (8.15), yields

$$\bar{\omega}_k(p)\bar{\omega}_l(q)+\bar{\omega}_k(q)\bar{\omega}_l(p) = \bar{\omega}_l(p)\bar{\omega}_l(q)+\bar{\omega}_k(q)\bar{\omega}_k(p),$$

from which it follows that

$$[\bar{\omega}_k(p)-\bar{\omega}_l(p)][\bar{\omega}_l(q)-\bar{\omega}_k(q)] = 0.$$

If $\bar{\omega}_k(p)=\bar{\omega}_l(p)$ then from (8.15) we have $\bar{\omega}_k(p)=\bar{\omega}'_k(p)$ contradicting (8.14). Similarly, if $\bar{\omega}_l(q)=\bar{\omega}_k(q)$ we obtain from (8.15) that $\bar{\omega}_k(q)=\bar{\omega}'_l(q)$, contradicting (8.14) again. This completes the discussion of Case 1.

Case 2. $\zeta \neq \zeta^g$, $\zeta' \neq (\zeta')^g$. In this case $\eta=\zeta^*$ and $\eta'=(\zeta')^*$ are both irreducible, and $\eta \mid W=\zeta+\zeta^g$, $\eta' \mid W=\zeta'+(\zeta')^g$.

(8.16) PROPOSITION. *Suppose* $(\zeta-\zeta', 1_P^W)=0$ *for all parabolic subgroups P of W. Then* $(\zeta^g-(\zeta')^g, 1_P^W)=0$ *for all such P.*

Proof. It is sufficient to prove that if P is a parabolic subgroup of W, then so is P^g, for $g=-(n)$. The proposition follows since $w_i^g=w_i$, $1 \leq i \leq n-2$, and $w_{n-1}^g=w_n$ $w_n^g=w_{n-1}$.

Proposition (8.16) implies that $(\eta-\eta', 1_P^*)=0$ for all parabolic subgroups P of W. Using Proposition (8.2) and the discussion in Case 1, we obtain either $\eta=\eta'$ or, by (8.3), $\eta=\varepsilon\eta'$ where $\varepsilon \mid W=1$. Restricting to W gives $\zeta=\zeta'$ or $\zeta^g=\zeta'$. Suppose $\zeta^g=\zeta'$. Then $(\zeta^g-\zeta, 1_P^W)=0$ for all parabolic subgroups P of W.

(8.17) PROPOSITION. *If* $(\zeta^g-\zeta, 1_P^W)=0$ *for all parabolic subgroups P of W, then* $\zeta=\zeta^g$.

Proof. If $\zeta(x) \neq \zeta^g(x)$, then all cycles of x are even and positive, by Carter [2, Proposition 25]. There exists an element y of the same cycle type as x in W_1 $=\langle w_1, \ldots, w_{n-1}\rangle$. By Carter [2, Proposition 25], x is conjugate in W either to y or y^g, since $W(B_n)=W \cup gW$. Now, W_1 and $W_1^g=\langle w_1, \ldots, w_{n-2}, w_n\rangle$ are both parabolic in W and are of type A_{n-1}. By an argument similar to Proposition (5.1), we conclude that if $\zeta(y) \neq \zeta^g(y)$, for $y \in W_1$, there exists a parabolic subgroup P_1 of W_1 such that $(\zeta-\zeta^g, 1_{P_1}^W) \neq 0$. This completes the proof of Proposition (8.17), and finishes the discussion of Case 2, since it now follows that $\zeta=\zeta'$.

Case 3. $\zeta \neq \zeta^g$, $\zeta'=(\zeta')^g$. In this case, Proposition (8.16) implies that $(\zeta^g-\zeta', 1_P^W)=0$ for all parabolic subgroups $P \subset W$. Thus $(\zeta^g-\zeta, 1_P^W)=0$ for all such P, and $\zeta=\zeta^g$ by Proposition (8.17), contradicting the hypothesis. This completes the proof of Theorem (2.8) for type D_n, $n \geq 4$.

9. Proof of Theorem (2.9) for D_n. The proof of Theorem (2.9) for $W=D_n$, $n \geq 4$, follows from Propositions (7.1), (9.1), and (9.6), together with the results in §§4 and 8.

