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RESTRICTED MEAN VALUES AND HARMONIC FUNCTIONS

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Abstract. A function h defined on a region R in $\mathbf{R}^n$ will be said to possess a restricted mean value property if the value of the function at each point is equal to the mean value of the function over one open ball in R, with centre at that point. It is proved here that this restricted mean value property implies h is harmonic under certain conditions.

1. **Introduction.** Let R be a region in $\mathbf{R}^n$. For each $x \in R$, let $S(x)$ be a ball in R with centre x. Suppose h is a function defined on R such that

$$h(x) = [\mu(S(x))]^{-1} \int_{S(x)} h(y)\, \mu(dy) \tag{1.1}$$

for each x in R, μ being the ordinary Lebesgue measure on $\mathbf{R}^n$. In this case h will be said to possess the restricted mean value property with respect to the balls $S(x)$. The purpose of this paper is to show that, under certain conditions, (1.1) implies that h is harmonic. It was proved in the paper of Akcoglu and Sharpe [1] that if h is bounded, if R is the unit disc in $\mathbf{R}^2$, and if for each $x \in R$, $S(x)$ is the largest disc in R with centre x, then (1.1) implies h is harmonic. The results in this paper give the following theorem.

THEOREM (1.2). *Suppose* (a) *h is bounded and measurable*;

(b) $\boldsymbol{R}$, *the closure of R in $\mathbf{R}^n$, is a compact C^1 manifold with boundary*;

(c) *the radius of $S(x)$ is a measurable function of x and is greater than some fixed fraction of the distance from x to R^c, for all x in R.*

Then (1.1) *implies h is harmonic on R.*

2. **Restatement of the problem.** Let $F(x, y)$ be a measurable function defined on $R \times R$. Suppose

(2.1) $F(x, y)$ is bounded in x for each fixed y, and

(2.2) $\int |F(x, y)|\, \mu(dy) \leqq 1$ for each x.

Define a linear operator T on $\mathscr{L}_1(R, \mu)$ by the equation

(2.3) $Tf(y) = \int f(x)F(x, y)\, \mu(dx)$ for each $y \in R$ and $f \in \mathscr{L}_1(R, \mu)$.

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It is easy to show that T is well defined and $\|T\| \leqq 1$. Clearly

(2.4) $T^*h(x) = \int h(y)F(x, y)\, \mu(dy)$ for each $x \in R$ and $h \in \mathscr{L}_\infty(R, \mu)$.

In particular, suppose

(2.5) $F(x, y) = [\mu(S(x))]^{-1}\chi_{S(x)}(y)$, where (1.2(c)) holds.

Then (2.1) and (2.2) are satisfied. (2.4) becomes

(2.6) $T^*h(x) = [\mu(S(x))]^{-1} \int_{S(x)} h(y)\, \mu(dy)$.

Thus (1.1) is equivalent to

(2.7) $T^*h = h$.

To prove Theorem (1.2) is therefore the same as showing that any function left unaltered by T^* is harmonic. A proof can be given for a more general $F(x, y)$ than that defined by (2.5). We will now list the properties which will be required of $F(x, y)$ in addition to (2.1) and (2.2). We require

(2.8) $T^*h = h$ for all h which are harmonic on R.

We require that $\exists \alpha > 0$, $c_1 > 0$ such that, $\forall x \in R$,

(2.9) $F(x, y) \geqq \alpha[\mu(S_{c_1 d(x, R^c)}(x))]^{-1}$ on $S_{c_1 d(x, R^c)}(x)$.

Here $S_r(p)$ denotes the ball of radius r and centre p, and $d(x, R^c)$ denotes the distance from x to R^c. In future we will write $\mu(S_c(p))$ as $m(c)$, for brevity.

Let $B^r = \{x \mid x \in R, d(x, R^c) < r\}$ for each $r > 0$.

The next property that $F(x, y)$ must have is best expressed in terms of T. It might be called the "boundary-strip" property. Intuitively, it means that if one chooses a sufficiently broad strip between a function f and the boundary of R, then the operator T cannot make the function jump over the strip. In precise terms:

For every $\varepsilon > 0$, a number $\lambda > 0$ can be found such that, for any number $d > 0$ and any $f \geqq 0$ in $\mathscr{L}_1(R, \mu)$,

(2.10) $\|\chi_{B^d - B^{\lambda d}} T \chi_{R - B^d} f\|_1 \geqq (1-\varepsilon)\|\chi_{B^d} T \chi_{R - B^d} f\|_1$.

It is easy to see that if (2.5) holds, then (2.10) is true. Using radial coordinates, one finds it is sufficient to have $(1-\lambda)^n(2\lambda)^{-n} \geqq 1-\varepsilon$.

THEOREM (2.11). *If* (1.2(b)), (2.1), (2.2), (2.8), (2.9), *and* (2.10) *are true, then any function left unaltered by* T^* *is harmonic.*

The proof of Theorem (2.11) is completed in §5, where the main lemma, Lemma (5.4), is given. Some simpler lemmas make up the other sections. To conclude this section, some properties of T are proved, which depend only upon (2.1), (2.2), and (2.8).

We know that $\|T\| \leqq 1$ and $T^*h = h$ for every bounded harmonic function. Then, in particular,

(2.12) $\int f\, d\mu = \int Tf\, d\mu$ for every f in $\mathscr{L}_1(R, \mu)$.

Hence

(2.13) $\|T\| = 1$, and for every $f \geqq 0$ in $\mathscr{L}_1(R, \mu)$,

(2.14) $Tf \geqq 0$ and $\|f\|_1 = \|Tf\|_1$.

Each function $f \geqq 0$ in $\mathscr{L}_1(R, \mu)$ defines an absolutely continuous measure and may be pictured as having a certain mass. When T acts on f, it pushes its mass about but does not destroy any of its mass.

