

COMPLEX STRUCTURES ON REAL PRODUCT BUNDLES WITH APPLICATIONS TO DIFFERENTIAL GEOMETRY

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Abstract. The purpose of this paper is to classify holomorphic principal fibre bundles which admit a smooth section (i.e. are real product bundles). This is accomplished if the structure group is solvable of type (E). In the general case, a sufficient condition is obtained for a real product bundle to be equivalent to the complex product bundle. A necessary and sufficient condition for the existence of a holomorphic connection on a real product bundle is also obtained. Using this criterion in the case where the structure group is abelian, a generalization of a theorem due to Atiyah (in the case the structure group is C^*) is obtained.

CHAPTER 0: SUMMARY OF MAIN RESULTS

In his lecture to the International Congress of Mathematicians, 1950, H. Cartan [3] gave a formulation, due to Weil, of the classical second Cousin's problem. The formulation is as follows: Given a holomorphic principal fibre bundle ξ which, when considered as a real (C^∞) bundle, is the trivial (or product) bundle, how can we decide whether ξ is also trivial as a holomorphic bundle? Rephrasing this slightly, if $\text{Pic}(M, G)$ is the set of holomorphic equivalence classes of real trivial holomorphic principal fibre bundles over the complex manifold M with structure group the complex Lie group G , how can we describe $\text{Pic}(M, G)$? If G is abelian then J.-P. Serre showed that

$$\text{Pic}(M, G) = \mathcal{D}_{0,1}(M, \hat{G})/H^1(M, \hat{G}), \quad M \text{ is compact,}$$

where \hat{G} is the Lie algebra of G , and $\mathcal{D}_{0,1}$ denotes the Dolbeault $\bar{\partial}$ cohomology of M . In 1957, J. Frenkel [6] proved that if G is solvable and $\mathcal{D}_{0,1}(M, C) = 0$ then $\text{Pic}(M, G) = 0$. Both Serre's and Frenkel's results use the machinery of sheaf-theoretic cohomology. Our solution to this problem lies in a set $\text{Exp } D(M, G)$ and a map $\Xi: \text{Exp } D(M, G) \rightarrow \text{Pic}(M, G)$ which is always onto and is an isomorphism almost every time that G is solvable.

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We begin by giving some results of a general nature in the first chapter and then applying them to the above case in the second chapter. (We shall number the results as in this paper.)

Let M (resp. G) be a connected complex manifold (resp. complex Lie Group). Let \hat{G} be the Lie algebra of G , $\Lambda^{0,1}(M, \hat{G}) = \hat{G}$ -valued 1-forms on M of type $(0, 1)$. If P is an almost complex manifold with almost complex structure J_0 then $\xi_0: G \rightarrow P \xrightarrow{\pi} M$ is called an almost holomorphic principal fibre bundle (a.h.p.f.b.) if G acts on P in such a way that $\Phi: P \times G \rightarrow P \times P$ is an almost holomorphic diffeomorphism $(p, g) \rightsquigarrow (p, p^g)$ of $P \times G$ onto a closed subset of $P \times P$ and $M = P/G$ (via π). The almost complex structure J on P is said to be *coherent with* ξ_0 if $\xi: G \rightarrow P \xrightarrow{\pi} M$ is an a.h.p.f.b. Let $\text{Ker } \pi$ denote the sub-bundle of $T(P)$ given by $\text{Ker } \pi = \{(p, X) \mid X \in T_p P, p \in P \text{ and } \pi(X) = 0\}$.

THEOREM 1.2.1. *If $\xi_0: G \rightarrow P \xrightarrow{\pi} M$ is an a.h.p.f.b. then there is a 1-1 correspondence between almost complex structures on P coherent with ξ_0 and \hat{G} -valued $(0, 1)$ forms ω (with respect to J_0 -structure) on P such that*

- (1) $\omega(\text{Ker } \pi) = 0$,
- (2) ω is of adjoint type.

In fact, ω is defined by $(J - J_0)_p(X) = \theta_p(\omega_p(X))$ where θ_p is the canonical embedding of \hat{G} onto $(\text{Ker } \pi)_p \subseteq T_p(P)$ for each $p \in P$.

THEOREM 1.2.5. *Let ξ_0 be a holomorphic principal fibre bundle. Let J_1 be an almost complex structure coherent with J_0 . If J_1 corresponds to $\omega \in \Lambda^{0,1}(P, \hat{G})$ then J_1 is a complex structure if and only if the sum of the $(2, 0)$ and $(0, 2)$ components (with respect to J_1) of $d\omega$ is $-(i/4)[\omega, \omega]$. In particular, if J is complex then $\bar{\partial}_{J_1}\omega = -(i/4)[\omega, \omega]$.*

THEOREM 1.2.6. *There is a 1-1 correspondence between $\Lambda^{0,1}(M, \hat{G})$ and almost complex structures on $M \times G$ coherent with the product structure. In fact, this correspondence is given by: for each $\eta \in \Lambda^{0,1}(M, \hat{G})$, $\eta \rightsquigarrow J^\eta$ where*

$$J_{z,\lambda}^\eta(W, Z) = (J_M W, J_G Z + (dR_\lambda)_* \eta(W))$$

for $z \in M$, $\lambda \in G$, $W \in T_z(M)$, $Z \in T_\lambda(G)$.

Let \sim_{exp} be the equivalence relation generated by: for $\omega, \eta \in \Lambda^{0,1}(M, \hat{G})$, $\omega \sim_{\text{exp}} \eta \Leftrightarrow \exists h: M \rightarrow \hat{G}$ such that for all $A \in T_z(M)$

$$2i((e^{-\text{ad } h(z)} - I)/\text{ad } h(z))(\bar{\partial}h(A)) = \omega(A) - e^{-\text{ad } h(z)}\eta(A).$$

ω is *expexact* if $\omega \sim_{\text{exp}} 0$. Let $Z \exp D(M, \hat{G})$ be the set of all $(0, 1)$ forms ω such that ω is locally expexact. Let $\text{Exp } D(M, G) = Z \exp D(M, \hat{G}) / \sim_{\text{exp}}$. We say that M have the *exponential lift* property with respect to G if any map $s: M \rightarrow G$ can be factored through $\exp: \hat{G} \rightarrow G$.

THEOREM 2.2.4. (a) *There is a surjection for any M and G ,*

$$\Xi: \text{Exp } D(M, G) \rightarrow \text{Pic}(M, G)$$

given by $\Xi(\eta) = \{J^n\}$.

(b) *If M has exponential lift with respect to G then Ξ is an isomorphism.*

In §3 of Chapter II, we compute $\text{Exp } D$ under certain circumstances and so are able to prove Serre's, Frenkel's and a converse of Frenkel's result; to wit

THEOREM 2.3.1. *If G is abelian then $\text{Exp } D(M, G) = \mathcal{D}_{0,1}(M, \hat{G})$.*

THEOREM 2.3.2. *If $[\hat{G}, \hat{G}] \neq \hat{G}$ then $\text{Exp } D(M, G) = 0$ implies $\mathcal{D}_{0,1}(M, C) = 0$.*

THEOREM 2.3.4. *If G is solvable then $\mathcal{D}_{0,1}(M, C) = 0$ implies $\text{Exp } D(M, G) = 0$.*

THEOREM 2.3.5. *Let $\eta \in \Lambda^{0,1}(M, \hat{G})$ correspond to J^n as in Theorem 1.2.5. Then the following are equivalent:*

- (a) $\eta \in Z \text{ exp } D(M, G)$,
- (b) J^n is a complex structure,
- (c) $\bar{\partial}\eta = (i/4)[\eta, \eta]$.

In Chapter III, we first apply the above to the theory of holomorphic connections obtaining the following necessary and sufficient condition for the existence of a holomorphic connection on a real product bundle:

THEOREM 3.1.5. *The set of holomorphic connections on $G \rightarrow (M \times G)_J \rightarrow M$ is in 1-1 correspondence with the set of all $\beta \in \Lambda^{1,0}(M, \hat{G})$ such that $\bar{\partial}\beta = \partial\eta - (i/2)[\eta, \beta]$.*

THEOREM 3.1.7. *If G is abelian, M compact Kähler, then $G \rightarrow (M \times G)_J \rightarrow M$ possesses a holomorphic connection.*

The above theorem and also its converse were first proved by Atiyah [1] in the case when $G = C^*$. We prove the converse in the abelian case under the further assumption that M is simply connected.

In the last section we put a natural Hermitian metric on $(M \times G)_J$ and show that if G is a complex torus then $(M \times G)_J$ is Kähler in this metric if and only if $d\eta = 0$.

CHAPTER I: ALMOST HOLOMORPHIC PRINCIPAL FIBRE BUNDLES

1. Preliminaries. We shall first review some of the basic definitions about complex manifolds, mostly to fix terminology. We shall follow [10] as a standard reference. Throughout this work, a manifold (resp. Lie group) will always mean a connected, paracompact and Hausdorff manifold (resp. Lie group). We write \hat{G} for the Lie algebra of the Lie group G . \hat{G} is, in particular, a complex vector space with $\dim_C \hat{G} = \dim_C G$. Let M be a real manifold, $T_R(M)$ its (real) tangent bundle and $T(M) = T_R(M) \otimes C$ be the complex tangent bundle of M , i.e., $T(M) = \{X + iY \mid X, Y \in T_R(M)\}$. In the future by tangent vector, 1-form, etc. we mean complex tangent vector, 1-form, etc. Let J be an almost complex structure on M

(i.e., an endomorphism of $T_R(M)$ such that $J^2 = -I$). We shall occasionally write M_J to mean the almost complex manifold M with structure J . If $\psi: M \rightarrow N$ we shall write $d\psi_z$ or ψ'_z for the differential of ψ at the point $z \in M$; that is $d\psi_z = \psi'_z: T_z(M) \rightarrow T_{\psi(z)}N$. Let V be a complex vector space and let $\Lambda^p(M, V) = \Lambda^p(\text{Hom}_\mathbb{C}(T(M), V))$ be the complex vector space of V -valued p -forms on M and $\Lambda(M, V) = \bigoplus \Lambda^p(M, V)$. If

$$T^{1,0}(M) = \{X \in T(M) \mid JX = iX\} \quad \text{and} \quad T^{0,1}(M) = \{X \in T(M) \mid JX = -iX\},$$

then the eigenspace decomposition of $T(M) = T^{1,0}(M) \oplus T^{0,1}(M)$ induces a decomposition $\Lambda(M, V) = \bigoplus \Lambda^{p,q}(M, V)$. An element $\omega \in \Lambda^{p,q}(M, V)$ is called a form of type (p, q) . In particular, $\omega \in \Lambda^{0,1}(M, V)$ if and only if $\omega(JX) = -i\omega(X)$. Assume now that J is, in fact, a complex structure. Let d be the usual exterior derivative operator on $T_R(M)$ extended to $T(M)$, then for $\omega \in \Lambda^{p,q}(M, V)$ $d\omega$ may be written uniquely as $\partial\omega + \bar{\partial}\omega$ where $\partial\omega \in \Lambda^{p+1,q}(M, V)$, $\bar{\partial}\omega \in \Lambda^{p,q+1}(M, V)$. In particular, if $h: M \rightarrow V$ is a differentiable map then

$$(1.1.1) \quad -2i \bar{\partial}h(X) = dh(J_M X) - i(dh)(X) \quad \text{for all } X \in T(M).$$

Since $\bar{\partial}^2 = 0$ for a complex manifold, we have the Dolbeault cohomology groups $\mathcal{D}_{p,q}(M, V) = \text{Ker } \bar{\partial}_{p,q} / \text{Im } \bar{\partial}_{p,q-1}$ for all p and q .

Let M be a complex manifold, G be a complex Lie group and P an almost complex manifold. We say that $\xi = \{P, G, M; \pi: P \rightarrow M\}$ is an almost holomorphic principal fibre bundle if there is an almost holomorphic action $\Phi: P \times G \rightarrow P \times P$ given by $\Phi(p, g) = (p, p^g)$ which is (1) an almost holomorphic diffeomorphism of $P \times G$ onto $\Phi(P \times G)$; (2) $\Phi(P \times G)$ is a closed subspace of $P \times P$ and (3) $\pi: P \rightarrow M$ induces a biholomorphic homeomorphism of P/G onto M . We shall specify an almost holomorphic principal fibre bundle by writing $\xi: G \rightarrow P \xrightarrow{\pi} M$. If P is a complex manifold then this is the usual definition of a holomorphic principal fibre bundle as in [13]. The equivalence of the two definitions is E. Cartan's Theorem [11, Exposé 6]. For fixed $g \in G$ the map $P \rightarrow P$ given by $p \rightsquigarrow p^g$ will be denoted R_g .

We say that the almost holomorphic principal fibre bundle $\xi: G \rightarrow P \xrightarrow{\pi} M$ is *trivial as a real bundle* (or is a *real product*) if there is a C^∞ section (i.e., a C^∞ map $s: M \rightarrow P$ such that $\pi \circ s$ is the identity on M). Let $\tilde{\xi}: G \rightarrow \tilde{P} \xrightarrow{\tilde{\pi}} M$ be another almost holomorphic principal fibre bundle, then $\tilde{\xi}$ and ξ are (holomorphically) *equivalent* if there is an almost holomorphic diffeomorphism $\psi: \tilde{P} \rightarrow P$ such that

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\psi} & P \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ M & \xrightarrow{\text{identity}} & M \end{array}$$

commutes and $\psi(\tilde{p}^g) = \psi(\tilde{p})^g$ for all $\tilde{p} \in \tilde{P}$, $g \in G$.

Let $G \rightarrow P \xrightarrow{\pi} M$ be an almost holomorphic principal fibre bundle. As before, let $\ker \pi$ denote the sub-bundle of $T(P)$ given by $\ker \pi = \{(p, X) \mid X \in T_p(P), p \in P \text{ and } \dot{\pi}(X) = 0\}$.