(9.1) PROPOSITION. *Let ζ be an irreducible character of W of type D_n, $n \geq 4$. Then there exists a parabolic subgroup P of W and an irreducible character μ of P such that* $(\zeta, \mu^W)=1.$

Proof. Let $\eta = (\check{\psi}_{kl}\omega_k\omega_l)^*$ be an irreducible component of ζ^*. From Proposition (7.1), $\chi = \omega_k(\check{\psi}_{0l}\omega_l)$ is an irreducible character of $S_k \times B_l$ and $(\eta, \chi^*) = 1$. Let $P = S_k \times D_l$. Then P is a parabolic subgroup of W and $P^g = P$ for $g = -(n)$. Let λ be an irreducible component of $\chi|_P$. We consider four cases:

Case 1. $\zeta^g \neq \zeta$, $\lambda^g \neq \lambda$. Then $\eta = \zeta^*$ and

$$(9.2) \qquad\qquad \lambda^W = a\zeta + b\zeta^g + \xi,$$

where a and b are nonnegative integers and neither ζ nor ζ^g is an irreducible component of ξ. Thus $(\lambda^*, \eta) = (a+b)$. Also $(\lambda^*, \eta) = (\chi^*, \eta) = 1$ since $\lambda^{S_k \times B_l} = \chi$. If $a = 1, b = 0$ we set $\mu = \lambda$. If $a = 0, b = 1$ we set $\mu = \lambda^g$. Then $(\mu^W, \zeta) = 1$.

Case 2. $\zeta^g \neq \zeta$, $\lambda^g = \lambda$. We have (9.2) and

$$(9.3) \qquad\qquad \lambda^{S_k \times B_l} = \chi + \varepsilon\chi,$$

where $\varepsilon \neq 1$ is the linear character of $W(B_n)$ which is trivial on W. Since $\eta = \zeta^*$ and $\varepsilon|W = 1$, we have $\varepsilon\eta = \eta$ and thus $(\lambda^{B_n}, \eta) = 2$. Since $(\lambda^W)^g = (\lambda^g)^W = \lambda^W$, we have from (9.2) that $a = b$. Therefore, the proposition follows in this case with $\mu = \lambda$.

Case 3. $\zeta = \zeta^g$, $\lambda \neq \lambda^g$. In this case $\zeta^* = \eta + \varepsilon\eta$ and

$$(9.4) \qquad\qquad \lambda^W = a\zeta + \xi',$$

where a is a nonnegative integer and ζ is not an irreducible component of ξ'. Then $(\lambda^*, \eta) = a$. But $\lambda^{S_k \times B_l} = \chi$ and hence $(\lambda^*, \eta) = (\chi^*, \eta) = 1$. Again the proposition follows with $\mu = \lambda$.

Case 4. $\zeta = \zeta^g$, $\lambda = \lambda^g$. Both (9.3) and (9.4) hold in this case and $(\lambda^*, \eta) = (\lambda^*, \varepsilon\eta) = a$. From (9.3) and (7.2) we have

$$(9.5) \qquad \lambda^{T_{kl}} = \check{\psi}_{kl}\omega_k\omega_l + \varepsilon\check{\psi}_{kl}\omega_k\omega_l + \sum_{t \in T}\theta_t + \varepsilon\left(\sum_{t \in T}\theta_t\right)$$

where $\eta = (\check{\psi}_{kl}\omega_k\omega_l)^*$, and $\{\theta_t\}_{t \in T}$ are irreducible characters of T_{kl} which are irreducible components of $\psi_{J,l}^{T_{kl}}$, for $J \neq \varnothing$. From the discussion in §7, and (8.3), $(\sum_{t \in T}\theta_t)^*$ can contain $\varepsilon(\check{\psi}_{kl}\omega_k\omega_l)^* = (\check{\psi}_{lk}\omega_l\omega_k)^*$ only if $k > l$. We may assume, however, that in choosing η, $k \leq l$. Thus $(\lambda^*, \varepsilon\eta) = 1$ and thus $(\zeta, \mu^W) = 1$ for $\mu = \lambda$.