Let E be a measurable subset of R. *Define* T_E on $\mathscr{L}_1(R, \mu)$ by

(2.15) $T_E f = \chi_E f + T\chi_{E^c} f$ for each f in $\mathscr{L}_1(R, \mu)$.

One may regard T_E intuitively as that partial operation of T which holds fixed whatever mass of a function lies on E, and moves the remaining mass of the function as much as possible.

Clearly,

(2.16) $\chi_E T_E^k f$ increases to a limit, $\Gamma_E f$, for each $f \geqq 0$ in $\mathscr{L}_1(R, \mu)$, as $k \to \infty$.

Γ_E may be extended to a linear operator on $\mathscr{L}_1(R, \mu)$. Then

(2.17) $\|\Gamma_E\| \leqq 1$ and $\Gamma_E f = 0$ on E^c for any f.

If $f \geqq 0$, then $\Gamma_E f \geqq 0$.

Intuitively, $\Gamma_E f$ is the end result of pushing more and more of the mass of f onto E, to the greatest degree possible, and then disregarding that part of f which will not leave E^c.

LEMMA (2.18). *Let E and F be measurable subsets of R, with $E \subset F$. Then $\Gamma_E = \Gamma_E \Gamma_F$.*

Proof. It is sufficient to show $\Gamma_E f = \Gamma_E \Gamma_F f$ for $f \geqq 0$ in $\mathscr{L}_1(R, \mu)$.

It will be shown by induction that

(2.19) $\chi_E T_E^k \chi_F T_F^l f \geqq \chi_E T_E^k f$ whenever $l \geqq k$.

For $k=0$, (2.19) reduces to $\chi_E T_F^l f \geqq \chi_E f$, which is clearly true.

Assume (2.19) is true for some $k \geqq 0$. Then, for $l \geqq k+1$,

$$\begin{aligned}
\chi_E T_E^{k+1} \chi_F T_F^l f &= \chi_E T_E^{k+1} \chi_F T_F^l \chi_F f + \chi_E T_E^{k+1} \chi_F T_F^l \chi_{F^c} f \\
&= \chi_E T_E^{k+1} \chi_F f + \chi_E T_E^{k+1} \chi_F T_F^{l-1} T_E \chi_{F^c} f \\
&\geqq \chi_E T_E^{k+1} \chi_F f + \chi_E T_E^k \chi_F T_F^{l-1} T_E \chi_{F^c} f \\
&\geqq \chi_E T_E^{k+1} \chi_F f + \chi_E T_E^k T_E \chi_{F^c} f \quad \text{(using the induction hypothesis)} \\
&= \chi_E T_E^{k+1} f.
\end{aligned}$$

Thus (2.19) holds with k replaced by $k+1$, so (2.19) is true for all k.

Letting $l \to \infty$ and then $k \to \infty$ in (2.19),

(2.20) $\Gamma_E \Gamma_F f \geqq \Gamma_E f$.

On the other hand,

(2.21) $\chi_E T_E^{k+l} f \geqq \chi_E T_E^k \chi_F T_F^l f$ for all $k \geqq 0$, $l \geqq 0$.

In fact

(2.22) $\chi_E T_E^{k+l} f \geqq \chi_E T_E^k T_F^l f$ for all $k \geqq 0$, $l \geqq 0$.

(2.22) can be proved by induction on l.

Letting $k \to \infty$ and then $l \to \infty$ in (2.21),

(2.23) $\Gamma_E f \geqq \Gamma_E \Gamma_F f$.

Thus the lemma is proved.

The definitions of T and of the partial applications of T follow the paper of Akcoglu and Sharpe [1].

It is convenient to express (2.10) again in terms of Γ rather than T.

LEMMA (2.24). *For every $\varepsilon>0$, a number $\lambda>0$ can be found such that, for any number $d>0$ and any $f\geqq 0$ in $\mathscr{L}_1(R,\mu)$,*

$$\|\chi_{B^d-B^{\lambda d}}\Gamma_{B^d}\chi_{R-B^d}f\|_1 \geqq (1-\varepsilon)\|\Gamma_{B^d}\chi_{R-B^d}f\|_1.$$

Proof. Choose λ as given by (2.10). By induction,

$$\|\chi_{B^d-B^{\lambda d}}T^k_{B^d}\chi_{R-B^d}f\|_1 \geqq (1-\varepsilon)\|\chi_{B^d}T^k_{B^d}\chi_{R-B^d}f\|,$$

for all $k\geqq 1$. Letting $k\to\infty$, the result follows.

3. Dissipation and cancellation properties.

LEMMA (3.1). *Let E be a measurable subset of R with $d(E,R^c)=d>0$. Then, for each $f\geqq 0$ in $\mathscr{L}_1(R,\mu)$ such that $f=0$ outside E,*

$$\int f(y)|y|^2\,\mu(dy)+(\alpha(c_1d)^2/8)\|f\|_1 \leqq \int Tf(y)|y|^2\,\mu(dy).$$

Proof.

$$\begin{aligned}\int Tf(y)|y|^2\,\mu(dy) &= \int\int f(x)F(x,y)|y|^2\,\mu(dx)\,\mu(dy)\\ &= \int\int f(x)F(x,y)|y-x|^2\,\mu(dx)\,\mu(dy)+\int\int f(x)F(x,y)2x\cdot y\,\mu(dx)\,\mu(dy)\\ &\quad -\int\int f(x)F(x,y)|x|^2\,\mu(dx)\,\mu(dy)\\ &= \int f(x)\left[\int F(x,y)|y-x|^2\,\mu(dy)\right]\mu(dx)+\int T^*_y(f(x)2x\cdot y)(x)\,\mu(dx)\\ &\quad -\int T_x(|x|^2f(x))(y)\,\mu(dy).\end{aligned}$$

Here T^*_y means T^* acting on functions of y, etc.