THEOREM 1.1.1. *Let \hat{G} be the Lie algebra of G , $\exp: \hat{G} \rightarrow G$ be the usual exponential map, then there is a canonical embedding of \hat{G} onto $(\ker \pi)_p$ for each $p \in P$ given by*

$$(1.1.2) \quad \begin{aligned} \theta_p: \hat{G} &\rightarrow (\ker \pi)_p \subset T_p(P) \\ A &\rightsquigarrow \theta(A), \\ \theta(A)f &= \lim_{t \rightarrow 0} \frac{f(p^{\exp tA}) - f(p)}{t} \quad \text{for any } f: P \rightarrow C. \end{aligned}$$

Furthermore θ_p is of type $(1, 0)$ (that is, $\theta_p(JX) = i\theta_p(X)$ for each $X \in T_p(P)$) and

$$(1.1.3) \quad (R_g^* \theta)_p(A) = \theta_p(\text{ad } g^{-1}(A)) \quad \text{for all } p \in P, g \in G, A \in \hat{G}$$

where $\text{ad}: G \rightarrow \text{End}(\hat{G})$ is the usual adjoint representation ($\text{ad } g = L_g \circ R_{g^{-1}}$).

2. Coherent almost complex structures on fibre bundles. Let $\xi_0: G \rightarrow P_{J_0} \xrightarrow{\pi} M$ be an almost holomorphic principal fibre bundle, where the almost complex structure on P is J_0 . The almost complex structure J on P is said to be *coherent with ξ_0* if $\xi: G \rightarrow P_J \xrightarrow{\pi} M$ is also an almost holomorphic principal fibre bundle, i.e. the action of G on P is holomorphic with respect to both J_0 and J , and the projection π is holomorphic with respect to both J_0 and J .

THEOREM 1.2.1. *Let $\xi_0: G \rightarrow P_{J_0} \xrightarrow{\pi} M$ be an almost holomorphic principal fibre bundle, then there is a 1-1 correspondence between almost complex structures on P coherent with ξ_0 and \hat{G} -valued 1-forms, ω , on P such that*

- (1) $\omega(\ker \pi) = 0$,
- (2) $(R_g^* \omega)(X) = (\text{ad } g^{-1})\omega(X)$ for all $g \in G$ and $X \in T(P)$,
- (3) ω is of type $(0, 1)$ with respect to J_0 .

Proof. Let J be a coherent almost complex structure. If $X \in T_p(P)$, $p \in P$, then

$$\dot{\pi}((J - J_0)(X)) = \dot{\pi}(JX) - \dot{\pi}(J_0X) = J_M(\dot{\pi}X) - J_M(\dot{\pi}X) = 0$$

where J_M is the almost complex structure on M . We then see that $(J - J_0)(X) \in \ker \pi$ for each $X_p \in T(P)$, hence, by Theorem 1.1.1, there is a unique $A \in \hat{G}$ such that $(J - J_0)_p(X) = \theta_p(A)$. Let $\omega_J \in \Lambda^1(P, \hat{G})$ be defined by $(\omega_J)_p(X) = A$ or, more succinctly,

$$(1.2.1) \quad (J - J_0)(X) = \theta(\omega_J(X)) \quad \text{for all } X \in T(P).$$

We shall now show that the assignment $J \rightarrow \omega_J$ is a 1-1 correspondence satisfying the conditions of the theorem.

Firstly, if $X = \theta(A)$ then $(J - J_0)(\theta(A)) = \theta(iA) - \theta(iA) = 0$ so $\omega(X) = 0$ for $X \in \ker \pi$.

Secondly, if $\omega_p(X) = A$ then

$$J(R_g X) - J_0(R_g X) = R_g((J - J_0)X) = R_g \theta_p(A) = \theta_{p^g}((\text{ad } g^{-1})A)$$

where the last equality follows from equation (1.1.3).

Thirdly, we show that $\omega(J_0 X) = -i\omega(X)$. By definition,

$$(1.2.2) \quad \omega_i(X) = A \Leftrightarrow J(X) - J_0(X) = \theta(A),$$

$$(1.2.3) \quad \omega_i(J_0 X) = B \Leftrightarrow J(J_0 X) - J_0(J_0 X) = \theta(B).$$

Applying J to equation (1.2.2) we obtain $-X - JJ_0 X = J\theta(A) = \theta(iA)$ so

$$(1.2.4) \quad J(J_0 X) = -\theta(iA) - X.$$

Substituting (1.2.4) into (1.2.3) yields $(-\theta(iA) - X) + X = \theta(B)$. Hence $B = -iA$ and so $\omega(J_0 X) = -i\omega(X)$.

Conversely: Given any $\omega \in \Lambda^{0,1}(P, \hat{G})$ satisfying (1) and (2) of the proposition, let J^ω be defined at $p \in P$ by

$$(J^\omega)_p(X) = (J_0)_p(X) - \theta_p(\omega_p(X)) \quad \text{for all } X \in T(P).$$

J^ω is clearly linear.

$$\begin{aligned} (J^\omega)^2(X) &= J^\omega(J_0(X) - \theta(\omega(X))) \\ &= J_0(J_0(X) - \theta(\omega(X))) - \theta(\omega(J_0(X) - \theta(\omega(X)))) \\ &= -X - J_0\theta(\omega(X)) - \theta(\omega(J_0 X)) + \theta(\omega(\theta(\omega(X)))) \end{aligned}$$

But $\omega(\theta(\omega(X))) = 0$ since $\theta(\omega(X))$ is vertical; hence the last term is zero so

$$J^2(X) = -X - \theta(i\omega(X)) + \theta(i\omega(X)) = -X$$

where the first equality follows because ω is of type $(0, 1)$ with respect to J_0 .

To show that J^ω is coherent with ξ_0 we need show that (a) $J^\omega \dot{R}_g = \dot{R}_g J^\omega$ for all $g \in G$ and (b) π is holomorphic with respect to J^ω .

Let $X \in T_p(P)$, $p \in P$, then

$$J^\omega \dot{R}_g X = J_0(\dot{R}_g X) - \theta_{p^g} \omega_{p^g}(\dot{R}_g X) = \dot{R}_g J_0 X - \theta_{p^g}((\text{ad } g^{-1})(\omega_p(X)))$$

but

$$= \dot{R}_g J_0 X - \dot{R}_g \theta_p(\omega_p(X)) = \dot{R}_g J^\omega(X);$$

hence (a) follows.

For $X \in T_p(P)$, $\dot{\pi}(J^\omega X) = \dot{\pi}(J_0 X - \theta_p(\omega_p(X))) = \dot{\pi}(J_0 X) - \dot{\pi}(\theta_p(\omega_p(X)))$.

However the last term is zero because $\theta_p(\omega_p(X)) \in (\text{Ker } \pi)$; hence $\dot{\pi}(J^\omega X) = \dot{\pi}(J_0 X) = J_M \dot{\pi} X$ and so π is holomorphic with respect to J^ω .

COROLLARY 1.2.2. *Let G be abelian, then there is a 1-1 correspondence between almost complex structures coherent with $\xi: G \rightarrow P_{J_0} \xrightarrow{\pi} M$ and elements of $\Lambda^{0,1}(M, \hat{G})$.*

Proof. By Theorem 1.2.1 we need only show that there is a 1-1 correspondence between \hat{G} -valued $(0, 1)$ -forms on P such that (1) $\omega(\text{Ker } \pi) = 0$ and (2) $(R_g^* \omega)(X) = \text{ad } g^{-1}(\omega(X)) = \omega(X)$ and $\Lambda^{0,1}(M, \hat{G})$. This correspondence is given as follows: Let $\eta \in \Lambda^{0,1}(M, G)$ be defined by $\eta_m(\tilde{W}) = \omega_p(\tilde{W})$ where p is a point in $\pi^{-1}(m)$ and $\pi(\tilde{W}) = W$. Because of condition (1) above, if $\pi \tilde{W}' = W$ then $\omega_p(\tilde{W}) = \omega_p(\tilde{W}')$. The definition of η is independent of the choice of $p \in \pi^{-1}(m)$ because $\pi_p(\dot{R}_g \tilde{W}) = \dot{\pi}_p \tilde{W} = W$ but

$$\omega_{p^g}(\dot{R}_g \tilde{W}) = (R_g^* \omega)_p(\tilde{W}) = \omega_p(\tilde{W}).$$

It is usual to confuse the set of almost complex structures and equivalence classes of almost complex structures. Please note that Theorem 1.2.1 and the following corollaries assert nothing about *equivalence classes* of almost complex structures; that is, it is easy for ω_1 and $\omega_2 \in \Lambda^{0,1}(P, \hat{G})$ to represent equivalent almost complex structures without $\omega_1 = \omega_2$. We shall investigate the notion of equivalence in Chapter II.

We shall now see which forms ω satisfying conditions (1), (2), and (3) of Theorem 1.2.1 correspond to complex structures. Let N_i be the torsion of the almost complex structure J_i ($i=0, 1$); that is,

$$N_i(X, Y) = 2\{[J_i X, J_i Y] - [X, Y] - J_i[X, J_i Y] - J_i[J_i X, Y]\}.$$

The Newlander-Nirenberg Theorem [10, p. 321] asserts that J_i is a complex structure if and only if $N_i = 0$. We shall therefore compute $N_1 - N_0$ in terms of ω . Note that $N_1 - N_0$ is certainly vertical.

THEOREM 1.2.3. *Let $\xi_0: G \rightarrow P_{J_0} \rightarrow M$ be an almost holomorphic principal fibre bundle. Let J_1 be an almost complex structure corresponding to $\omega \in \Lambda^{0,1}(P, \hat{G})$ as in Theorem 1.2.1; then*

$$(1.2.5) \quad \frac{1}{2} \theta^{-1}(N_1(X, Y) - N_0(X, Y)) = 2 d\omega(J, X, Y) + 2 d\omega(X, J, Y) + [\omega X, \omega Y] \\ \text{for all } X, Y \in T(P).$$

Before proving the above theorem, we shall need some notation. $A^* \in T(P)$ is called a fundamental vector field if there is an $A \in \hat{G}$ such that $A_p^* = \theta_p(A)$ for all $p \in P$. Let $\omega_p^*(X) = \theta_p(\omega(X)) \in \text{Ker } \pi$ for each $X \in T(P)$.

LEMMA 1.2.4. *For all $X, Y \in T(P)$*

- (a) *Let A^* be a fundamental vector field, then $J_0[A^*, X] = [A^*, J_0 X]$.*
- (b) *$\theta^{-1}([\omega^* X, J_0 Y] - J_0[\omega^* X, Y]) = iY(\omega(X)) - (J_0 Y)(\omega(X))$.*
- (c) *$2 d\omega(J_0 X, Y) + 2 d\omega(X, J_0 Y) = -\omega([X, J_0 Y] + [J_0 X, Y]) + (J_0 X)(\omega(Y)) - J_0 Y(\omega(X)) + iY(\omega(X)) - iX(\omega(Y))$.*
- (d) *$2 d\omega(\omega^* X, Y) + 2 d\omega(X, \omega^* Y) = -\omega([\omega^* X, Y] + [X, \omega^* Y]) + (\omega^* X)(\omega Y) - (\omega^* Y)(\omega X)$.*
- (e) *$\theta^{-1}([\omega^* X, \omega^* Y]) = (\omega^* X)(\omega Y) - (\omega^* Y)(\omega X) + [\omega X, \omega Y]$.*

Proof. (a) Let $\phi_t(p) = p^{\exp tA}$ for all $p \in P$, $A \in \hat{G}$, $t \in \mathbb{R}$, then ϕ_t is the one parameter group of diffeomorphisms of P generated by A^* ; hence [9, p. 15]

$$J_0[A^*, X] = J_0\left(\lim_{t \rightarrow 0} \frac{1}{t}(X - \phi_t X)\right).$$

Since the action of G on P is almost holomorphic we have $J_0 \dot{\phi}_t = \dot{\phi}_t J_0$; hence

$$J_0[A^*, X] = \lim_{t \rightarrow 0} \frac{1}{t}(J_0 X - \dot{\phi}_t(J_0 X)) = [A^*, J_0 X].$$

(b) Let $\omega^* X = \sum f^i A_i^*$ where f^i are smooth functions on P , then (using part (a))

$$\begin{aligned} \theta^{-1}([\omega^* X, J_0 Y] - J_0[\omega^* X, Y]) &= \sum \theta^{-1}\{[f^i A_i^*, J_0 Y] - J_0[f^i A_i^*, Y]\} \\ &= \sum \theta^{-1}\{f^i[A_i^*, J_0 Y] - (J_0 Y)f^i A_i^* + f^i(J_0 Y)A_i^* \\ &\quad - J_0(f^i[A_i^*, Y] - Yf^i A_i^* + f^i Y A_i^*)\} \\ &= -(J_0 Y)\omega(X) + iY\omega(X). \end{aligned}$$

(c) $2 d\omega(J_0 X, Y) + 2 d\omega(X, J_0 Y)$

$$\begin{aligned} &= J_0 X \omega(Y) - Y \omega(J_0 X) - \omega([J_0 X, Y]) + X \omega(J_0 Y) - J_0 Y \omega(X) - \omega([X, J_0 Y]) \\ &= J_0 X \omega(Y) + iY \omega(X) - iX \omega(Y) - J_0 Y \omega(X) - \omega([J_0 X, Y] + [X, J_0 Y]) \end{aligned}$$

where the last equality follows because $\omega \in \Lambda^{0,1}(P, \hat{G})$.

(d) $2 d\omega(\omega^* X, Y) + 2 d\omega(X, \omega^* Y)$

$$\begin{aligned} &= (\omega^* X)\omega(Y) - Y\omega(\omega^* X) - \omega([\omega^* X, Y]) \\ &\quad + X\omega(\omega^* Y) - (\omega^* Y)\omega(X) - \omega([X, \omega^* Y]) \\ &= -\omega([\omega^* X, Y] + [X, \omega^* Y]) + (\omega^* X)(\omega Y) - (\omega^* Y)(\omega X) \end{aligned}$$

since $\omega(\text{Ker } \pi) = 0$.