(9.6) PROPOSITION. *Let μ and P be as in the proof of Proposition (9.1). Then there exists a parabolic subgroup P_1 of P such that $(\mu, 1_{P_1}^P) = 1$.*

Proof. Let R_k, R_l, and χ be as in the proof of Proposition (7.3). Then we have $(\chi, 1_{R_k \times R_l}^{S_k \times B_l}) = 1$. If $\mu^{S_k \times B_l} = \chi$, then $(\chi|_{R_k \times R_l}, 1_{R_k \times R_l}) = ((\mu + \mu^g)|_{R_k \times R_l}, 1_{R_k \times R_l}) = 1$. Now both $R_k \times R_l$ and $(R_k \times R_l)^g$ are parabolic subgroups of W, and by setting P_1 equal to one of these groups we may ensure $(\mu, 1_{P_1}^P) = 1$.
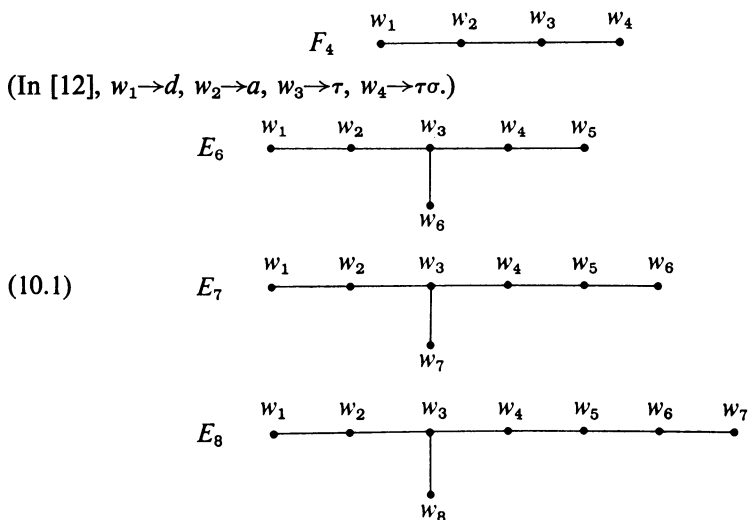
If $\mu^{S_k \times B_l} = \chi + \varepsilon\chi$, we have $(\varepsilon\chi, 1_{R_k \times R_l}^{S_k \times B_l}) = (\chi, 1_{R_k \times R_l}^{S_k \times B_l}) = 1$. Therefore $(\mu^{S_k \times B_l}, 1_{R_k \times R_l}^{S_k \times B_l}) = 2$ and since $\mu^{S_k \times B_l}|_P = 2\mu$, we have $(2\mu|_{R_k \times R_l}, 1_{R_k \times R_l}) = 2$. Thus, setting $P_1 = R_k \times R_l$ we have $(\mu|_{P_1}, 1_{P_1}) = (\mu, 1_{P_1}^P) = 1$. This completes the proof of Theorem (2.9) for W of type D_n.

10. Proofs of Theorems (2.8) and (2.9) for the exceptional groups.

If (W, R) is of type G_2, F_4, E_6, E_7, or E_8, it can be verified directly that the irreducible characters,

ζ, are determined by the multiplicities, $(\zeta, 1_P^W)$, P parabolic, with the exception of the two characters of degree 2 in the case of G_2, and the two characters of degree 512 in the case of E_7. If (W, R) is of type G_2 with $R = \{r, s\}$, it is shown in [4] that the representation of the generic ring corresponding to the reflection representation W is in $Q(u^{(c_r + c_s)/2})$. Since we have assumed that $c_r + c_s$ is even, this representation is rational, and hence so is the representation corresponding to the other irreducible character of W of degree 2. Thus Theorem (2.8) is established.

To prove Theorem (2.9) for the exceptional groups, it is sufficient, by the remarks at the end of §4, to show that for each irreducible character ζ of W, ζ is either of parabolic type or there exists a parabolic subgroup P such that $(\zeta, \varepsilon_P^W) = 1$, where ε_P is the alternating character ε of W, restricted to P. Since for each irreducible character ζ of W, we have $(\varepsilon\zeta, 1_P^W) = (\zeta, \varepsilon_P^W)$, it is sufficient to show that either ζ or $\varepsilon\zeta$ is of parabolic type.