Of these three terms, the first is

$$\begin{aligned}&\geqq \int f(x)\left[\int_{S_{c_1d(x,R^c)}(x)}\alpha|y-x|^2m(c_1\,d(x,R^c))^{-1}\mu(dy)\right]\mu(dx)\\ &\geqq \int_E (f(x)\alpha(c_1\,d(x,R^c))^2/8)\,\mu(dx) \geqq (\alpha(c_1\,d)^2/8)\|f\|_1,\end{aligned}$$

using (2.9). Since $f(x)2x\cdot y$ is harmonic in y, $T^*_y(f(x)2x\cdot y)(z)=f(x)2x\cdot z$, for all z. Thus the second of the three terms is $\int f(x)2|x|^2\,\mu(dx)$. Since $\|Tg\|_1=\|g\|_1$ for any $g\geqq 0$ in $\mathscr{L}_1(R,\mu)$, the third of the three terms is $\int f(x)|x|^2\,\mu(dx)$. Substituting, the lemma is proved.

COROLLARY (3.2). $\int Tf(y)|y|^2\,\mu(dy)\geqq\int f(y)|y|^2\,\mu(dy)$, *for every $f\geqq 0$ in $\mathscr{L}_1(R,\mu)$.*

COROLLARY (3.3). $\int Tf(y)|y|^2\,\mu(dy) \geqq \int f(y)|y|^2\,\mu(dy)+(\alpha(c_1\,d)^2/8)\|\chi_Ef\|_1$, *for every $f\geqq 0$ in $\mathscr{L}_1(R,\mu)$.*

COROLLARY (3.4).

$$\int T^k f(y)|y|^2\,\mu(dy) \geqq \int f(y)|y|^2\,\mu(dy) + (\alpha(c_1 d)^2/8)[\|\chi_E f\|_1 + \|\chi_E Tf\|_1 + \cdots + \|\chi_E T^{k-1}f\|_1],$$

for every $f \geqq 0$ *in* $\mathscr{L}_1(R, \mu)$ *and every* $k \geqq 1$.

Since $|y^2|$ is bounded for $y \in R$, Corollary (3.4) shows that T is dissipative. In particular,

(3.5) $\lim_{k\to\infty} \|\chi_E T^k f\|_1 = 0$ for every f in $\mathscr{L}_1(R, \mu)$.

LEMMA (3.6). *Let* $E \subset R$ *be a measurable set such that* $d(E^c, R^c) > 0$. *Then* $\Gamma_E = \lim_{k\to\infty} T_E^k$ *(strongly).*

Proof. Follows at once from (3.5).

COROLLARY (3.7). *If* $f \geqq 0$, $f \in \mathscr{L}_1(R, \mu)$, *then* $\|\Gamma_E f\|_1 = \|f\|_1$.

COROLLARY (3.8). Γ_E^* *leaves unaltered any function left unaltered by* T^*.

LEMMA (3.9). *Let* $\lambda > 0$ *be fixed. Then a number* $\gamma > 0$ *and an integer* $k_0 > 0$ *exist such that for any ball* $S_c(p) \subset R$ *with* $d(S_c(p), R^c) = \lambda c$, *and for any* $f \geqq 0$ *in* $\mathscr{L}_1(R, \mu)$,

$$(T\chi_{S_c(p)})^{k_0} f \geqq \gamma m(c)^{-1} \|\chi_{S_c(p)} f\|_1$$

on $S_c(p)$.

Proof. Fix $c > 0$ and $p \in \mathbf{R}^n$. Define $U = U(c, p)$ on $\mathscr{L}_1(S_c(p), \mu)$ by

$$Uf(y) = \int_{S_c(p)} f(x)\alpha m(c_1(\lambda+2)c)^{-1} \chi_{S_{c_1\lambda c}(x)}(y)\,\mu(dx), \tag{3.10}$$

for each y in $S_c(p)$, $f \in \mathscr{L}_1(S_c(p), \mu)$. From (3.10) and (2.9), $T\chi_{S_c(p)} \geqq U(c, p)$ whenever $S_c(p) \subset R$ and $d(S_c(p), R^c) = \lambda c$. We will establish the inequality for the operator $U(c, p)$. Clearly

$$Uf(y) = \alpha[c_1^n(\lambda+2)^n m(c)]^{-1} \int_{S_c(p)\cap S_{c_1\lambda c}(y)} f(x)\,\mu(dx). \tag{3.11}$$

Hence U defines a continuous map from $\mathscr{L}_1(S_c(p), \mu)$ into $\mathscr{C}(\mathbf{S}_c(p))$, the Banach space of continuous functions on $\mathbf{S}_c(p)$.

Also, if f is continuous on $\mathbf{S}_c(p)$,

$$\begin{aligned} &\{y \mid y \in \mathbf{S}_c(p),\ Uf(y) > 0\} \\ &\quad = \{y \mid y \in \mathbf{S}_c(p),\ |y-x| < c_1\lambda c \text{ for some } x \text{ in } S_c(p) \text{ with } f(x) > 0\}. \end{aligned} \tag{3.12}$$

So by induction, for any $k \geqq 1$,

$$\begin{aligned} &\{y \mid y \in \mathbf{S}_c(p),\ U^k f(y) > 0\} \\ &\quad = \{y \mid y \in \mathbf{S}_c(p),\ |y-x| < kc_1\lambda c \text{ for some } x \text{ in } S_c(p) \text{ with } f(x) > 0\}. \end{aligned} \tag{3.13}$$

Choose k such that $kc_1\lambda > 2$. Then for any continuous function f on $S_c(p)$ which is nonnegative and nonzero at some point,

(3.14) $U^k f > 0$ everywhere on $S_c(p)$.