(e) Let $\omega^* X = \sum_i f^i A_i^*$ and $\omega^* Y = \sum_j g^j A_j^*$ then $\omega X = \sum_i f^i A_i$ and $\omega Y = \sum_j g^j A_j$; hence

$$\begin{aligned} &(\omega^* X)(\omega Y) - (\omega^* Y)(\omega X) + [\omega X, \omega Y] \\ &= \sum_{i,j} \{(f^i A_i^*)(g^j A_j) - (g^j A_j^*)(f^i A_i) - [f^i A_i, g^j A_j]\}; \end{aligned}$$

hence, since the last bracket is in \hat{G} ,

$$= \sum_{i,j} \{f^i(A_i^* g^j) A_j - g^j(A_j^* f^i) A_i - f^i g^j [A_i, A_j]\}.$$

Thus

$$\begin{aligned} &\theta((\omega^* X)(\omega Y) - (\omega^* Y)(\omega X) + [\omega X, \omega Y]) \\ &= \sum_{i,j} \{f^i(A_i^* g^j) A_j^* - g^j A_j^* f^i A_i^* - f^i g^j [A_i^*, A_j^*]\} = [\omega^*(X), \omega^*(Y)]. \end{aligned}$$

We shall now prove Theorem 1.2.3. Because $(J_1 - J_0)(X) = \theta(\omega(X)) = \omega^*(X)$ we have

$$\begin{aligned}
 \frac{1}{2}N_1(X, Y) &= [J_0X + \omega^*X, J_0Y + \omega^*Y] - [X, Y] - J_1[X, J_0Y + \omega^*Y] - J_1[J_0X + \omega^*X, Y] \\
 &= [J_0X, J_0Y] + [\omega^*X, \omega^*Y] + [J_0X, \omega^*Y] + [\omega^*X, J_0Y] \\
 &\quad - [X, Y] - J_0[X, J_0Y] - J_0[X, \omega^*Y] - \omega^*([X, J_0Y]) \\
 &\quad - \omega^*([X, \omega^*Y]) - J_0[J_0X, Y] - J_0[\omega^*X, Y] - \omega^*[J_0X, Y] - \omega^*[\omega^*X, Y]. \\
 \frac{1}{2}N_1(X, Y) &= \frac{1}{2}N_0(X, Y) + \{[\omega^*X, \omega^*Y] - \omega^*[X, \omega^*Y] - \omega^*[\omega^*X, Y]\} \\
 (1.2.6) \quad &\quad + \{[\omega^*X, J_0Y] - J_0[\omega^*X, Y]\} + \{[J_0X, \omega^*Y] - J_0[X, \omega^*Y]\} \\
 &\quad - \{\omega^*([X, J_0Y]) + \omega^*([J_0X, Y])\}.
 \end{aligned}$$

We shall now reduce the right-hand side of equation (1.2.5) to the above form.

$$\begin{aligned}
 2 d\omega(J_1X, Y) + 2 d\omega(X, J_1Y) + [\omega X, \omega Y] \\
 = 2 d\omega(\omega^*X, Y) + 2 d\omega(X, \omega^*Y) + 2 d\omega(J_0X, Y) + 2 d\omega(X, J_0Y) + [\omega X, \omega Y].
 \end{aligned}$$

Applying Lemma 1.2.4(c), (d):

$$\begin{aligned}
 &= -\omega\{[\omega^*X, Y] + [X, \omega^*Y]\} + (\omega^*X)(\omega Y) - (\omega^*Y)(\omega X) \\
 &\quad - \omega\{[X, J_0Y] + [J_0X, Y]\} + (J_0X)\omega(Y) - (J_0Y)\omega(X) \\
 &\quad + iY(\omega X) - iX\omega Y + [\omega X, \omega Y].
 \end{aligned}$$

Applying Lemma 1.2.4(b), (e) yields:

$$\begin{aligned}
 &= -\omega\{[\omega^*X, Y] + [X, \omega^*Y]\} + \theta^{-1}([\omega^*X, \omega^*Y]) \\
 &\quad - \omega\{[X, J_0Y] + [J_0X, Y]\} + \theta^{-1}([J_0X, \omega^*Y] - J_0[X, \omega^*Y]) \\
 &\quad + \theta^{-1}([\omega^*X, J_0Y] - J_0[\omega^*X, Y]).
 \end{aligned}$$

Since $\omega = \theta^{-1}\omega^*$ we have

$$\begin{aligned}
 2 d\omega(J_1X, Y) + 2 d\omega(X, J_1Y) + [\omega X, \omega Y] \\
 = \theta^{-1}\{[\omega^*X, \omega^*Y] - \omega^*[X, \omega^*Y] - \omega^*[\omega^*X, Y] + [\omega^*X, J_0Y] \\
 \quad - J_0[\omega^*X, Y] + [J_0X, \omega^*Y] - J_0[X, \omega^*Y] - \omega^*[X, J_0Y] - \omega^*[J_0X, Y]\}
 \end{aligned}$$

which is exactly θ^{-1} applied to the right-hand side of equation (1.2.6).

THEOREM 1.2.5. Let $\xi_0: G \rightarrow P_{J_0} \rightarrow M$ be a holomorphic principal fibre bundle. Let J_1 be an almost complex structure coherent with J_0 . If J_1 corresponds to $\omega \in \Lambda^{0,1}(P, \hat{G})$ then J_1 is a complex structure if and only if $\bar{\partial}\omega + (d\omega)_{2,0} = -(i/4)[\omega, \omega]$. In particular, if J_1 is complex then $\bar{\partial}_{J_1}\omega = -(i/4)[\omega, \omega]$.

Proof. Assume that J_1 is a complex structure, then equation (1.2.5) implies that $2 d\omega(J_1X, Y) + 2 d\omega(X, J_1Y) = -[\omega(X), \omega(Y)]$. Let

$$-\psi(X, Y) = 2 d\omega(J_1X, Y) + 2 d\omega(X, J_1Y);$$

then it suffices to show $\psi = 4i \bar{\partial}\omega$. Since J_1 is complex, $d\omega \in \Lambda_{J_1}^{0,2}(P, \hat{G}) \oplus \Lambda_{J_1}^{1,1}(P, \hat{G})$. If $X, Y \in T_{J_1}^{1,0}(P)$ then $\psi(X, Y) = 4i d\omega(X, Y) = 0$. If $X \in T_{J_1}^{1,0}(P)$ and $Y \in T_{J_1}^{0,1}(P)$ then $\psi(X, Y) = 2i d\omega(X, Y) - 2i d\omega(X, Y) = 0$, therefore $\psi \in \Lambda^{(0,2)}(P, \hat{G})$. Now $d\omega - \psi/4i$ has no $(0, 2)$ component because if $X, Y \in T_{J_1}^{0,1}(P)$ then

$$\begin{aligned} d\omega(X, Y) - \psi(X, Y) &= d\omega(X, Y) - (2i d\omega(X, Y) + 2i d\omega(X, Y))/4i \\ &= d\omega(X, Y) - d\omega(X, Y) = 0. \end{aligned}$$

Since $d\omega = (d\omega - \psi/4i) + \psi/4i$, $\psi/4i$ is of type $(0, 2)$ and $d\omega - \psi/4i$ has no component of type $(0, 2)$ we conclude that $\psi/4i$ is the $(0, 2)$ component of $d\omega$, i.e. $\bar{\partial}\omega = \psi/4i$ because ω is of type $(0, 1)$ with respect to J_1 as well as J_0 .

We shall now prove that if $(i/4)[\omega, \omega]$ is the sum of the $(0, 2)$ and $(2, 0)$ components of $d\omega$ then J_1 is in fact a complex structure. By a procedure similar to the previous one we see that $2 d\omega(J_1 X, Y) + 2 d\omega(X, J_1 Y)$ is $4i$ times the sum of the $(0, 2)$ and $(2, 0)$ components of $d\omega$ hence $2 d\omega(J_1 X, Y) + 2 d\omega(X, J_1 Y) + [\omega X, \omega Y] = 0$ and so $N_1 = 0$ by Theorem 1.2.2.

We shall now examine the real trivial case. If $\xi: G \rightarrow P \xrightarrow{\pi} M$ is an almost holomorphic principal fibre bundle then we say that ξ is *real product* if ξ admits a C^∞ section; that is, a smooth map $s: M \rightarrow P$ such that $\pi \circ s$ is the identity map on M . Clearly if ξ is real product then P is diffeomorphic to $M \times G$. Let J be the complex structure on $M \times G$ given by $J_{z,\lambda}(W, Z) = (J_M W, J_G Z)$ for all $m \in M$, $\lambda \in G$, $W \in T_m(M)$ and $Z \in T_\lambda(G)$; then J is called the *product complex structure* on $M \times G$. Clearly, a bundle ξ is a real product if and only if it is coherent with the product structure on $M \times G$. We now determine all real trivial almost holomorphic principal fibre bundles.

THEOREM 1.2.6. *There is a one-to-one correspondence between $\Lambda^{0,1}(M, \hat{G})$ and almost complex structures on $M \times G$ coherent with the product structure. In fact, this correspondence is given by $\eta \in \Lambda^{0,1}(M, \hat{G})$, $\eta \rightsquigarrow J^\eta$, where*

$$(1.2.9) \quad J_{z,\lambda}^\eta(W, Z) = (J_M W, J_G Z + (dR_\lambda)_e \eta(W))$$

for $z \in M$, $\lambda \in G$, $W \in T_z(M)$, $Z \in T_\lambda(G)$.

Proof. The canonical embedding θ of \hat{G} into $T(P) = T(M \times G) = T(M) \times T(G)$ takes the form $\theta_{z,\lambda}(A) = (0, (dR_\lambda)(A))$, $A \in \hat{G}$, $z \in M$, $\lambda \in G$. Thus let $X = (W, Z) \in T_{z,\lambda}(M \times G)$; then equation (1.2.1) reduces to (with $J_0 = \text{product structure}$)

$$(1.2.10) \quad \omega_f(X) = A \Leftrightarrow J(X) - (J_M W, J_G Z) = (0, (dR_\lambda)_e(A)),$$

i.e. by Theorem 1.2.1 there is a 1-1 correspondence between those $\omega \in \Lambda^{0,1}(P, \hat{G})$ such that $\omega(\text{Ker } \pi) = 0$ and $R_g^* \omega = \text{ad } g^{-1} \omega$ with almost complex structures on $M \times G$ coherent with the product structure and this correspondence is given by equation (1.2.10). We need only show that there is a 1-1 correspondence between those $\omega \in \Lambda^{0,1}(P, \hat{G})$ such that $\omega(\text{Ker } \pi) = 0$ and $R_g^* \omega = \text{ad } g^{-1} \omega$ with $\Lambda^{0,1}(M, \hat{G})$. This is provided by the following lemma.

LEMMA 1.2.7. *If $\xi: G \rightarrow P \rightarrow M$ is a real trivial holomorphic principal fibre bundle then there is a one-to-one correspondence between those elements $\omega \in \Lambda^{0,1}(P, \hat{G})$ such that (a) $\omega(\text{Ker } \pi) = 0$, (b) $R_g^* \omega = (\text{ad } g^{-1})\omega$ and the set of all $\eta \in \Lambda^{0,1}(M, \hat{G})$.*

Proof. Let $s: M \rightarrow P$ be a C^∞ section on ξ . We may assume P is diffeomorphic to $M \times G$, and $s: M \rightarrow M \times G$ is given by $s(m) = (m, e)$ where e is the identity of G . Thus $s^*: \Lambda^{0,1}(P, \hat{G}) \rightarrow \Lambda^{0,1}(M, \hat{G})$. We shall prove that this map is a 1-1 correspondence as in the lemma.

We first show that $s^*(\Lambda^{0,1}(P, \hat{G})) \subset \Lambda^{0,1}(M, \hat{G})$. Let $\omega \in \Lambda^{0,1}(P, \hat{G})$; then

$$(s^*\omega)(J_M A) = \omega(ds(J_M A)) = \omega((J_M A, 0)) = -i\omega(A, 0) = -i(s^*\omega)(A),$$

so $s^*\omega \in \Lambda^{0,1}(M, \hat{G})$.

Suppose $s^*\omega = 0$, then $\omega_{z,e}((A, 0)) = 0$ for all $A \in T_z(M)$. However, by assumption $\omega_{z,e}((0, B)) = 0$; hence $\omega_{z,e}(A, B) = \omega_{z,e}(A, 0) + \omega_{z,e}(0, B) = 0$. However,

$$\omega_{z,\lambda}(A, B) = (\text{ad } \lambda^{-1})\omega_{z,e}(A, B) = 0.$$

We shall now show that s^* is onto. Let $\eta \in \Lambda^{0,1}(M, \hat{G})$. Let $\omega_{z,\lambda}(A, B) = (\text{ad } \lambda^{-1})\eta_z(A)$; then ω satisfies all of the necessary conditions.

CHAPTER II: COMPLEX STRUCTURES ON REAL PRODUCT BUNDLES

1. Statement of the problem and solution in the abelian case. The aim of this chapter is to obtain information on the set of (holomorphic) equivalence classes of principal fibre bundles with structure group G and base M which are trivial as smooth bundles, i.e. equivalence classes of real product bundles.

We shall call this set $\text{Pic}(M, G)$. If M is a Riemann surface and $G = \mathbb{C}^*$ then $\text{Pic}(M, G)$ is called the Picard variety of M [7]. We shall now give the sheaf-theoretic interpretation of $\text{Pic}(M, G)$. Let $H^1(M, \mathcal{G}_h)$ (resp. $H^1(M, \mathcal{G}_\infty)$) denote the first cohomology set of M with coefficients in the sheaf of germs of holomorphic (resp. C^∞) maps from M to G . If $\alpha: \mathcal{G}_h \rightarrow \mathcal{G}_\infty$ is the map obtained by viewing a holomorphic map as a C^∞ map, then α induces a map, also called $\alpha: H^1(M, \mathcal{G}_h) \rightarrow H^1(M, \mathcal{G}_\infty)$. It is well known [4] that the set of holomorphic (resp. C^∞) equivalence classes of holomorphic (resp. C^∞) principal fibre bundles over M with structure group G is equal to $H^1(M, \mathcal{G}_h)$ (resp. $H^1(M, \mathcal{G}_\infty)$) where the trivial holomorphic (resp. C^∞) bundle is denoted by O_h (resp. O_∞); that is, $O_h: G \rightarrow (M \times G) \rightarrow_{\text{Product structure}} M$. The set of equivalence classes of real trivial holomorphic principal fibre bundles is therefore $\ker \alpha = \alpha^{-1}(O_\infty)$.

We will first explain why $\text{Pic}(M, G)$ need not be zero.

Topologically. $\xi: G \rightarrow (M \times G)_J \rightarrow M$ is the general form of a real product bundle where the action of G on $(M \times G)_J$ is, of course, the second coordinate action (i.e. $(z, g)^\lambda = (z, g\lambda)$ for all $z \in M, g, \lambda \in G$). In order for ξ to be holomorphically trivial, the complex structure J must be equivalent to the product of the complex structures of M and G . We will see below that $M \times G$ can be given complex structures which are not equivalent to the product structure.