In the following tables we list one of the irreducible characters ζ, or $\varepsilon\zeta$, of W followed by the Coxeter element w_P of a parabolic subgroup P such that $(\zeta, 1_P^W)$ or $(\varepsilon\zeta, 1_P^W) = 1$. We omit the case of G_2 where it may be verified that every irreducible character is of parabolic type. In the case of F_4 all irreducible characters are listed. In labelling the irreducible characters we follow the notation in [12], [8], and [9], except that we write ζ' for $\varepsilon\zeta$ and in the case of F_4 we write $\chi_{d,i}$ for the ith character of degree d given in the character table in [12]. The generators in R are labeled as in (10.1):

F_4

(In [12], $w_1 \to d$, $w_2 \to a$, $w_3 \to \tau$, $w_4 \to \tau\sigma$.)

E_6

(10.1)

E_7

E_8

(10.2) F_4:

ζ	w_P	ζ	w_P
χ_{11}	$w_1 w_2 w_3 w_4$	χ_{61}	$w_1 w_2$
χ_{12}	$w_1 w_2 w_3 w_4$	χ_{62}	$w_2 w_3$
χ_{13}	$w_1 w_2 w_3 w_4$	$\chi_{12,1}$	$w_2 w_3$
χ_{14}	$w_1 w_2 w_3 w_4$	χ_{42}	$w_1 w_2 w_3$

χ_{21}	w_3	χ_{43}	w_3
χ_{22}	w_3	χ_{44}	w_1
χ_{23}	w_1	χ_{45}	w_1
χ_{24}	w_1	χ_{81}	w_3w_4
χ_{41}	w_2w_3	χ_{82}	w_3w_4
χ_{91}	$w_1w_2w_3$	χ_{83}	w_1w_2
χ_{92}	w_2w_3	χ_{84}	w_1w_3
χ_{93}	w_2w_3	$\chi_{16,1}$	$w_1w_3w_4$
χ_{94}	w_1w_3		

(10.3) E_6:

ζ	w_P	ζ	w_P
1_p	$w_1w_2w_3w_4w_5w_6$	$24'_p$	$w_1w_3w_5$
$6'_p$	w_1	60_p	$w_1w_2w_3w_4$
15_p	$w_2w_3w_4w_6$	20_s	$w_1w_3w_5$
20_p	$w_1w_2w_3w_4w_5$	90_s	$w_1w_2w_3w_5$
$30'_p$	$w_1w_3w_5$	80_s	$w_1w_2w_3w_5$
$64'_p$	$w_1w_2w_4$	60_s	$w_1w_2w_3$
81_p	$w_1w_2w_3w_4$	10_s	w_1w_2
$15'_q$	$w_1w_3w_5$		

(10.4) E_7:

ζ	w_P	ζ	w_P
1	$w_1w_2w_3w_4w_5w_6w_7$	405_a	$w_1w_2w_3w_4w_5$
7_a	w_1	$168'_a$	$w_1w_2w_4w_6$
$27'_a$	w_1w_3	56_a	$w_1w_3w_5$
$21'_a$	w_1w_3	$120'_a$	$w_1w_2w_4$
35_a	$w_1w_3w_5$	$210'_a$	$w_1w_2w_4w_6$
105_a	$w_1w_2w_4$	280_a	$w_1w_2w_4w_5$
$189'_a$	$w_1w_2w_4w_5$	336_a	$w_1w_2w_3w_4$
21_b	w_1w_3	216_a	$w_1w_2w_3w_5$
$35'_b$	$w_1w_3w_5$	512_a	$w_1w_2w_3w_4w_6$
189_b	$w_1w_2w_4w_6$	378_a	$w_2w_4w_5w_6w_7$
189_c	$w_1w_2w_4w_6$	$84'_a$	$w_1w_2w_3$
15_a	$w_1w_3w_5$	$420'_a$	$w_1w_2w_3w_4$
$105'_b$	$w_1w_2w_4w_6$	280_b	$w_1w_2w_3w_4w_5$
$105'_c$	$w_1w_2w_4w_5$	210_b	$w_1w_2w_3w_4$
$315'_a$	$w_1w_2w_3w_4w_5$	70_a	$w_1w_2w_4w_5$