Hence

(3.15) $U^{k+1}f > 0$ everywhere on $S_c(p)$ for any $f \geqq 0$ in $\mathscr{L}_1(S_c(p), \mu)$, $f \neq 0$.

For each $z \in S_c(p)$, $y \in S_c(p)$, let

$$G_z(y) = \alpha m(c_1(\lambda+2)c)^{-1}\chi_{S_{c_1\lambda c}(z)}(y). \tag{3.16}$$

Then $z \to G_z$ defines a continuous map from $S_c(p)$ into $\mathscr{L}_1(S_c(p), \mu)$. Hence $z \to U^{k+1}G_z$ defines a continuous map from $S_c(p)$ into $\mathscr{C}(S_c(p))$. So, finally,

$$z \to \inf_{y \in S_c(p)} [U^{k+1}G_z](y)$$

defines a continuous map from $S_c(p)$ into $\mathbf{R}$. By (3.15) this last map is never zero. Hence there exists $\beta > 0$ such that

(3.17) $U^{k+1}G_z > \beta$ on $S_c(p)$ for all $z \in S_c(p)$.

Let $f \geqq 0$ be in $\mathscr{L}_1(S_c(p), \mu)$. Let $h \geqq 0$ be in $\mathscr{L}_\infty(S_c(p), \mu)$.

$$\begin{aligned}
\int_{S_c(p)} h(y)U^{k+2}f(y)\mu(dy) &= \int_{S_c(p)} U^{k+1*}h(y)Uf(y)\mu(dy) \\
&= \int_{S_c(p)}\int_{S_c(p)} U^{k+1*}h(y)f(z)G_z(y)\mu(dz)\mu(dy) \\
&= \int_{S_c(p)} f(z)\int_{S_c(p)} U^{k+1*}h(y)G_z(y)\mu(dy)\mu(dz) \\
&\geqq \int_{S_c(p)} f(z)\mu(dz)\int_{S_c(p)} h(y)\beta\mu(dy) \\
&= \beta\|f\|_1 \int_{S_c(p)} h(y)\mu(dy).
\end{aligned} \tag{3.18}$$

It follows from (3.18) that

(3.19) $U^{k+2}f \geqq \beta\|f\|_1$ on $S_c(p)$.

Let $k_0 = k+2$. Let $\omega = \omega(c, p)$ be the largest number such that

(3.20) $U^{k_0}f \geqq \omega\|f\|_1$ on $S_c(p)$ for each $f \geqq 0$ in $\mathscr{L}_1(S_c(p), \mu)$. By (3.19), $\omega \geqq \beta > 0$. We wish to show that $\omega(c, p)\mu(S_c(p))$ is a constant, γ.

Obviously $\omega(c, p)$ is independent of p, $\omega(c, p) = \omega(c)$. For any $c > 0$, $\sigma > 0$, $f \in \mathscr{L}_1(R, \mu)$, $f \geqq 0$,

$$\begin{aligned}
U(\sigma c, 0)f(y) &= \alpha[c_1^n(\lambda+2)^n m(\sigma c)]^{-1}\int_{S_{\sigma c}(0)\cap S_{c_1\lambda\sigma c}(y)} f(x)\mu(dx) \\
&= \alpha[c_1^n(\lambda+2)^n m(c)\sigma^n]^{-1}\int_{S_c(0)\cap S_{c_1\lambda c}(y/\sigma)} f(\sigma z)\sigma^n\,\mu(dz) = U(c, 0)g(y/\sigma),
\end{aligned} \tag{3.21}$$

where $g(x) = f(cx)$, and $g \in \mathscr{L}_1(S_c(0), \mu)$.

Hence $U^k(\sigma c, 0)f(y) = U^k(c, 0)g(y/\sigma)$ for all $k \geqq 0$. Hence by (3.20),

(3.22) $U^{k_0}(\sigma c, 0)f \geqq \omega(c)\|g\|_1$ on $S_{\sigma c}(0)$, so $U^{k_0}(\sigma c, 0)f \geqq \omega(c)\sigma^{-n}\|f\|_1$. Hence $\omega(\sigma c) \geqq \omega(c)\sigma^{-n}$. Replacing σ by $1/\sigma$ and c by σc, $\omega(c) \geqq \omega(\sigma c)\sigma^n$. Hence $\omega(c) = \omega(\sigma c)\sigma^n$ or $\omega(c)c^n$ is constant. Thus

(3.23) $\omega(c)\mu(S_c(p))$ is constant, $= \gamma > 0$.

Thus, rewriting (3.20), for any $c > 0$, $p \in R$, $f \geqq 0$ in $\mathscr{L}_1(\mathbf{R}^n, \mu)$, $U^{k_0}f \geqq \gamma m(c)^{-1}\|\chi_{S_c(p)}f\|_1$ on $S_c(p)$. Hence the lemma is proved.