Analytically. We now view a bundle ξ in terms of its transition functions [13]. ξ is trivial as a C^∞ -bundle if and only if there is an open cover $\{U_\alpha\}$ of M such that the transition functions $\{t_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow G\}$ of ξ with respect to $\{U_\alpha\}$ take the form

$$(2.1.1) \quad t_{\alpha\beta}(z) = s_\alpha(z)s_\beta(z)^{-1}, \quad z \in U_\alpha \cap U_\beta,$$

where $s_\alpha: U_\alpha \rightarrow G$ and $s_\beta: U_\beta \rightarrow G$ are smooth functions. Because ξ is a holomorphic bundle the $t_{\alpha\beta}$ are holomorphic functions but, of course, the s_α need not be holomorphic. In fact, there are holomorphic functions satisfying (2.1.1) exactly when ξ is trivial as a holomorphic bundle. In the case when $G = \mathbf{C}^*$, the problem of finding holomorphic functions s_α satisfying (2.1.1) (given the holomorphic functions $t_{\alpha\beta}$) is called the Cousin problem of several complex variables, and the phrasing of this problem in fibre bundle terminology was first published by H. Cartan [3].

Recall that there is an isomorphism (due to Dolbeault-Serre, [4])

$$\text{dolbeault}: \mathcal{D}_{p,q}(M, V) \xrightarrow{\sim} H^q(M, \Omega^p(V))$$

where V is any vector bundle over M and $\Omega^p(V)$ is the sheaf of germs of holomorphic sections of $\Lambda^{p,0}(M, V)$. For any details about the arguments in sheaf-theoretic cohomology which follow see [4].

When G is abelian, we can obtain the following information: (If M is a compact Riemann surface and $G = \mathbf{C}^* = \{z \in \mathbf{C} \mid z \neq 0\}$ then this result is in [7].)

THEOREM 2.1.1. *Let M be a complex manifold, G an abelian complex Lie group; then the following is an exact sequence of abelian groups*

$$\begin{array}{ccccccc} \xrightarrow{i} & \mathcal{D}_{0,1}(M, \hat{G}) & \xrightarrow{\beta} & H^1(M, G_h) & \xrightarrow{\alpha} & H^1(M, G_\infty) & \\ & & & & & & \\ & & & i \circ \delta_\infty & \longrightarrow & \mathcal{D}_{0,2}(M, \hat{G}) & \longrightarrow \end{array}$$

where $\beta = \exp \circ \text{dolbeault}$.

Proof. Because $\exp: \hat{G} \rightarrow G$ is a covering map (recall all Lie groups are assumed to be connected), we see that the sequence of abelian groups $0 \rightarrow \pi_1(G) \rightarrow \hat{G} \xrightarrow{\exp} G \rightarrow 0$ is exact; hence so are the sequences in cohomology:

$$\begin{array}{ccccccccccc} \longrightarrow & H^1(M, \pi_1(G)) & \longrightarrow & H^1(M, \hat{G}_\infty) & \xrightarrow{\exp} & H^1(M, G_\infty) & \xrightarrow{\delta_\infty} & H^2(M, \pi_1(G)) & \longrightarrow & H^2(M, \hat{G}_\infty) & \longrightarrow \\ & \uparrow \text{identity} & & \uparrow \alpha & & \uparrow \alpha & & \uparrow \text{identity} & & \uparrow \alpha & \\ \longrightarrow & H^1(M, \pi_1(G)) & \longrightarrow & H^1(M, \hat{G}_h) & \xrightarrow{\exp} & H^1(M, G_h) & \xrightarrow{\delta_h} & H^2(M, \pi_1(G)) & \longrightarrow & H^2(M, \hat{G}_h) & \longrightarrow \end{array}$$

However \hat{G} is isomorphic as a Lie group to \mathbf{C}^n , hence \hat{G}_∞ is a fine sheaf and so $H^k(M, \hat{G}_\infty) = 0$ for all $k \geq 1$, hence δ_∞ is an isomorphism. The Dolbeault isomorphism and some diagram chasing completes the proof.

COROLLARY 2.1.2. *Let G be abelian; then $\text{Pic}(M, G)$ has the natural structure of the product of a torus with \mathbf{C}^r .*

Proof. The proof is by diagram chasing.

$$(2.1.2) \quad \text{Pic}(M, G) = \ker \alpha = \text{image } \beta = \mathcal{D}_{0,1}(M, \hat{G})/i(H^1(M, \pi_1(G))).$$

Since $\pi_1(G)$ is free, $i(H^1(M, \pi_1(G)))$ is a lattice in the vector space $\mathcal{D}_{0,1}(M, G)$, i.e. $\text{Pic}(M, G)$ is the quotient of \mathbb{C}^n (n might be infinite) by a discrete subgroup, hence has the structure of a torus crossed with \mathbb{C}^r .

COROLLARY 2.1.3. *If $\pi_1(M)$ is a torsion group and G is abelian then $\text{Pic}(M, G)$ is isomorphic to $\mathcal{D}_{0,1}(M, \hat{G})$.*

Proof. The universal coefficient theorem for cohomology [12] states that

$$0 \rightarrow \text{Ext}(H_0(M), \pi_1(G)) \rightarrow H^1(M, \pi_1(G)) \rightarrow \text{Hom}(H_1(M), \pi_1(G)) \rightarrow 0$$

is exact. However $H_0(M)$ is a free group, so $\text{Ext}(H_0(M), \pi_1(G))=0$ and $H_1(M)$ is torsion (by the Hurewicz isomorphism); hence we conclude $H^1(M, \pi_1(G))=0$ because there are no nonzero homomorphisms from a torsion group to a free group.

Note that if $\pi_1(M)$ is a torsion group and G is abelian then $\text{Pic}(M, G)$ has a natural structure of a (infinite-dimensional) vector space. We shall see that this will contrast violently with the simplest nonabelian case in which $\text{Pic}(M, G)$ is either zero or is not even Hausdorff!

2. Statement and proof of the fundamental theorem. Before starting the main theorem (Theorem 2.2.4), we shall require some definitions. As always, M will be a connected complex manifold and G a connected complex Lie group with Lie algebra \hat{G} . We say that G has the *exponential lift property* with respect to M if for any (C^∞) map $s: M \rightarrow G$ there is a (C^∞) map $h: M \rightarrow \hat{G}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \hat{G} \\ & \nearrow h & \downarrow \exp \\ M & \xrightarrow{s} & G \end{array}$$

We say that a solvable Lie group is of *type (E)* if the exponential map is a covering map. (G is of type (E) if G is nilpotent. If G is solvable there is a criterion due to Dixmier; see [2] for details.)

PROPOSITION 2.2.2. *If $\pi_1(M)$ is a torsion group and G is of type (E), then G has the exponential lift property with respect to M .*

Proof. It is well known, [12], that given $s: M \rightarrow G$ as in (2.2.1) there is $h: M \rightarrow \hat{G}$ such that (2.2.1) commutes if and only if $s_\#(\pi_1(M)) \subset \exp_\#(\pi_1(\hat{G}))=0$ where $\#$ denotes the map induced on the fundamental group π_1 . Since G is solvable, $\pi_1(G)$ is free, but the image under $s_\#$ of the torsion group $\pi_1(M)$ is torsion and hence is zero, so $s: M \rightarrow G$ always lifts.

If $\omega, \eta \in \Lambda^{0,1}(M, \hat{G})$, then we say that ω is *exponentially cohomologous* (or, more briefly, *expcohomologous*) to η (and write $\omega \sim_{\text{exp}} \eta$) if there is a map $h: M \rightarrow \hat{G}$ such that

$$(2.2.2) \quad 2i((I - e^{-\text{ad } h(z)})/\text{ad } h(z))(\bar{\partial}h(A)) = \omega(A) - e^{-\text{ad } h(z)}\eta(A) \quad \text{for all } A \in T_z(M), z \in M.$$

The first term is to be interpreted (for each $z \in M$) as the convergent series

$$\sum_{K=0}^{\infty} \frac{(-1)^K (\text{ad } h(z))^K}{(K+1)!} \in \text{End } (G),$$

all forms are evaluated at $z \in M$ and $(\text{ad } h(u))(X) = [h(u), X]$ for all $X \in \hat{G}$. Note that \sim_{exp} is reflexive ($h=0$) and symmetric (let $f = -h$). I have not been able to show that \sim_{exp} is transitive. We shall use the same symbol to denote the equivalence relation generated by the relation \sim_{exp} of equation (2.2.2). ω is *exponentially exact* (or *expexact*) if $\omega \sim_{\text{exp}} 0$. ω is *exponentially closed* (or *expclosed*) if there is an open cover $\{U_\alpha\}$ of M such that $i_\alpha^* \omega$ is expexact ($i_\alpha: U_\alpha \rightarrow M$ is the obvious inclusion). Let $Z \exp(M, \hat{G})$ be the set of expclosed $(0, 1)$ -forms. We define

$$A \exp D(M, G) = \Lambda^{0,1}(M, \hat{G})/\sim_{\text{exp}}, \quad \exp D(M, G) = Z \exp(M, \hat{G})/\sim_{\text{exp}},$$

then both $A \exp D$ and $\exp D$ are sets with a distinguished element 0 (being the equivalence class of the $(0, 1)$ -form, $\omega \equiv 0$).

PROPOSITION 2.2.3. *Let G be abelian; then $\exp D(M, G) = \mathcal{D}_{0,1}(M, \hat{G})$.*

Proof. If G is abelian then $\text{ad } h(z) = 0$, hence equation (2.2.2) becomes $2i\bar{\partial}h = \omega - \eta$; hence $\omega \sim_{\text{exp}} \eta$ if and only if ω is $\bar{\partial}$ -cohomologous to η . $Z \exp(M, \hat{G})$ is just the $\bar{\partial}$ -cocycles by the Dolbeault-Grothendieck Lemma.

We shall now state the main theorem of this chapter. We shall refer to this as the fundamental theorem.

THEOREM 2.2.4. *Let M be a complex manifold, G a complex Lie group. Let*

$$\Xi: \exp D(M, \hat{G}) \rightarrow \text{Pic } (M, G) \\ [\omega] \rightsquigarrow J^\omega$$

be defined by

$$(2.2.3) \quad J_{z,\lambda}^\omega(A, B) = (J_M A, J_G B + (dR_\lambda)\omega(A))$$

(as in equation (1.2.9)); then

(1) Ξ is onto.

(2) *If G has the exponential lift property with respect to M , then Ξ is an isomorphism.*

It will also be clear from the proof of this theorem that if G has the exponential lift property with respect to M then \sim_{exp} is a true equivalence relation.

We begin with a lemma. Let $\xi: G \rightarrow (M \times G)_{J_1} \rightarrow M$ and $\psi: G \rightarrow (M \times G)_{J_2} \rightarrow M$ be real trivial holomorphic principal fibre bundles. Let J_1 correspond to ω and J_2 correspond to η via Theorem 1.2.6, i.e. $J_1 = J^\omega$ and $J_2 = J^\eta$ in the notation of equations (1.2.9) and (2.2.3).

LEMMA 2.2.5. ξ is equivalent to ψ if and only if there exists a smooth map $\gamma: M \rightarrow G$ such that

$$(2.2.4) \quad J_G d\gamma_z(A) + dR_{\gamma(z)}(\eta_z(A)) = d\gamma_z(J_G A) + dL_{\gamma(z)}(\omega_z(A))$$

for all $A \in T_z(M)$ and all $z \in M$.

Proof. ξ is equivalent to ψ if and only if $\exists \phi: (M \times G)_{J_1} \rightarrow (M \times G)_{J_2}$ which is an equivariant almost holomorphic diffeomorphism which induces the identity on M , i.e. the following diagram commutes:

$$(I) \quad \begin{array}{ccc} (M \times G)_{J_1} & \xrightarrow{\phi} & (M \times G)_{J_2} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\text{identity}} & M \end{array}$$

$$(II) \quad \phi(m, g_1)g_2 = \phi(m, g_1g_2) \quad \text{for all } m \in M, g_1, g_2 \in G.$$

$$(III) \quad \phi J_1 = J_2 \phi.$$

Condition (I) holds $\Leftrightarrow \phi(m, g) = (m, \beta(m, g))$ for some $\beta: M \times G \rightarrow G$. Now condition (II) says $\phi(m, g_1)g_2 = (m, \beta(m, g_1)g_2) = \phi(m, g_1g_2) = (m, \beta(m, g_1g_2))$ so $\beta(m, g_1)g_2 = \beta(m, g_1g_2)$. If $\beta(m, e) = \gamma(m)$, then letting $g_1 = e$, $g_2 = g$ shows that $\beta(m, e)g = \beta(m, g)$ and so ϕ satisfies (I) and (II) if and only if there is a $\gamma: M \rightarrow G$ such that $\phi(m, g) = (m, \gamma(m)g)$. We shall now calculate $\dot{\phi}$ and test condition (III):

$$\dot{\phi}_{z,\lambda}(A, B) = (A, \dot{\beta}_{z,\lambda}(A, B)), \quad A \in T_z(M), B \in T_\lambda(G),$$

where $\beta(m, g) = \gamma(m)g$. Let ${}^z\beta: G \rightarrow G$ be ${}^z\beta(g) = \gamma(z)g$ and $\beta^\lambda: M \rightarrow M$ be $\beta^\lambda(m) = \gamma(m)\lambda$; then ${}^z\beta = L_{\gamma(z)}$ and $\beta^\lambda = R_\lambda \circ \gamma$. The Leibniz formula [9] dictates:

$$\dot{\beta}_{z,\lambda}(A, B) = ({}^z\dot{\beta})_\lambda(B) + (\dot{\beta}^\lambda)_z(A)$$

or

$$(2.2.5) \quad \dot{\beta}_{z,\lambda}(A, B) = \dot{L}_{\gamma(z)}(B) + (R_\lambda \circ \gamma)_z(A).$$

Now

$$\dot{\phi}_{z,\lambda}(J_1(A, B)) = \dot{\phi}_{z,\lambda}(J_M A, J_G B + \dot{R}_\lambda \omega(A))$$

so

$$\dot{\phi}_{z,\lambda}(J_1(A, B)) = (J_M A, \dot{\beta}_{z,\lambda}(J_M A, J_G B + \dot{R}_\lambda \omega(A)))$$

and

$$J_2 \dot{\phi}_{z,\lambda}(A, B) = J_2(A, \dot{\beta}_{z,\lambda}(A, B)) = (J_M A, J_G \dot{\beta}_{z,\lambda}(A, B) + \dot{R}_{\gamma(z)\lambda}(\gamma(A))).$$