(10.5) E_8:

ζ	w_P	ζ	w_P
1_x	$w_1w_2w_3w_4w_5w_6w_7w_8$	4200_y	$w_1w_2w_3w_4w_5w_7$
$28'_x$	w_1w_3	2688_y	$w_1w_2w_3w_4w_5w_7$

$35'_x$	w_1w_3	2100_y	$w_1w_2w_3w_4w_5$
$84'_x$	$w_1w_3w_5$	1400_y	$w_1w_2w_3w_4w_5$
$50'_x$	$w_1w_3w_5w_7$	4536_y	$w_1w_2w_3w_4w_5w_7$
$350'_x$	$w_1w_2w_4w_5$	5670_y	$w_1w_2w_3w_4w_5w_7$
$300'_x$	$w_1w_2w_4w_5$	4480_y	$w_1w_2w_3w_4w_5w_7$
$567'_x$	$w_1w_2w_3$	$8'_z$	w_1
$210'_x$	$w_1w_2w_4$	56_z	$w_1w_2w_3w_4w_5$
840_x	$w_1w_2w_3w_4w_5$	$160'_z$	$w_1w_2w_4$
$700'_x$	$w_1w_2w_4w_5$	$112'_z$	w_1w_2
$175'_x$	$w_1w_2w_4w_5$	$840'_z$	$w_1w_2w_3w_5w_7$
$1400'_x$	$w_1w_2w_3w_5w_7$	$1296'_z$	$w_1w_2w_3w_4$
$1050'_x$	$w_1w_2w_3w_5w_7$	$1400'_z$	$w_1w_2w_3w_5$
$1575'_x$	$w_1w_2w_3w_5w_7$	$1008'_z$	$w_1w_2w_3w_5$
$1344'_x$	$w_1w_2w_3w_5$	$560'_z$	$w_1w_2w_4w_6$
$2100'_x$	$w_1w_2w_3w_4w_6$	$1400'_{zz}$	$w_1w_2w_3w_5w_6$
$2268'_x$	$w_1w_2w_3w_4$	4200_z	$w_1w_2w_3w_4w_5w_6$
$525'_x$	$w_1w_2w_3w_5$	$400'_z$	$w_1w_2w_4w_6w_8$
700_{xx}	$w_1w_2w_3w_4w_5w_7$	$3240'_z$	$w_1w_2w_3w_5w_6$
$972'_x$	$w_1w_2w_3w_5w_7$	$4536'_z$	$w_1w_4w_5w_6w_7w_8$
$4096'_x$	$w_1w_2w_3w_4w_6$	$2400'_z$	$w_1w_4w_5w_6w_7w_8$
$4200'_x$	$w_1w_4w_5w_6w_7w_8$	$3360'_z$	$w_1w_4w_5w_6w_7w_8$
$2240'_x$	$w_1w_2w_5w_6w_7w_8$	$2800'_z$	$w_1w_2w_3w_4w_6$
2835_x	$w_1w_2w_3w_4w_5w_6$	$4096'_z$	$w_1w_2w_3w_4w_6$
$6075'_x$	$w_1w_2w_3w_5w_6w_7$	$5600'_z$	$w_1w_2w_3w_4w_5$
$3200'_x$	$w_1w_2w_3w_4w_5$	$448'_z$	$w_1w_2w_4w_5w_7$
70_y	$w_1w_2w_4w_5$	448_w	$w_1w_2w_3w_4w_6$
1134_y	$w_1w_4w_5w_6w_7w_8$	1344_w	$w_1w_2w_3w_5w_6w_7w_8$
1680_y	$w_1w_2w_3w_4w_5$	5600_w	$w_1w_2w_3w_4w_5w_7$
168_y	$w_1w_2w_5w_6w_7w_8$	2016_w	$w_1w_2w_3w_4w_5$
420_y	$w_1w_2w_3w_4$	7168_w	$w_1w_2w_3w_5w_6w_7w_8$
3150_y	$w_1w_2w_3w_4w_5$		

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