Lemma (3.24). *Let $\lambda > 0$ be fixed. Then a number $\gamma > 0$ exists such that for any ball $S_c(p) \subset R$ with $d(S_c(p), R^c) \geqq \lambda c$, and for any functions f and g in $\mathscr{L}_1(R, \mu)$ with $f = 0 = g$ outside $S_c(p)$, and $\int f\, d\mu = \int g\, d\mu$; if $r \leqq \lambda c$ then $\|\Gamma_{B^r}(f-g)\|_1 \leqq (1-\gamma)\|f-g\|_1$.*

Proof. We may assume $d(S_c(p), R^c) = \lambda c$, and also $f \geqq 0$, $g \geqq 0$, and $fg = 0$.

$$\|\Gamma_{B^r}(f-g)\|_1 = \lim_{k\to\infty} \|\chi_{B^r}T^k_{B^r}(f-g)\|_1.$$

Let $\gamma > 0$ and k_0 be as in Lemma (3.9). Then

$$\begin{aligned} \|\Gamma_{B^r}(f-g)\|_1 &\leqq \|T^{k_0}_{B^r}(f-g)\|_1 \\ &\leqq \|[T^{k_0}_{B^r}f - \gamma m(c)^{-1}\|f\|_1\chi_{S_c(p)}]\|_1 + \|[T^{k_0}_{B^r}g - \gamma m(c)^{-1}\|g\|_1\chi_{S_c(p)}]\|_1, \end{aligned}$$

where each function in square brackets is nonnegative for $r \leqq \lambda c$. Hence

$$\|\Gamma_{B^r}(f-g)\|_1 \leqq \|f\|_1 - \gamma\|f\|_1 + \|g\|_1 - \gamma\|g\|_1 = \|f-g\|_1 - \gamma\|f-g\|_1.$$

Lemma (3.24) is the desired cancellation property of T.

4. Consequences of the smoothness of the boundary.

It is assumed from now on that $\boldsymbol{R}$ is a compact C^1 manifold with boundary. More concretely,

(4.1) For each x in ∂R, the boundary of R, there exists an open set U containing x, and a C^1 function f defined on U, such that $|\nabla f|^2 > 0$ on U, and $R \cap U = \{y \mid y \in U, f(y) > 0\}$.

Lemma (4.2). *For any $\varepsilon > 0$, a number $\delta > 0$ can be found such that, if $0 < d < \delta$ and $x \in \partial R$, then a coordinate system $(y_1, \ldots, y_n)$ can be chosen, with x as origin, such that $|y-x| < d$, $y_n > \varepsilon d$ imply $y \in R$, and $|y-x| < d$, $y_n < -\varepsilon d$ imply $y \in R^c$.*

Lemma (4.2) is a direct consequence of assumption (4.1).

Lemma (4.3). *Let $c_2 > 0$ and $\lambda > 0$ be fixed. Then numbers $\delta > 0$ and $c > 0$ can be found such that, if $0 < d < \delta$ and $x \in \partial R$, and $E = \{y \mid y \in B^d - B^{\lambda d}, |y-x| < c_2 d\}$, then a ball S of radius cd can be found with $E \subset S \subset R$ and $d(S, R^c) \geqq (1/2)\lambda d$.*

Lemma (4.3) follows from Lemma (4.2) and the compactness of $\boldsymbol{R}$.

Lemma (4.4). *Let $\varepsilon > 0$ be given. Then a number $c_2 > 0$ can be found such that, if θ is a function harmonic on R, continuous on $\boldsymbol{R}$, $0 \leqq \theta \leqq 1$ on $\boldsymbol{R}$, and if for some x in R, $\theta(y) = 0$ whenever y is in ∂R and $|y-x| < c_2\, d(x, R^c)$, then $\theta(x) < \varepsilon$.*

Proof. Let $U=\{z \mid z_1^2+\cdots+z_{n-1}^2<1, 0<z_n<1\}$. Let ϕ be a function harmonic on U, continuous and nonnegative on $\boldsymbol{U}$, such that

(4.5) $\phi=1$ on $\partial U \cap \{z \mid z_n>0\}$ and

(4.6) $\phi(0)=0$.

Then a number $\sigma>0$ exists such that $\phi(z)<\varepsilon$ for $|z|<\sigma$. Let

(4.7) $c_2=4/\sigma+3$.

Choose $\delta>0$ such that, for any $d<\delta$, and any $x\in R$ with $d(x, R^c)=d$, a coordinate system $(y_1,\ldots,y_n)$ can be chosen with $x=(0,\ldots,0,x_n)$, $|x_n|<2d$, and so that, for any y with $|y|<5d/\sigma$,

(4.8) $y_n>d$ implies y is in R and $y_n<d/2$ implies y is in R^c.

Suppose θ is a function harmonic on R, continuous on $\boldsymbol{R}$, and $0\leqq\theta\leqq 1$ on $\boldsymbol{R}$; and, for some x in R, $\theta(y)=0$ for all points y in ∂R with $|y-x|<c_2 d$, $d<\delta$, where $d=d(x, R^c)$. It will be shown that $\theta(x)<\varepsilon$.

Using the coordinate system $y=(y_1,\ldots,y_n)$ just described, let

$$U' = \{y \mid y = (2d/\sigma)z, z\in U\}.$$

Clearly if $y\in U'$, then

(4.9) $|y|<4d/\sigma$ and $|y-x|<4d/\sigma+2d<c_2 d$.

Hence $\theta(y)=0$ for all y in $\partial R\cap \boldsymbol{U}'$.

Let ϕ' be defined on U' by $\phi'(y)=\phi(\sigma y/2d)$. θ and ϕ' are both defined on $\boldsymbol{U}'\cap\boldsymbol{R}$. Consider $\partial(\boldsymbol{U}'\cap\boldsymbol{R})=(\partial U'\cap\boldsymbol{R})\cup(\boldsymbol{U}'\cap\partial R)$. Clearly

$$\partial U'\cap \boldsymbol{R}\subset \partial U'\cap\{y \mid y_n \geqq d/2\}. \tag{4.10}$$

Therefore $\phi'=1$ on $\partial U'\cap\boldsymbol{R}$. Also $\theta=0$ on $\boldsymbol{U}'\cap\partial R$. Hence $\phi'\geqq\theta$ on $\partial(\boldsymbol{U}'\cap\boldsymbol{R})$. Therefore $\phi'\geqq\theta$ on $U'\cap R$.

$$\phi'(x) = \phi((0,\ldots,0,\sigma x_n/2d))$$

and $\sigma x_n/2d<\sigma$. Therefore $\phi'(x)<\varepsilon$ and, accordingly, $\theta(x)<\varepsilon$.