Thus $\dot{\phi}J_1 = J_2\dot{\phi}$ if and only if

$$i\dot{\beta}_{z,\lambda}(A, B) + \dot{R}_{\gamma(z)\lambda}\eta(A) = \dot{\beta}_{z,\lambda}(J_MA, J_GB + \dot{R}_\lambda\omega(A)).$$

Hence from (2.2.5), $\dot{\phi}J_1 = J_2\dot{\phi}$ if and only if

$$J_G(\dot{L}_{\gamma(z)}(B) + (R_\lambda \circ \gamma)_z(A)) + \dot{R}_{\gamma(z)\lambda}\eta(A) = \dot{L}_{\gamma(z)}(J_GB + \dot{R}_\lambda\omega(A)) + (R_\lambda \circ \gamma)_z(J_MA).$$

Since left-translation is holomorphic and right and left translation commute we have ξ is equivalent to ψ if and only if there exists $\gamma: M \rightarrow G$ such that

$$J_G\dot{\gamma}_z(A) + \dot{R}_{\gamma(z)}\eta(A) = \dot{L}_{\gamma(z)}\omega(A) + \dot{\gamma}_z(J_MA).$$

PROPOSITION 2.2.6. *In the above notation:*

- (1) *If $\omega \sim_{\exp} \eta$ then ξ is equivalent to ψ .*
- (2) *If G has the exponential lift property with respect to M then $\omega \sim_{\exp} \eta$ if and only if ξ is equivalent to ψ .*

Proof. We shall prove (1) and (2) simultaneously. If $\omega \sim_{\exp} \eta$ then there is an $h: M \rightarrow \hat{G}$ satisfying equation (2.2.2). If $\gamma(z) = \exp(h(z))$ then we claim that γ satisfies equation (2.2.4) of Lemma 2.2.5. If, on the other hand, G has exponential lift with respect to M and ξ is equivalent to ψ then we may find an $h: M \rightarrow \hat{G}$ and a $\gamma: M \rightarrow G$ such that γ satisfies (2.2.4) of Lemma 2.2.5 and $\gamma = \exp \circ h$. In this case we must show that h satisfies equation (2.2.2). These assertions are proven in the following computation: Let $\gamma(z) = \exp h(z)$ then for $X = h(z)$ equation (2.2.4) becomes

$$J_G d(\exp)_X(dh(A)) - d(\exp)_X dh(J_MA) = dL_{\exp X}\omega(A) - dR_{\exp X}\eta(A)$$

or (because \exp is holomorphic)

$$d(\exp)_X(i dh(A) - dh(J_MA)) = dL_{\exp X}\omega(A) - dR_{\exp X}\eta(A)$$

thus from equation (1.1.1) we have

$$2i d(\exp)_X(\bar{\partial}h(A)) = dL_{\exp X}(\omega(A)) - dR_{\exp X}(\eta(A))$$

but [8, p. 95] $d(\exp)_X = d(L_{\exp X})_e \circ ((1 - e^{-\text{ad } X})/\text{ad } X)$ so

$$\begin{aligned} 2i d(L_{\exp X})_e \circ ((1 - e^{-\text{ad } X})/\text{ad } X) \bar{\partial}h(A) &= d(L_{\exp X})\omega(A) - dR_{\exp X}\eta(A), \\ 2i((I - e^{-\text{ad } X})/\text{ad } X) \bar{\partial}h(A) &= \omega(A) - d(L_{\exp(-X)} \circ R_{\exp X})\eta(A). \end{aligned}$$

However $d(L_{\exp(-X)} \circ R_{\exp X}) = \text{ad}(\exp X)$ which is known [5] to be $e^{\text{ad } X}$; hence, after identifying $T(G)$ and \hat{G} , we obtain

$$2i((I - e^{-\text{ad } h(z)})/\text{ad } h(z))(\bar{\partial}h(A)) = \omega(A) - e^{-\text{ad } h(z)}\eta(A)$$

which is equation (2.2.2).

COROLLARY 2.2.7. *Let M be a complex manifold, G a complex Lie group. Let*

$$\Xi_A: A \exp D(M, \hat{G}) \rightarrow \text{Equivalence classes of real product bundles}$$

$$[\omega] \rightsquigarrow J^\omega$$

where J^ω is as in equation (2.2.3); then

(1) Ξ_A is onto.

(2) If G has the exponential lift property with respect to M then Ξ_A is an isomorphism.

The maps Ξ and Ξ_A are well defined by Proposition 2.2.6 and onto by Corollary 1.2.2. Part (2) of Theorem 2.2.4 and Corollary 2.2.7 follow from part (2) of Proposition 2.2.6; hence all we need to prove our fundamental theorem is that J^ω is a complex structure if and only if ω is expclosed. To establish this fact we need the following:

LEMMA 2.2.8. *Let J be an almost complex structure on $M \times G$ coherent with the trivial bundle and let V be an open set in G containing the identity, then J is a complex structure if and only if there is an open cover $\{U_\alpha\}$ of M and maps $s_\alpha: U_\alpha \rightarrow V$ such that $\Sigma_\alpha: U_\alpha \rightarrow (U_\alpha \times G)_J$ given by $\Sigma_\alpha(u) = (u, s_\alpha(u))$ is almost holomorphic for each α .*

Proof. Assume that Σ_α are given. Define Γ_α via

$$\begin{array}{ccc} (U_\alpha \times G)_{\text{Product}} & \xrightarrow{\Sigma_\alpha \times 1} & (U_\alpha \times G)_J \times G \\ & \searrow \Gamma_\alpha & \downarrow \text{action} \\ & & (U_\alpha \times G)_J \end{array}$$

i.e. $\Gamma_\alpha(u, g) = (u, s_\alpha(u)g)$. Then Γ_α is an almost holomorphic diffeomorphism of $(U_\alpha \times G)_{\text{Product}}$ onto $(U_\alpha \times G)_J$ which says that J is locally equivalent to the product structure. Because the product structure is a complex structure we have shown that J is a complex structure.

If, on the other hand, J is complex then there is an open cover $\{U'_\beta\}$ of M and local sections $\Sigma'_\beta: U'_\beta \rightarrow U_\beta \times G$ defined by $\Sigma'_\beta(u) = (u, s'_\beta(u))$ which are J -holomorphic. Let z be a given point in M . Choose any U'_β such that $z \in U'_\beta$. Let $\alpha = (z, \beta)$ and $s'_\alpha = L_{s'_\beta(z)}^{-1} \circ s'_\beta$. Let $U_\alpha = U'_\beta \cap (s'_\alpha)^{-1}(V)$ and $s_\alpha = s'_\alpha$ restricted to U_α ; then $s'_\alpha(z) = L_{s'_\beta(z)}^{-1} \circ s'_\beta(z) = (s'_\beta(z))^{-1} s'_\beta(z) = e \in V$, hence $z \in U_\alpha$. Clearly s_α is almost holomorphic as the restriction of the composition of almost holomorphic maps and $s_\alpha(U_\alpha) = s_\alpha(U'_\beta \cap (s'_\alpha)^{-1}(V)) \subset s_\alpha(s'_\alpha)^{-1}(V) = V$.

The following proposition completes the proof of Theorem 2.2.4.

PROPOSITION 2.2.9. *Let $\omega \in \Lambda^{0,1}(M, \hat{G})$, then ω is expclosed if and only if J^ω is a complex structure.*

Proof. If ω is expclosed then there is an open cover $\{U_\alpha\}$ of M and $h_\alpha: U_\alpha \rightarrow \hat{G}$ such that equation (2.2.2) holds for each α . Let $s_\alpha = \exp \circ h_\alpha$, then (from the proof of Proposition 2.2.6) s_α is almost holomorphic.

Conversely, suppose J^ω is a complex structure. It is well known that there is an open set \tilde{V} of zero in \hat{G} such that the restriction of the exponential map to \tilde{V} is a

diffeomorphism [5]. Let $V = \exp(\tilde{V})$ and apply Lemma 2.2.8 to obtain $s_\alpha: U_\alpha \rightarrow V \subseteq G$. Then s_α can be written as $s_\alpha(z) = \exp h_\alpha(z)$ for some $h_\alpha: U_\alpha \rightarrow \hat{G}$ (because \exp^{-1} exists on V). Since s_α gives an equivalence between J and the product structure, then h_α yields the fact that ω restricted to U_α is expexact for each α , hence ω is expclosed.

3. Computation of $\text{Exp } D(M, G)$. In this section we shall apply the theory developed in the first two sections of this chapter. We shall first dispose of the abelian case by reproving Corollary 2.1.3 and exhibiting the isomorphism between $\mathcal{D}_{0,1}(M, \hat{G})$ and $\text{Pic}(M, G)$ in a more concrete manner. We shall then examine the case in which G is solvable. In [6], Frenkel (using the techniques of sheaf-theoretic cohomology) proves that if G is solvable then the vanishing of $\mathcal{D}_{0,1}(M, C)$ implies the vanishing of $\text{Pic}(M, G)$. We shall prove both a generalization of Frenkel's theorem and its converse (Theorems 2.3.2 and 2.3.4). We shall then calculate the exponentially closed $(0, 1)$ -forms for arbitrary G .

Let $\langle \eta \rangle$ be the $\bar{\partial}$ -cohomology class of $\eta \in \Lambda^{0,1}(M, \hat{G})$.

THEOREM 2.3.1. *Let $\pi_1(M)$ be a torsion group and G be abelian, then the map $z: \mathcal{D}_{0,1}(M, \hat{G}) \rightarrow \text{Pic}(M, G)$ given by $z(\langle \eta \rangle) = G \rightarrow (M \times G)_J \rightarrow M$ is an isomorphism.*

Proof. By the fundamental theorem, it suffices to show that $\text{Exp } D(M, G) = \mathcal{D}_{0,1}(M, \hat{G})$ but this is exactly Proposition 2.2.3.

Let $h: M \rightarrow \hat{G}$, then for the remainder of this section we shall write

$$A_z(h) = 2i \sum_{n=0}^{\infty} \frac{(-1)^n (\text{ad } h(z))^n}{(n+1)!} = 2i \frac{e^{-\text{ad } h(z)} - I}{\text{ad } h(z)}$$

for all $z \in M$. We shall also write $A(h)$ for $A_z(h)$ if there is no chance of confusion.

THEOREM 2.3.2. *If $[\hat{G}, \hat{G}] \neq \hat{G}$ then $\text{Exp } D(M, G) = 0$ implies $\mathcal{D}_{0,1}(M, C) = 0$.*

Proof. Let $0 \neq e_1 \in \hat{G}$ such that $e_1 \notin [\hat{G}, \hat{G}]$. Let ω_1 be a $\bar{\partial}$ -cocycle. Let $\omega = \omega_1 e_1$. We may find a locally defined function $h_1: U \rightarrow C$ such that $\omega_1 = \bar{\partial} h_1$. Let $h = -ih_1 e_1/2$; then

$$A(h)\bar{\partial}h = 2i\bar{\partial}h + 2 \sum_{n=1}^{\infty} \frac{(-1)^n (\text{ad } h(u))^n}{(n+1)!} \bar{\partial}h = \bar{\partial}h_1 e_1 = \omega$$

because $(\text{ad } h)\bar{\partial}h = [h_1 e_1, \bar{\partial}h_1 e_1] = 0$. ω is therefore expclosed and hence by assumption expexact. Let $l: M \rightarrow \hat{G}$ be a function such that

$$(2.3.1) \quad A(l)\bar{\partial}l = \omega.$$

Let $\{e_i\}$ be any basis of \hat{G} whose first element is e_1 ; then we may write $l = \sum l_i e_i$ where $l_i: M \rightarrow C$. Hence equation (2.3.1) becomes $\omega = A(l)\bar{\partial}l = \sum A(l)(\bar{\partial}l_i)e_i = (2i\bar{\partial}l_1)e_1 + \alpha$ where $\alpha \in \hat{G} - Ce_1$. Therefore $2i\bar{\partial}l_1 = \omega_1$ and so ω_1 is a $\bar{\partial}$ -coboundary.

The above theorem is the converse of Frenkel's theorem. Before proving a generalization of Frenkel's theorem by our techniques, we shall need a lemma.

LEMMA 2.3.3. *Let $\phi: \hat{G}_1 \rightarrow \hat{G}_2$ be a Lie algebra homomorphism. If ω is expexact then so is $\phi(\omega)$.*

Proof. Let $\omega \in \Lambda^{0,1}(M, \hat{G}_1)$ with $\omega = A(h)\bar{\partial}h$ where $h: M \rightarrow \hat{G}_1$. Let $l = \phi \circ h: M \rightarrow \hat{G}_2$; then

$$\begin{aligned}\phi(\omega)(X) &= \phi(\omega(X)) = \phi(A(h)\bar{\partial}h(X)) \\ &= \phi\left(2i \sum_{n=0}^{\infty} \frac{(-1)^n (\text{ad } h)^n}{(n+1)!} \bar{\partial}h(X)\right) \\ &= 2i \sum_{n=0}^{\infty} \frac{(-1)^n (\text{ad } l)^n}{(n+1)!} \bar{\partial}l(X) = A(l)\bar{\partial}l(X).\end{aligned}$$

THEOREM 2.3.4. *If G is solvable, then $\mathcal{D}_{0,1}(M, C) = 0$ implies that $\text{Exp } D(M, \hat{G}) = 0$.*

Proof. We proceed by induction on the dimension of G . We remark that the theorem clearly holds if G is abelian (by Proposition 2.2.3), which enables us to start the induction. Let Z be the center of the nil-radical of \hat{G} ; then Z is an abelian ideal of \hat{G} with $\dim Z \geq 1$. Let V be any vector space such that $\hat{G} = V \oplus Z$ (where \oplus is understood to be the direct sum as vector spaces, not necessarily as Lie algebras). Let $\pi: \hat{G} \rightarrow \hat{G}/Z$ be the canonical projection.