It was seen that the numbers c_2 and δ chosen are such that c_2 satisfies the requirements of the lemma, provided that we restrict our attention to x in R with $d(x, R^c)<\delta$. By choosing a new c_2, larger than the old, such that $c_2\delta$ is greater than the diameter of R, the proof is finished.

5. A cancellation property for equivalent functions.

(5.1) Two measures ω and ν on $\boldsymbol{R}$ will be called equivalent if $\int h\,d\omega=\int h\,d\nu$ for every function h which is harmonic on R, and continuous on $\boldsymbol{R}$. Two functions f and g in $\mathscr{L}_1(R,\mu)$ will be called equivalent if they are equivalent when regarded as (absolutely continuous) measures.

LEMMA (5.2). *For every measure ν on $\boldsymbol{R}$, there is a unique measure $\hat{\nu}$ on ∂R which is equivalent to ν.*

Proof. Since the Dirichlet problem is soluble for R (because of (4.1)), Lemma (5.2) follows at once from the Riesz representation theorem.

LEMMA (5.3). *Let $(X, \mathscr{F}, m)$ be a measure space, $m(X)<\infty$. Suppose for any set $A\in\mathscr{F}$ and any β, $0\leqq\beta\leqq 1$, that a set $B\in\mathscr{F}$ can be found such that $B\subset A$ and $m(B)=\beta m(A)$. Now let $A_1,\dots,A_k$ be fixed sets in $\mathscr{F}$. Let E be the set of all points in $\boldsymbol{R}^k$ of the form $(m(P_1),\dots,m(P_k))$, where $P_1,\dots,P_k$ are mutually disjoint sets in $\mathscr{F}$ with $P_i\subset A_i$ for $i=1,\dots,k$, $\bigcup P_i=\bigcup A_i$. Then E is compact.*

Proof. The proof follows by induction on k, or by consideration of the partition generated by the sets $A_1,\dots,A_k$.

Actually the same result holds for any finite measure space, but only the special case is needed here.

LEMMA (5.4). *A number $\gamma_0>0$ exists such that, if f and g are in $\mathscr{L}_1(R,\mu)$ and f and g are equivalent, then*

$$\lim_{r\to 0}\|\Gamma_{B^r}(f-g)\|_1 \leqq \|f-g\|_1(1-\gamma_0).$$

Proof.

(5.5) By Lemma (4.4), a number $c_2>1$ can be chosen such that, if θ is a function harmonic on R, continuous on $\boldsymbol{R}$, $0\leqq\theta\leqq 1$ on $\boldsymbol{R}$, and if for some x in R, $\theta(y)=0$ whenever y is in ∂R and $|y-x|<c_2\, d(x,R^c)$, then $\theta(x)<1/20$.

(5.6) By Lemma (2.24) a number $\lambda>0$ can be chosen such that, for any number $d>0$ and any $f\geqq 0$ in $\mathscr{L}_1(R,\mu)$,

$$\|\chi_{B^d-B^{\lambda d}}\Gamma_{B^d}\chi_{R-B^d}f\|_1 \geqq (39/40)\|\Gamma_{B^d}\chi_{R-B^d}f\|_1 = (39/40)\|\chi_{R-B^d}f\|_1.$$

(5.7) By Lemma (4.3), numbers $\delta>0$ and $c>0$ can be chosen such that, if $0<d<\delta$, and $y\in\partial R$, and $E=\{z\mid z\in B^d-B^{\lambda d},\ |z-y|<2c_2d\}$, then a ball S of radius cd can be found such that $E\subset S\subset R$ and $d(S,R^c)\geqq(1/2)\lambda d$.

(5.8) By Lemma (3.24) (with λ replaced by $\lambda/2c$, a number $\gamma>0$ can be chosen such that, for any ball $S\subset R$ of radius cd with $d(S,R^c)\geqq(1/2)\lambda d$, and for all functions f and g in $\mathscr{L}_1(R,\mu)$ with $f=0=g$ outside S and $\int f\,d\mu=\int g\,d\mu$; if $r\leqq(1/2)\lambda d$, then $\|\Gamma_{B^r}(f-g)\|_1\leqq(1-\gamma)\|f-g\|_1$.

(5.9) Let $\gamma_0=(1/20)\gamma$.

Now let f and g be equivalent functions in $\mathscr{L}_1(R,\mu)$. To prove the lemma, it clearly can be assumed that $f\geqq 0$, $g\geqq 0$, and $fg=0$. Choose $d>0$, $d<\delta$ such that

$$\text{(5.10)}\qquad \|\chi_{B^d}f\|_1 \leqq (1/39)\|f\|_1, \quad\text{and}\quad \|\chi_{B^d}g\|_1 \leqq (1/39)\|g\|_1.$$

It will be shown that for $r\leqq(1/2)\lambda d$, $\|\Gamma_{B^r}(f-g)\|_1\leqq\|f-g\|_1(1-\gamma_0)$, which will prove the lemma.

By (5.10) and (5.6),

$$\text{(5.11)}\qquad \|\chi_{B^d-B^{\lambda d}}\Gamma_{B^d}f\|_1 \geqq (19/20)\|f\|_1 \quad\text{and}\quad \|\chi_{B^d-B^{\lambda d}}\Gamma_{B^d}g\|_1 \geqq (19/20)\|f\|_1.$$

Let $\hat\nu$ be the measure on ∂R which is equivalent to f and g. Let $p_1,\dots,p_k$ be points in ∂R such that the collection of balls $S_{c_2d}(p_i)$, $i=1,\dots,k$, covers ∂R. Then the collection $S_{2c_2d}(p_i)$, $i=1,\dots,k$, covers B^d, since $c_2>1$.

Let E_i, $i=1,\ldots,k$, be an ordered partition of (all of) ∂R such that

(5.12) $E_i \subset S_{c_2 d}(p_i)$, $i=1,\ldots,k$.

Let $\mathscr{C}$ be the class of all ordered partitions $\mathscr{P}=\{P_1,\ldots,P_k\}$ of (all of) B^d such that

(5.13) $P_i \subset S_{2c_2 d}(p_i)$, $i=1,\ldots,k$.

Because of the compactness property stated in Lemma (5.3), the sum

$$\sum_{i=1}^{k} \left| \int_{P_i} \Gamma_{B^d} f\, d\mu - (1/2)\hat{\nu}(E_i) \right|^2 \tag{5.14}$$

defined for partitions $\mathscr{P} \in \mathscr{C}$ is minimal when $\mathscr{P}$ equals some partition $\mathscr{Q} =\{Q_1,\ldots,Q_k\}$.

Let

$$I^+ = \left\{ i \,\middle|\, \int_{Q_i} \Gamma_{B^d} f\, d\mu > (1/2)\hat{\nu}(E_i) \right\}; \quad I^- = \left\{ i \,\middle|\, \int_{Q} \Gamma_{B^d} f\, d\mu \leqq (1/2)\hat{\nu}(E_i) \right\}. \tag{5.15}$$

Because of the minimality of (5.14) when $\mathscr{P}=\mathscr{Q}$, it follows that, for any i in I^+ and any j in I^-,

$$\int_{Q_i \cap S_{2c_2 d}(p_j)} \Gamma_{B^d} f\, d\mu = 0. \tag{5.16}$$

Let θ be a function which is harmonic on R, continuous on $\boldsymbol{R}$, $0 \leqq \theta \leqq 1$ on $\boldsymbol{R}$, such that

(5.17) $\theta=0$ on ∂R outside $\bigcup_{i\in I^-} S_{c_2 d}(p_i)$, and

(5.18) $\int \theta\, d\hat{\nu} \geqq \frac{3}{4} \sum_{i\in I^-} \hat{\nu}(E_i)$.

From (5.5),

(5.19) $\theta < (1/20)$ on B^d outside $\bigcup_{i\in I^-} S_{2c_2 d}(p_i)$.

Hence from (5.16)

$$\int_{Q_i} \theta \Gamma_{B^d} f\, d\mu \leqq (1/20) \int_{Q_i} \Gamma_{B^d} f\, d\mu, \tag{5.20}$$

for each i in I^+.

Because $\Gamma_{B^d} f$ and $\hat{\nu}$ are equivalent,

$$\int_{B^d} \theta \Gamma_{B^d} f\, d\mu \geqq \frac{3}{4} \sum_{i\in I^-} \hat{\nu}(E_i), \tag{5.21}$$

by (5.18). But also

$$\begin{aligned} \int_{B^d} \theta \Gamma_{B^d} f\, d\mu &= \sum_{i=1}^{k} \int_{Q_i} \theta \Gamma_{B^d} f\, d\mu \\ &= \sum_{i\in I^+} \int_{Q_i} \theta \Gamma_{B^d} f\, d\mu + \sum_{i\in I^-} \int_{Q_i} \theta \Gamma_{B^d} f\, d\mu \leqq (1/20)\|f\|_1 + \frac{1}{2} \sum_{i\in I^-} \hat{\nu}(E_i), \end{aligned}$$

using (5.20) and (5.15).

Then from (5.21)

$$\sum_{i\in I^-} \hat{\nu}(E_i) \leqq (1/5)\|f\|_1. \tag{5.22}$$

Let I^* denote the subset of I^+, consisting of those indices i in I^+ such that

$$\int_{Q_i\cap(B^d-B^{\lambda d})} \Gamma_{B^d}f\,d\mu \geqq (1/4)\hat{\nu}(E_i). \tag{5.23}$$

For $i\in I^+-I^*$, clearly

$$\int_{Q_i\cap B^{\lambda d}} \Gamma_{B^d}f\,d\mu \geqq (1/4)\hat{\nu}(E_i).$$

Therefore

$$\frac{1}{4}\sum_{i\in I^+-I^*} \hat{\nu}(E_i) \leqq \int_{B^{\lambda d}} \Gamma_{B^d}f\,d\mu. \tag{5.24}$$

From (5.11) follows

(5.25) $\sum_{i\in I^+-I^*} \hat{\nu}(E_i)\leqq\frac{1}{5}\|f\|_1$.

From (5.22) and (5.25),

(5.26) $\sum_{i\in I^*} \hat{\nu}(E_i)\geqq\frac{3}{5}\|f\|_1$.

Repeating all these arguments for g instead of f, a partition $\mathscr{Q}'=\{Q_1',\dots,Q_k'\}$ of B^d can be found such that

(5.27) $Q_i'\subset S_{2c_2d}(p_i)$, $i=1,\dots,k$,

and a set of indices $I^{*\prime}$ can be found such that

(5.28) $\sum_{i\in I^{*\prime}} \hat{\nu}(E_i)\geqq(3/5)\|g\|_1$ and

(5.29) $\int_{Q_i'\cap(B^d-B^{\lambda d})} \Gamma_{B^d}g\,d\mu\geqq(1/4)\hat{\nu}(E_i)$ for each $i\in I^{*\prime}$.

Then

(5.30) $\sum_{i\in I^*\cap I^{*\prime}} \hat{\nu}(E_i)\geqq(1/5)\|f\|_1=(1/5)\|g\|_1$.