If ω is any expclosed $(0, 1)$ -form on M with values in \hat{G} then $\omega^* = \pi\omega$ is an expclosed $(0, 1)$ -form with values in \hat{G}/Z by Lemma 2.3.3. By induction, we know that there is a $k^*: M \rightarrow \hat{G}/Z$ such that $A(k^*)\bar{\partial}k^* = \omega^*$. Now, $\pi|_V: V \rightarrow \hat{G}/Z$ is an isomorphism, hence there is a $k: M \rightarrow V$ such that $\pi k = k^*$. Let $\theta = A(k)\bar{\partial}k$, then $\omega - \theta$ is an expclosed \hat{G} -valued $(0, 1)$ -form over M . Because $\pi(\omega - \theta) = \pi\omega - \pi\theta = \omega^* - \omega^* = 0$ we have $\omega - \theta$ is, in fact, a Z -valued $(0, 1)$ -form over M , hence there is $l: M \rightarrow Z$ such that

$$(2.3.2) \quad \omega - \theta = A(l)\bar{\partial}l.$$

Now for any $Y \in Z$ we have $(\text{ad } Y)\hat{G} \subset Z$, hence the map which $\text{ad } Y$ induces on \hat{G}/Z is trivial; hence we may rewrite equation (2.3.2) as $\omega - e^{-\text{ad } l}\theta = A(l)\bar{\partial}l$ and so $\omega \sim_{\text{exp}} \theta \sim_{\text{exp}} 0$ which completes the proof.

We shall now give a simple criterion for $\omega \in \Lambda^{0,1}(M, \hat{G})$ to be expclosed. Recall that if $\alpha, \beta \in \Lambda^1(M, \hat{G})$ then $[\alpha, \beta]$ is defined to be an element of $\Lambda^2(M, \hat{G})$ via

$$[\alpha, \beta](X, Y) = \frac{1}{2}([\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]) \quad \text{for all } X, Y \in T(M).$$

THEOREM 2.3.5. *Let $\eta \in \Lambda^{0,1}(M, \hat{G})$ correspond to the almost complex structure J^n on $M \times G$; then the following are equivalent:*

- (a) η is expclosed,
- (b) J^n is a complex structure,
- (c) $\bar{\partial}\eta = (i/4)[\eta, \eta]$.

We shall first prove a preliminary proposition. Let $\gamma \in \Lambda^r(M, \hat{G})$, define $\gamma^\# \in \Lambda^r(M \times G, \hat{G})$ via

$$\gamma_{z,\lambda}^\#((A_1, B_1), (A_2, B_2), \dots, (A_r, B_r)) = (\text{ad } \lambda^{-1})\gamma_z(A_1, \dots, A_r).$$

Let $\tau \in \Lambda^1(M \times G, \hat{G})$ be given by $\tau_{z,\lambda}(A, B) = L_{\lambda^{-1}}B$.

PROPOSITION 2.3.6. *For any $\gamma \in \Lambda^1(M, \hat{G})$, $d(\gamma^\#) = (d\gamma)^\# - [\tau, \gamma^\#]$.*

Proof. For any $A_1, A_2 \in T_z(M)$, $B_1, B_2 \in T_\lambda(G)$

$$\begin{aligned} & 2 d(\gamma^\#)((A_1, B_1), (A_2, B_2)) \\ &= (A_1, B_1) \text{ad } \lambda^{-1}\gamma(A_2) - (A_2, B_2)(\text{ad } \lambda^{-1})\gamma(A_1) - \text{ad } \lambda^{-1}[\gamma(A_1), \gamma(A_2)] \\ &= 2(d\gamma)^\# + (0, B_1) \text{ad } \lambda^{-1}\gamma(A_2) - (0, B_2)(\text{ad } \lambda^{-1})\gamma(A_1). \end{aligned}$$

We must therefore prove

$$\begin{aligned} (2.3.3) \quad & \frac{1}{2}\{(0, B_1) \text{ad } \lambda^{-1}\gamma(A_2) - (0, B_2) \text{ad } \lambda^{-1}\gamma(A_1)\} \\ &= \frac{1}{2}\{[L_{\lambda^{-1}}B_1, \text{ad } \lambda^{-1}\gamma(A_2)] - [L_{\lambda^{-1}}B_2, \text{ad } \lambda^{-1}\gamma(A_1)]\}. \end{aligned}$$

It suffices to prove equation (2.3.3) when B_1 and B_2 are left-invariant vector fields. Let $B = B_1$ or B_2 and $A = A_1$ or A_2 . Let $\alpha(t) = \lambda_0 \exp(tB)$ and $f(\lambda) = (\text{ad } \lambda^{-1})\gamma(A)$.

$$\begin{aligned} (0, B_1) \text{ad } \lambda^{-1}\gamma(A) &= \lim_{t \rightarrow 0} \frac{f(\alpha(t)) - f(\lambda_0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\text{ad } (\lambda_0^{-1} \exp(-tB))\gamma(A) - \text{ad } \lambda_0^{-1}\gamma(A)}{t} = \lim_{t \rightarrow 0} \frac{(\text{ad } e^{-tB} - I) \text{ad } \lambda_0^{-1}\gamma(A)}{t}, \end{aligned}$$

hence

$$(0, B_1) \text{ad } \lambda^{-1}(A) = -[L_{\lambda^{-1}}B, \text{ad } \lambda_0^{-1}\gamma(A)].$$

Substituting this last result into equation (2.3.3) yields the proposition.

We now proceed with the proof of Theorem 2.3.5. Firstly, note that the equivalence of (a) and (b) is exactly Proposition 2.2.9. Let ω be the element of $\Lambda^{0,1}(M \times G, \hat{G})$ which corresponds to $\eta \in \Lambda^{0,1}(M, \hat{G})$; then by the construction in Lemma 1.2.7 we have $\omega = \eta^\#$. By Theorem 1.2.3, we have that J^η is complex if and only if

$$(2.3.4) \quad 0 = 2 d\eta^\#(J^\eta X, Y) + 2 d\eta^\#(X, J^\eta Y) + [\eta^\# X, \eta^\# Y] \quad \text{for all } X, Y \in T(M \times G).$$

Let $X = (A, B)$, $Y = (C, D)$, where $A, C \in T_z(M)$, $B, D \in T_\lambda(G)$; then

$$J_{z,\lambda}^\eta(X) = (J_M A, J_G B + dR_\lambda \eta(A)) \quad \text{and} \quad J_{z,\lambda}^\eta(Y) = (J_M C, J_G D + dR_\lambda \eta(C));$$

hence,

$$2[\tau, \eta^\#](J^\eta X, Y) = [\tau(J^\eta X), \eta^\# Y] - [\tau(Y), \eta^\#(J^\eta X)],$$

hence,

$$2[\tau, \eta^\#](J^\eta X, Y) = [L_{\lambda^{-1}}J_G B + \text{ad } \lambda^{-1}\eta(A), \text{ad } \lambda^{-1}\eta(C)] - [L_{\lambda^{-1}}D, \text{ad } \lambda^{-1}\eta(J_M A)];$$

similarly,

$$2[\tau, \eta^\#](X, J^\eta Y) = [L_{\lambda^{-1}}B, \text{ad } \lambda^{-1}\eta(J_M C)] - [L_{\lambda^{-1}}J_G D + \text{ad } \lambda^{-1}\eta(C), \text{ad } \lambda^{-1}\eta(A)].$$

Adding yields

$$(2.3.5) \quad [\tau, \eta^\#](J^n X, Y) + [\tau, \eta^\#](X, J^n Y) = [\eta^\#, \eta^\#](X, Y)$$

because $\eta(J_M C) = -i\eta(C)$. Applying Proposition 2.3.6 to equation (2.3.4) we find:
 J^n is complex \Leftrightarrow

$$0 = 2(d\eta)^\#(J^n X, Y) - 2[\tau, \eta^\#](J^n X, Y) + (d\eta)^\#(X, J^n Y) \\ - 2[\tau, \eta^\#](X, J^n Y) + [\eta^\# X, \eta^\# Y];$$

hence from equation (2.3.5)

$$0 = 2\{(d\eta)^\#(J^n X, Y) + (d\eta)^\#(X, J^n Y)\} - [\eta^\# X, \eta^\# Y].$$

Thus J^n is complex \Leftrightarrow for all $A \in T_x(M)$, $C \in T_\lambda(G)$

$$(2.3.6) \quad d\eta(J_M A, C) + d\eta(A, J_M C) = \frac{1}{2}[\eta(A), \eta(C)].$$

We shall now show that equation (2.3.6) holds if and only if $\bar{\partial}\eta = (i/4)[\eta, \eta]$ where $\bar{\partial}$ is, of course, taken with respect to the complex structure, J_M , of M .

Assume that equation (2.3.6) holds; then, since $(i/4)[\eta, \eta]$ is of type $(0, 2)$, we need only show that $(d\eta - (i/4)[\eta, \eta])(A, C) = 0$ for all $A, C \in T^{0,1}(M)$ because $(J_M \text{ being complex})$ $d\eta$ has no $(2, 0)$ -component. Now if $A, C \in T^{0,1}(M)$ then equation (2.3.6) dictates

$$(i/4)[\eta(A), \eta(C)] = (i/2)(d\eta(J_M A, C) + d\eta(A, J_M C)) \\ = (i/2)(-2i d\eta(A, C)) = d\eta(A, C),$$

hence $d\eta - (i/4)[\eta, \eta]$ is of type $(1, 1)$ and so $\bar{\partial}\eta = (i/4)[\eta, \eta]$.

Conversely, if $\bar{\partial}\eta = (i/4)[\eta, \eta]$ then $(d\eta - (i/4)[\eta, \eta])(A, C) = 0$ if $A, C \in T^{0,1}(M)$. However,

$$d\eta(J_M A, C) + d\eta(A, J_M C) = -2i d\eta(A, C) \\ = -2i \bar{\partial}\eta(A, C) = -(2i^2/4)[\eta, \eta] = \frac{1}{2}[\eta, \eta].$$

Thus $2 d\eta(J_M A, C) + 2 d\eta(A, J_M C) = [\eta(A), \eta(C)]$ if $A, C \in T^{0,1}(M)$. If $A, C \in T^{1,0}M$ or $A \in T^{0,1}M$ and $C \in T^{1,0}M$ then both sides of the above equation are zero.

In the interests of completeness we shall now write down explicitly the equations governing $\omega \sim_{\exp} \eta$ if

$$G = \left\{ \left(\begin{array}{ccc} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{array} \right) \middle| z_i \in C \right\}.$$

Let

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then we may write $\omega = \sum_{i=1}^3 \omega_i e_i$, $\eta = \sum_{i=1}^3 \eta_i e_i$ where $\omega_i, \eta_i \in \Lambda^{0,1}(M, C)$. A simple computation shows: $\omega \sim_{\text{exp}} \eta \Leftrightarrow$ there are smooth functions $h_i: M \rightarrow C$ such that

$$(2.3.7) \quad \begin{aligned} \omega_1 - \eta_1 &= 2i\bar{\partial}h_1, \\ \omega_2 - \eta_2 &= 2i\bar{\partial}h_2, \\ \omega_3 - \eta_3 + h_1\eta_2 - h_2\eta_1 &= 2i\bar{\partial}h_3 - ih_1\bar{\partial}h_2 + ih_2\bar{\partial}h_1. \end{aligned}$$

Of course if $\mathcal{D}_{0,1}(M, C) = 0$ then $\text{Exp } D(M, G) = 0$; however we can make the following amusing remark: if $\mathcal{D}_{0,1}(M, C) \neq 0$ then in any topology which $\text{Exp } D(M, G)$ inherits from $\Lambda^{0,1}(M, \hat{G})$ it is not Hausdorff. Consider $\omega^\lambda = \lambda\alpha e_1$; $\eta^\lambda = \lambda\alpha e_1 + \alpha e_3$ where α is a $\bar{\partial}$ -cocycle which is not a $\bar{\partial}$ -coboundary. Clearly ω^λ and η^λ are expclosed for all λ by Theorem 2.3.5. Furthermore, if $\lambda \neq 0$ $h_1 = 0$, $h_2 = 1/\lambda$ and $h_3 = 0$ satisfy equations (2.3.7) hence $\omega^\lambda \sim_{\text{exp}} \eta^\lambda$ for $\lambda \neq 0$. However, $\lim_{\lambda \rightarrow 0} \omega^\lambda = 0$, $\lim_{\lambda \rightarrow 0} \eta^\lambda = \alpha e_3$ and so $\lim_{\lambda \rightarrow 0} \omega^\lambda$ is not expcohomologous to $\lim_{\lambda \rightarrow 0} \eta^\lambda$. By passing to equivalence classes we see that we have a sequence in $\text{Exp } D(M, \hat{G})$ which has 2 limits; hence $\text{Exp } D(M, \hat{G})$ cannot be Hausdorff!

CHAPTER III: APPLICATIONS TO DIFFERENTIAL GEOMETRY

1. Applications to the theory of holomorphic connections. The following is immediate from Theorem 2, Theorem 5 and Proposition 12 of [1]:

THEOREM 3.1.1 (ATIYAH). *A holomorphic principal fibre bundle with compact Kähler base whose structure group is C^* possesses a holomorphic connection if and only if it is real trivial.*

Atiyah's proof is algebraic. In this section we attempt to give a constructive proof of Atiyah's theorem if the structure group is abelian. We are successful in one direction but in the other direction we must assume that the compact Kähler base is simply connected. We also give a necessary and sufficient condition for a real trivial holomorphic fibre bundle with arbitrary structure group and base to admit a holomorphic connection (Corollary 3.1.6).

We begin by recalling some definitions referring to [1] for all details. Unless otherwise noted, we shall maintain the notation of the earlier chapters. Let $\xi: G \rightarrow P \xrightarrow{\pi} M$ be a holomorphic principal fibre bundle. Let $Q = T(P)/G$ and $L(P) = (\text{Ker } \pi)/G$. Let λ be the inclusion $L(P) \subset Q$.

Q and $L(P)$ are both vector bundles over M and $\mathcal{A}(\xi): 0 \rightarrow L(P) \xrightarrow{\lambda} Q \xrightarrow{\pi} T(M) \rightarrow 0$ is an exact sequence of complex vector bundles called the *fundamental sequence* of ξ . A *holomorphic connection* for ξ is a splitting of the exact sequence $\mathcal{A}(\xi)$; that is, a holomorphic map $\Gamma: Q \rightarrow L(P)$ such that $\Gamma \circ \lambda$ is the identity on Q . Recall that a *connection form* on $\xi: G \rightarrow P \rightarrow B$ is \hat{G} -valued 1-form ψ on P such that

- (i) $\psi_p(\theta_p(A)) = A$ for all $A \in \hat{G}$ and $p \in P$, and
- (ii) $(R_g^* \psi)_p(X) = (\text{ad } g^{-1})\psi_p(X)$ for all $X \in T_p(P)$ and $p \in P$.