For each $i\in I^*\cap I^{*\prime}$, let S_i be a ball of radius cd such that $S_i\subset R$, S_i contains $S_{2c_2d}(p_i)\cap B^d-B^{\lambda d}$, and $d(S_i, R^c)\geqq(1/2)\lambda d$. Such a ball exists by (5.7).

Then

(5.31) $Q_i\cap(B^d-B^{\lambda d})\subset S_i$ and $Q_i'\cap(B^d-B^{\lambda d})\subset S_i$, for each i in $I^*\cap I^{*\prime}$. Let

(5.32) $f_i=\chi_{Q_i\cap(B^d-B^{\lambda d})}\Gamma_{B^d}f$ and $g_i=\chi_{Q_i'\cap(B^d-B^{\lambda d})}\Gamma_{B^d}g$, for each $i\in I^*\cap I^{*\prime}$.

From (5.23) and (5.29),

(5.33) $\|f_i\|_1\geqq(1/4)\hat{\nu}(E_i)$ and $\|g_i\|_1\geqq(1/4)\hat{\nu}(E_i)$.

Then by (5.8) and (5.33), for any $r\leqq(1/2)\lambda d$,

$$\|\Gamma_{B^r}(f_i-g_i)\|_1 \leqq \|f_i\|_1+\|g_i\|_1-(1/2)\gamma\hat{\nu}(E_i), \tag{5.34}$$

for each i in $I^*\cap I^{*\prime}$. Then

$$\|\Gamma_{B^r}\Gamma_{B^d}(f-g)\|_1 \leqq \|\Gamma_{B^d}f\|_1+\|\Gamma_{B^d}g\|_1-(1/2)\gamma\sum_{i\in I^*\cap I^{*\prime}}\hat{\nu}(E_i). \tag{5.35}$$

Hence

$$\|\Gamma_{B^r}(f-g)\|_1 \leqq \|f\|_1+\|g\|_1-(1/10)\gamma\|f\|_1 \tag{5.36}$$

by (5.30) or

$$\|\Gamma_{B^r}(f-g)\|_1 \leqq \|f-g\|_1-(1/20)\gamma\|f-g\|_1. \tag{5.37}$$

Since $\gamma_0=(1/20)\gamma$, the proof is finished.

COROLLARY (5.38). *If f and g are equivalent functions in $\mathscr{L}_1(R,\mu)$, then* $\lim_{r\to 0}\|\Gamma_{B^r}(f-g)\|_1=0$.

(5.39) **Proof of Theorem (2.11).** For each ε, $0<\varepsilon\leqq 1$, let

$$G_\varepsilon(x,y) = m(\varepsilon d(x,R^c))^{-1}\chi_{S_{\varepsilon d(x,R^c)}(x)}(y). \tag{5.40}$$

Clearly $G_\varepsilon(x,y)$ satisfies (2.1) and (2.3). Define U_ε on $\mathscr{L}_1(R,\mu)$ by

$$U_\varepsilon f(y) = \int f(x)G_\varepsilon(x,y)\mu(dx) \tag{5.41}$$

for each $y\in R$, $f\in\mathscr{L}_1(R,\mu)$.

If f is continuous and bounded, then clearly

(5.42) $\lim_{\varepsilon\to 0} U_\varepsilon f=f$ ($\mathscr{L}_1$-norm).

Since $\|U_\varepsilon\|\leqq 1$ for all ε, and U_ε converges strongly to I on a dense subset of $\mathscr{L}_1(R,\mu)$, it follows that

(5.43) $\lim_{\varepsilon\to 0} U_\varepsilon f=f$ for every $f\in\mathscr{L}_1(R,\mu)$.

Now let h be in $\mathscr{L}_\infty(R,\mu)$ such that $T^*h=h$. Then

(5.44) $\Gamma^*_{B^r}h=h$ for any $r>0$,

so

(5.45) $\int h\Gamma_{B^r}f\,d\mu=\int hf\,d\mu$ for any f in $\mathscr{L}_1(R,\mu)$, any $r>0$.

Let ε_1 and ε_2 be fixed, $0<\varepsilon_1$, $\varepsilon_2\leqq 1$. Clearly $G_{\varepsilon_1}(x,y)$ and $G_{\varepsilon_2}(x,y)$ are equivalent functions of y, for any fixed x. Then by Corollary (5.38),

(5.46) $\lim_{r\to 0}\|\Gamma_{B^r}(G_{\varepsilon_1}-G_{\varepsilon_2})\|_1=0$.

Hence from (5.45),

(5.47) $\int h(y)G_{\varepsilon_1}(x,y)\mu(dy)=\int h(y)G_{\varepsilon_2}(x,y)\mu(dy)$ for every x.

From (5.41), (5.47) is just another way of writing

(5.48) $U^*_{\varepsilon_1}h=U^*_{\varepsilon_2}h$.

Letting $\varepsilon_2\to 0$ and using (5.43),

(5.49) $\int U^*_{\varepsilon_1}hf\,d\mu=\int hf\,d\mu$ for any f in $\mathscr{L}_1(R,\mu)$.

Hence

(5.50) $U^*_{\varepsilon_1}h=h$.

Hence, from the definition of U^*_ε, h is continuous and

(5.51) $\int h(y)G_\varepsilon(x,y)\mu(dy)=h(x)$ for any x in R, or

$$h(x) = m(\varepsilon d(x,R^c))^{-1}\int h(y)\chi_{S_{\varepsilon d(x,R^c)}(x)}(y)\mu(dy). \tag{5.52}$$

Since ε can have any value, $0<\varepsilon\leqq 1$, (5.52) says h has the (ordinary) mean value property at every point. Hence by a well-known result, h is harmonic.

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Reference

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