A connection is of type $(1, 0)$ if $\psi(JX) = i\psi(X)$.

PROPOSITION 3.1.2. *There is a holomorphic connection Γ on $G \rightarrow P_J \rightarrow M$ if and only if there exists a connection form ψ of type $(1, 0)$ such that $\bar{\partial}_J \psi = 0$.*

Proof. Let $0 \rightarrow L(P) \xrightarrow{\lambda} Q(P) \xrightarrow{\pi} T(M) \rightarrow 0$ be the fundamental sequence for ξ . If $p \in P$ and $X \in T_p(P)$, let $[p, X]$ = image of $(p, X) \in T(P)$ in $Q(P)$ under the projection $T(P) \rightarrow T(P)/G = Q(P)$. If ψ is a connection 1-form then define $\Gamma([p, X]) = \langle p, \theta_p(\psi(X)) \rangle$ where $\theta_p: \hat{G} \rightarrow (\text{Ker } \pi)_p$ is the usual isomorphism and $\langle p, Y \rangle$ denotes the equivalence class of $(p, Y) \in \text{Ker } \pi$ in $L(P)$. Γ is well defined because

$$\begin{aligned} \Gamma([p^g, \dot{R}_g X]) &= \langle p^g, \theta_p(\psi(\dot{R}_g X)) \rangle = \langle p^g, \theta_{p^g}((\text{ad } g^{-1})\psi(X)) \rangle \\ &= \langle p^g, R_g \theta_p \psi(X) \rangle = \langle p, \theta_p(\psi(X)) \rangle. \end{aligned}$$

Γ is then holomorphic as the composition of holomorphic maps. (We have used here that a $(1, 0)$ -form P with values in \hat{G} is holomorphic as a $(1, 0)$ -form if and only if it is holomorphic as a map of the complex manifold $T(P)$ to the complex manifold \hat{G} .) The converse follows from a similar construction.

Note that Proposition 3.1.2 says that ψ is a holomorphic $(1, 0)$ -form on P .

PROPOSITION 3.1.3. *Let ψ be a \hat{G} -valued connection 1-form on $(M \times G)_J$, where J corresponds to $\omega \in \Lambda^{0,1}(M, \hat{G})$ via the fundamental theorem; then ψ is of type $(1, 0)$ with respect to J if and only if there is $\beta \in \Lambda^{(1,0)}(M, \hat{G})$ such that*

$$(3.1.1) \quad \psi_{z,\lambda}(A, B) = L_{\lambda^{-1}}B + (i/2)(\text{ad } \lambda^{-1})\pi^*((\beta - \omega)(A, B))$$

for all $A \in T_z(M)$, $B \in T_{\lambda}(G)$.

Proof. Clearly any form ψ satisfying equation (3.1.1) is a $(1, 0)$ connection. Assume now that ψ is a connection and let $\psi_{z,\lambda}^1(A, B) = \psi_{z,\lambda}(A, 0)$; then from the second condition in the definition of a connection we have

$$\begin{aligned} (\text{ad } \lambda^{-1})\psi_{z,e}(A, \dot{R}_{\lambda^{-1}}B) &= (R_{\lambda}^*\psi)_{z,e}(A, \dot{R}_{\lambda^{-1}}B) \\ &= \psi_{z,\lambda}(A, \dot{R}_{\lambda}\dot{R}_{\lambda^{-1}}B) = \psi_{z,\lambda}(A, B). \end{aligned}$$

But condition (1) of a connection demands $\psi_{z,e}(0, \dot{R}_{\lambda^{-1}}B) = \dot{R}_{\lambda^{-1}}B$; thus

$$\psi_{z,\lambda}(A, B) = (\text{ad } \lambda^{-1})\{\psi_{z,e}(A, 0) + \psi_{z,e}(0, \dot{R}_{\lambda^{-1}}B)\},$$

hence

$$= (\text{ad } \lambda^{-1})(\psi_{z,e}^1(A)) + L_{\lambda^{-1}}B.$$

$\psi_{z,e}^1(A)$ may be written $\pi^*\alpha$ for some $\alpha \in \Lambda^1(M, G)$ because $(\pi^*\alpha)_{z,\lambda}(A, B)$ is independent of both B and λ .

$$(3.1.2) \quad \begin{aligned} \psi_{z,\lambda}(A, B) &= (\text{ad } \lambda^{-1})\pi^*\alpha(A, B) + L_{\lambda^{-1}}B, \\ \psi_{z,\lambda}(J(A, B)) &= \psi_{z,\lambda}(J_M A, J_G B + \dot{R}_{\lambda}(\omega(A))); \end{aligned}$$

hence

$$\psi_{z,\lambda}(J(A, B)) = L_{\lambda^{-1}}(J_G B + \dot{R}_{\lambda}\omega(A)) + (\text{ad } \lambda^{-1})\alpha(J_M A).$$

However

$$i\psi_{z,\lambda}(A, B) = iL_{\lambda^{-1}}B + i(\text{ad } \lambda^{-1})\alpha(A).$$

Thus $\psi(J(A, B)) = i\psi(A, B) \Leftrightarrow$

$$iL_{\lambda^{-1}}B + i(\text{ad } \lambda^{-1})(\alpha(A)) = L_{\lambda^{-1}}J_G B + \text{ad } \lambda^{-1}\omega(A) + (\text{ad } \lambda^{-1})(\alpha(J_M A))$$

or $\psi(J(A, B)) = i\psi(A, B) \Leftrightarrow i\alpha(A) = \omega(A) + \alpha(J_M A).$

However for any $A \in T_z(M)$

$$\begin{aligned}\alpha(A) &= (i\alpha(A) - \alpha(J_M A) + i\alpha(A) + \alpha(J_M A))/2i \\ &= -i\omega(A)/2 + (i\alpha(A) + \alpha(J_M A))/2i,\end{aligned}$$

so

$$\alpha(A) = -i\omega(A)/2 + i\beta(A)/2$$

where β is an arbitrary element of $\Lambda^{(1,0)}(M, G)$. Substituting this last equation into equation (3.1.2) yields equation (3.1.1).

We shall now calculate the set of holomorphic connection forms on $G \rightarrow (M \times G)_I \rightarrow M$. Recall that for any $\gamma \in \Lambda^p(M, \hat{G})$ we may define $\gamma^\# \in \Lambda^p(M \times G, \hat{G})$ by $\gamma_{z,\lambda}^\# = (\text{ad } \lambda^{-1})(\pi^* \gamma)$ as in §3 of Chapter II. We also define $\tau: T(M \times G) \rightarrow \hat{G}$ by $\tau_{z,\lambda}(A, B) = L_{\lambda^{-1}}B$. Equation (3.1.1) now becomes (with $\psi_0 = \tau - i\omega^\# / 2$)

$$(3.1.1) \quad \psi = \tau + (i/2)(\beta - \omega)^\# = \psi_0 + i\beta^\# / 2.$$

LEMMA 3.1.4. (a) $\partial\omega^\# = (\partial\omega)^\# - [\psi_0, \omega^\#]$.

(b) $\bar{\partial}\beta^\# = (\bar{\partial}\beta)^\# - (i/2)[\omega^\#, \beta^\#]$.

Proof. From Proposition 2.3.6 we have

$$d\omega^\# = (d\omega)^\# - [\tau, \omega^\#] = (d\omega)^\# - [\psi_0, \omega^\#] - (i/2)[\omega^\#, \omega^\#]$$

and

$$d\beta^\# = (d\beta)^\# - [\tau, \beta^\#] = (d\beta)^\# - [\psi_0, \beta^\#] - (i/2)[\omega^\#, \beta^\#],$$

hence the result follows by comparing type.

THEOREM 3.1.5. *The set of holomorphic connections on $G \rightarrow (M \times G)_I \rightarrow M$ is in 1-1 correspondence with the set of all $\beta \in \Lambda^{1,0}(M, \hat{G})$ such that*

$$(3.1.3) \quad \bar{\partial}\beta = \partial\omega - (i/2)[\omega, \beta].$$

Proof. We know from Proposition 3.1.3 that the set of $(1, 0)$ connections is in 1-1 correspondence with the set of $\beta \in \Lambda^{1,0}(M, \hat{G})$ satisfying equation (3.1.1); hence we need only show that ψ is holomorphic if and only if β satisfies equation (3.1.3). We shall first calculate $\bar{\partial}\psi_0$. Since $d\tau = -\frac{1}{2}[\tau, \tau]$ we have

$$\begin{aligned}d\psi_0 &= -\frac{1}{2}[\tau, \tau] - (i/2)d\omega^\# = -\frac{1}{2}[\psi_0 + (i/2)\omega^\#, \psi_0 + (i/2)\omega^\#] - (i/2)d\omega^\# \\ &= -\frac{1}{2}([\psi_0, \psi_0] - \frac{1}{4}[\omega^\#, \omega^\#] + i[\psi_0, \omega^\#] - (i/2)d\omega^\#)\end{aligned}$$

hence

$$\bar{\partial}\psi_0 = -(i/2)[\psi_0, \omega^\#] - (i/2)\partial\omega^\#;$$

note that $\beta^\# \in \Lambda^{1,0}(M \times G, \hat{G})$ hence $\bar{\partial}\psi = \bar{\partial}\psi_0 + (i/2)\bar{\partial}\beta^\#$, so $\bar{\partial}\psi = -(i/2)[\psi_0, \omega^\#] - (i/2)\partial\omega^\# + (i/2)\bar{\partial}\beta^\#$. Applying Lemma 3.1.4,

$$\bar{\partial}\psi = -(i/2)[\psi_0, \omega^\#] - (i/2)((\partial\omega)^\# - [\psi_0, \omega^\#]) + (i/2)((\bar{\partial}\beta)^\# - (i/2)[\omega^\#, \beta^\#]);$$

hence

$$(3.1.4) \quad \bar{\partial}\psi = (i/2)(\bar{\partial}\beta - \partial\omega - (i/2)[\omega, \beta])^\#.$$

COROLLARY 3.1.6. $\xi: G \rightarrow (M \times G)^{\omega} \rightarrow M$ admits a holomorphic connection if and only if there is a $\beta \in \Lambda^{1,0}(M, G)$ such that $\bar{\partial}\beta = \partial\omega - (i/2)[\omega, \beta]$.

THEOREM 3.1.7. If M is a compact Kähler manifold then any real trivial holomorphic principal fibre bundle over M with abelian structure group admits a holomorphic connection.

Proof. By Corollary 3.1.6 it suffices to produce $\beta \in \Lambda^{1,0}(M, \hat{G})$ such that $\bar{\partial}\beta = \partial\omega$.

We shall use the machinery of harmonic forms. Let $\bar{\partial}^t$ (resp. d^t) be the transpose of $\bar{\partial}$ (resp. d) with respect to the Hodge inner product on $\Lambda^p(M, \hat{G})$. Let $H(\eta)$ denote the harmonic part of $\eta \in \Lambda^p(M, \hat{G})$ and let $\Delta = 2(\bar{\partial}\bar{\partial}^t + \bar{\partial}^t\bar{\partial})$ be the Laplacian and G be the Green's operator. Since the space of harmonic forms is $(\text{Ker } d \cap \text{Ker } d^t)$, we have that $\partial H(\gamma) = 0$ for all α . We have the decomposition $\omega = H(\omega) + G\Delta\omega$ or $\omega = H(\omega) + 2G(\bar{\partial}\bar{\partial}^t + \bar{\partial}^t\bar{\partial})\omega$; hence $\partial\omega = -2\bar{\partial}G\partial\bar{\partial}^t\omega - 2G\bar{\partial}^t\partial\bar{\partial}\omega$ or (since J^ω is complex, $\bar{\partial}\omega = 0$)

$$(3.1.5) \quad \partial\omega = \bar{\partial}\beta \quad \text{if } \beta = -2G\partial\bar{\partial}^t\omega.$$

Note that the above proof is constructive; the connection defined above is, for $A \in T_x(M)$, $B \in T_\lambda(G)$,

$$(3.1.6) \quad \psi_{z,\lambda}(A, B) = L_{\lambda^{-1}}B - (i/2)\pi^*(G\partial\bar{\partial}^t\omega + \omega)(A, B).$$

We now examine the converse of Theorem 3.1.7 in the abelian case.

THEOREM 3.1.8. Let $\xi: G \rightarrow P \rightarrow M$ be a holomorphic principal fibre bundle with G abelian and M compact Kähler. If ξ has a holomorphic connection then ξ has a flat connection.

Proof. Let $\omega \in \Lambda^{0,1}(M, \hat{G})$ corresponding to ξ via Proposition 1.2.4. We shall use the notation of the proof of Theorem 3.1.7. Let $\alpha_1 = -2\bar{\partial}\bar{\partial}^tG\omega$ then equation (3.1.5) yields $\bar{\partial}\alpha_1 = \partial\omega$ and $\partial\alpha_1 = 0$. If ψ is a holomorphic connection on ξ then $\psi = \pi^*\gamma$ for some $\gamma \in \Lambda^{1,0}(M, \hat{G})$ such that $\bar{\partial}\gamma = 0$. Let $\alpha_2 = 2\bar{\partial}^t\delta G\gamma$; then

$$\begin{aligned} \partial\alpha_2 &= 2\partial\bar{\partial}^t\delta G\gamma = 2(\partial\bar{\partial}^t\delta G\gamma + \partial\bar{\partial}\bar{\partial}^tG\gamma) + \partial H(\gamma) \\ &= \partial(\Delta G\gamma + H(\gamma)) = \partial\gamma \end{aligned}$$

and

$$\bar{\partial}\alpha_2 = 2\bar{\partial}^t\delta G\bar{\partial}\gamma = 0.$$

Let $\beta = \alpha_1 + (2/i)\alpha_2 \in \Lambda^{1,0}(M, G)$, then

$$\partial\beta = (2/i)\partial\alpha_2 = (2/i)\partial\gamma;$$

hence

$$(3.1.7) \quad (i/2)\pi^*\partial\beta = \partial\psi,$$

$$(3.1.8) \quad \bar{\partial}\beta = \bar{\partial}\alpha_1 = \partial\omega.$$

Now, let $\tilde{\psi} = \psi + (i/2)\pi^*(\omega - \beta)$, then if $\tilde{\Omega}$ is the curvature of $\tilde{\psi}$ we have $d\tilde{\psi} = -\frac{1}{2}[\tilde{\psi}, \tilde{\psi}] + \tilde{\Omega} = \tilde{\Omega}$, hence $\tilde{\Omega}^{1,1} = \bar{\partial}\psi + (i/2)\pi^*(\partial\omega - \bar{\partial}\beta) = 0$ because equation (3.1.8) says $\partial\omega = \bar{\partial}\beta$. Also

$$\tilde{\Omega}^{2,0} = \partial\psi + (i/2)\pi^*(\bar{\partial}\omega - \partial\beta) = \partial\psi + (i/2)\pi^*\partial\beta = 0$$

where the last equality is equation (3.1.7). We therefore see that $\tilde{\psi}$ is a flat connection.

COROLLARY 3.1.9. *Let M be a simply connected compact Kähler manifold and G be an abelian complex Lie group. Then if $\xi: G \rightarrow P \rightarrow M$ is a holomorphic principal fibre bundle which admits a holomorphic connection, then ξ is a real trivial holomorphic principal fibre bundle.*

Proof. This result is immediate from [9, p. 92].

2. Examples of Kähler manifolds. Given the manifold $P_J = (M \times G)_J$ where J corresponds to $\eta \in \Lambda^{0,1}(M, \hat{G})$ as in Theorem 1.2.2 we shall now put a natural Hermitian metric on P_J and obtain a necessary and sufficient condition in terms of η for P_J to be Kähler with respect to this metric. Recall that a Riemannian metric on P is a smooth inner product on the tangent space of P at each point. An (almost) *Hermitian metric* on the (almost) complex manifold P_J is a Riemannian metric on P such that $g(JX, JY) = g(X, Y)$ for all $X, Y \in T(M)$ where J is the (almost) complex structure on P . Let $\Omega(X, Y) = g(X, JY)$ for $X, Y \in T(M)$, then Ω is a 2-form on M , called the *Kähler form* of g . If $d\Omega = 0$ then the (almost) Hermitian metric is said to be (almost) *Kähler*.

PROPOSITION 2.4.1. *Let $\xi_0: G \rightarrow P_{J_0} \rightarrow M$ be an almost holomorphic principal fibre bundle and let J be an almost complex structure on P coherent with ξ_0 . Then g is a Hermitian metric on P where g is defined by*

$$\begin{aligned} g_p(X, Y) = & 2g_0(X, Y) + g_0(J_0X, \theta_p(\omega(Y))) \\ & + g_0(J_0Y, \theta_p(\omega(X))) + g_0(\theta_p(\omega(X)), \theta_p(\omega(Y))) \end{aligned}$$

where g_0 is any metric on P which is Hermitian with respect to J_0 , J corresponds to ω via Theorem 1.2.1 and θ_p is the usual embedding $\hat{G} \rightarrow (\text{Ker } \pi)_p$.

Proof. Let $\tilde{g}(X, Y) = g_0(X, Y) + g_0(JX, JY)$, then since

$$\tilde{g}(JX, JY) = g_0(JX, JY) + g_0(X, Y) = \tilde{g}(X, Y),$$

\tilde{g} is Hermitian with respect to J . The correspondence of Theorem 1.2.1 is given by $J_p(X) = (J_0)_p(X) + \theta_p(\omega_p(X))$ for $p \in P$. Hence

$$\begin{aligned}\tilde{g}(X, Y) &= g_0(X, Y) + g_0(J_0 X + \theta(\omega(X)), J_0 Y + \theta(\omega(Y))) \\ &= g_0(X, Y) + g_0(J_0 X, J_0 Y) \\ &\quad + g_0(J_0 X, \theta(\omega(Y))) + g_0(\theta(\omega(X)), J_0 Y) + g_0(\theta(\omega(X)), \theta(\omega(Y)))\end{aligned}$$

and so the result follows because g_0 is Hermitian with respect to J_0 .

We shall now calculate the fundamental form of g . We shall write $\Omega^0(X, Y) = g_0(X, J_0 Y)$.

$$\begin{aligned}\Omega_J(X, Y) &= 2g_0(X, JY) + g_0(J_0 X, \theta(\omega(JY))) \\ &\quad + g_0(J_0 JY, \theta(\omega(X))) + g_0(\theta(\omega(X)), \theta(\omega(JY))) \\ &= 2\{\Omega^0(X, Y) + g_0(X, \theta(\omega(Y)))\} + g_0(J_0 X, \theta(\omega(J_0 Y) + \theta(\omega(Y)))) \\ &\quad + g_0(J_0^2 Y + J_0 \theta(\omega(Y)), \theta(\omega(X))) + g_0(\theta(\omega(X)), \theta(\omega(J_0 Y) + \theta(\omega(Y)))).\end{aligned}$$

However since $\omega(\text{Ker } \pi) = 0$ we obtain

$$\begin{aligned}\Omega_J(X, Y) &= 2\Omega^0(X, Y) + 2g_0(X, \theta(\omega(Y))) + g_0(J_0 X, \theta(\omega(J_0 Y))) \\ &\quad - g_0(Y, \theta(\omega(X))) + g_0(J_0 \theta(\omega(Y)), \theta(\omega(X))) \\ &\quad + g_0(\theta(\omega(X)), \theta(\omega(J_0 Y))).\end{aligned}$$

Since $\omega(J_0 X) = -i\omega(X)$ and $\theta(iA) = J_0 \theta(A)$,

$$\begin{aligned}\Omega_J(X, Y) &= 2\Omega^0(X, Y) + 2g_0(X, \theta(\omega(Y))) - g_0(J_0 X, J_0 \theta(\omega(Y))) \\ &\quad - g_0(Y, \theta(\omega(X))) + g_0(J_0 \theta(\omega(Y)), \theta(\omega(X))) \\ &\quad - g_0(\theta(\omega(X)), J_0 \theta(\omega(Y))),\end{aligned}$$

and so we have proved

PROPOSITION 2.4.2. *Let J be an almost complex structure on P coherent with ξ_0 ; then the Kähler form of P_J in the metric of Proposition 2.4.1 is*

$$\Omega_J(X, Y) = 2\Omega^0(X, Y) + g_0(X, \theta(\omega(Y))) - g_0(Y, \theta(\omega(X))).$$

If the g^0 metric on P_{J_0} is Kähler we still cannot expect a priori that the Hermitian metric g (with respect to J) on P will be Kähler. The Hopf-Calabi-Eckmann manifolds furnish examples of bundles in which the base (a product of projective spaces) and fibre (a torus) are Kähler but the total spaces do not even possess a Kähler metric (see [10] for details).

We shall now specialize the results of Proposition 2.4.2 to the real product case. Let η be a \hat{G} -valued 1-form on M such that $\eta(J_0 X) = i\eta(X)$. We define the Kähler defect 2-form of η , Γ^η , by for $z \in M$, $\lambda \in G$, $A, A' \in T_z(M)$, $B, B' \in T_\lambda(G)$,

$$\Gamma_{z,\lambda}^\eta((A, B), (A', B)) = g_G(B, (dR_\lambda)_* \eta(A')) - g_G(B', (dR_\lambda)_* \eta(A))$$

where g_G is a right invariant Hermitian metric on G .

THEOREM 2.4.3. *Let g_M be a Hermitian metric on M , g_G a right invariant Hermitian metric on G . Let*

$$g_{z,\lambda}((A, B), (A', B')) = 2\{g_M(A, A') + g_G(B, B')\} + g_G(iB, (dR_\lambda)_e \eta(A')) \\ + g_G(iB', (dR_\lambda)_e \eta(A)) + g_G(\eta(A), \eta(B))$$

and let $J=J^n$ be an almost complex structure on $M \times G$ as in Theorem 1.2.6; then g is a Hermitian metric on $(M \times G)_J$ which is Kähler if and only if $d\Gamma^n=0$.

Proof. Let g_0 be the product Riemannian metric on $M \times G$ (i.e. $g_0((A, B), (A', B')) = g_M(A, A') + g_G(B, B')$). If $p=(z, \lambda)$ then $\theta_p(C)=(0, (dR_\lambda)_e(C))$ for all $C \in \hat{G}$ and so g is Hermitian by Proposition 2.4.1. However, if $X=(A, B)$, $Y=(A', B')$ then

$$g_0(X, \theta(\omega(Y))) = g_0((A, B), (0, dR_\lambda(\eta(A')))) = g_G(B, dR_\lambda(\eta(A'))),$$

hence

$$g_0(X, \theta\omega(Y)) - g_0(Y, \theta\omega(X)) = g_G(B, dR_\lambda(\eta(A'))) - g_G(B', dR_\lambda(\eta(A))) \\ = \Gamma_{z,\lambda}((A, B), (A', B')).$$

We shall now compute the Kähler defect of $\eta \in \Lambda^1(M, \hat{G})$ if G is a complex torus. We may then identify \hat{G} with C^r and $R_\lambda: G \rightarrow G$ is given by $R_\lambda(g)=g+\lambda$ for $g, \lambda \in G$, hence $(dR_\lambda)_e$ is the identity for any $\lambda \in G$. Let $\eta = \sum_{i=1}^{2r} \eta_i e_i$ where $\{e_1, \dots, e_r\}$ is the usual basis for $R^{2r}=C^r$. Locally we have the coordinate chart $(x^1, \dots, x^n, y^1, \dots, y^n)$. Let $m^i = x^i$, $i \leq n$, $m^{i+n} = y^i$, for $i > n$; then $\{dm^i\}_{i=1}^{2n}$ form a local basis for $\Lambda^1(M)$, $\{\delta/\delta m^i\}$ form a local basis for $T(M)$. Let $\{dg^i\}_{i=1}^{2r}$ and $\{\delta/\delta g^i\}$ be defined similarly for G . We may then write

$$\Gamma(z, \lambda) = \sum_{i,j} \{\Gamma_{ij} dm^i \wedge dm^j + \Gamma_{i,r+j} dm^i \wedge dg^j + \Gamma_{n+i,r+j} dg^i \wedge dg^j\},$$

$$\Gamma_{ij}(z, \lambda) = \Gamma_{z,\lambda}\left(\frac{\delta}{\delta m^i}, \frac{\delta}{\delta m^j}\right) = g\left(0, \eta\left(\frac{\delta}{\delta m^j}\right)\right) - g\left(0, \eta\left(\frac{\delta}{\delta m^i}\right)\right) = 0;$$

hence $\Gamma_{ij}=0$ for $1 \leq i, j \leq n$. Similarly $\Gamma_{n+i,r+j}=0$.

$$\Gamma_{i,n+j}(z, \lambda) = \Gamma_{z,\lambda}\left(\frac{\delta}{\delta m^i}, \frac{\delta}{\delta g^j}\right) = g(0, \eta(0)) - g\left(\frac{\delta}{\delta g^j}, \eta\left(\frac{\delta}{\delta m^i}\right)\right) \\ = -g\left(\frac{\delta}{\delta g^j}, \sum_k \eta_k \left(\frac{\delta}{\delta m^i}\right) e_k\right) = \sum_k g_{jk} \eta_k \left(\frac{\delta}{\delta m^i}\right) = -\eta_j \left(\frac{\delta}{\delta m^i}\right) = -\eta_{ji}$$

where $\eta_k = \sum_l \eta_{kl} dm^l$, so

$$\Gamma(z, \lambda) = -\sum_{i,j} \eta_{ji}(z) dm^i \wedge dg^j = -\sum_j \eta_j \wedge dg^j.$$

Observing that Γ is independent of $\lambda \in G$, we have $d\Gamma(z, \lambda) = -\sum_j d\eta_j \wedge dg^j$, hence

THEOREM 2.4.4. *Let G be any complex torus, then the following are equivalent:*

- (a) $(M \times G)_J$ is Kähler in the metric of Theorem 2.4.3.
- (b) $(M \times G)_J$ is almost Kähler in the metric of Theorem 2.4.4.
- (c) $d\eta=0$.

Added in proof. The author now has an example of a real product bundle with compact base and abelian structure group which does not admit a holomorphic connection, showing that the assumption of Kähler base is necessary in Theorem 3.1.7.

BIBLIOGRAPHY

1. M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207. MR **19**, 172.
2. L. Auslander, L. Green and F. Hahn, *Flows on homogeneous spaces*, Ann. of Math. Studies, no. 53, Princeton Univ. Press, Princeton, N. J., 1963. MR **29** #4841.
3. H. Cartan, *Problèmes globaux dans la théorie des fonctions analytiques de plusieurs variables complexes*, Proc. Internat. Congress of Mathematicians (Cambridge, Mass., 1950), vol. 1, Amer. Math. Soc., Providence, R. I., 1952, pp. 152–164. MR **13**, 548.
4. S. S. Chern, *Complex manifolds without potential theory*, Van Nostrand Math. Studies, no. 15, Van Nostrand, Princeton, N. J., 1967. MR **37** #940.
5. C. Chevalley, *Theory of Lie groups*. I, Princeton Math. Series, vol. 8, Princeton Univ. Press, Princeton, N. J., 1946. MR **7**, 412.
6. J. Frenkel, *Cohomologie non abélienne et espaces fibrés*, Bull. Soc. Math. France **85** (1957), 135–220. MR **20** #4662.
7. R. Gunning, *Lectures on Riemann surfaces*, Princeton Math. Notes, Princeton Univ. Press, Princeton, N. J., 1966. MR **34** #7789.
8. S. Helgason, *Differential geometry and symmetric spaces*, Pure and Appl. Math., vol. 12, Academic Press, New York, 1962. MR **26** #2986.
9. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. Vol. I, Interscience, New York, 1963. MR **27** #2945.
10. ———, *Foundations of differential geometry*. Vol. II, Interscience Tracts in Pure and Appl. Math., no. 15, vol. 2, Interscience, New York, 1969. MR **38** #6501.
11. Séminaire de topologie algébrique de l'École Normale Supérieure, 1950–1951, Exposés 14–20 par H. Cartan. MR **14**, 670.
12. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR **35** #1007.
13. N. E. Steenrod, *The topology of fibre bundles*, Princeton Math. Series, vol. 14, Princeton Univ. Press, Princeton, N. J., 1951. MR **12**, 522.

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