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SOME THEOREMS ON THE $\cos \pi\lambda$ INEQUALITY[1]

BY
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Abstract. In this paper we consider subharmonic functions $u \leqq 1$ in the unit disk whose minimum modulus and maximum modulus satisfy a certain inequality. We show the existence of an extremal member of this class with largest maximum modulus. We then obtain an upper bound for the maximum modulus of this function in terms of the logarithmic measure of a certain set. We use this upper bound to prove theorems about subharmonic functions in the plane.

1. **Introduction.** Let n be a positive integer. Let $\Gamma = \bigcup_{i=1}^{n} [-r_i', -r_i]$, where $0 \leqq r_i < r_i' \leqq r_{i+1} \leqq 1$, $1 \leqq i \leqq n$, and $r_{n+1} = 1$. Put $\Gamma^+ = \{r : -r \in \Gamma\}$.

Let u be subharmonic in $\{|z| < 1\}$, and put $m(r) = \inf_{|z|=r} u(z)$, $M(r) = \max_{|z|=r} u(z)$, for $0 < r < 1$. In this paper for given $\lambda \in (0, 1)$ we shall consider subharmonic functions u in $\{|z| < 1\}$ which satisfy

$$m(r) \leqq \cos \pi\lambda \, M(r), \qquad r \in \Gamma^+ - \{0, 1\}, \tag{1.1}$$

$$u \leqq 1. \tag{1.2}$$

For such a fixed Γ and λ we shall prove

THEOREM 1. *There exists a function* $U = U(\cdot, \Gamma, \lambda)$ *which has the following properties*:

(i) *U is bounded, continuous and subharmonic in* $\{|z| < 1\}$,

(ii) *U is harmonic in* $\{|z| < 1\} - \Gamma$,

(iii) $\lim_{z \to e^{i\theta}} U(z) = 1$, $|\theta| < \pi$,

(iv) $U(-r) = \cos \pi\lambda \, U(r)$, $r \in \Gamma^+ - \{1\}$,

(v) *if u is subharmonic in* $\{|z| < 1\}$, *and if u satisfies* (1.1) *and* (1.2), *then* $M(r, u) \leqq U(r)$, $0 < r < 1$,

(vi) *U is the unique function which satisfies* (i)–(v),

(vii) *if* $r \in [0, 1)$, *then*

$$U(r) \leqq C(\lambda) \exp \left[-\lambda \int_{\Gamma^+ \cap [r, 1]} \frac{dt}{t} \right],$$

where $C(\lambda)$ is a positive constant which depends only on λ.

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(1) This paper is based upon the author's doctoral dissertation, University of Illinois, Urbana, Illinois, 1970, directed by Professor Maurice Heins.

We mention that Beurling (see [4, pp. 44–64]) proved Theorem 1 for $\lambda=\frac{1}{2}$. Moreover, Hellsten, Kjellberg, and Norstad [7] proved the following theorem which will be used in §6.

THEOREM A. *If* $\lambda\in(0, 1)$, *if* $\Gamma=[-1, 0]$ *and if* $r\in[0, 1)$, *then*

$$U(r) \leqq (2/\pi\lambda)\tan(\pi\lambda/2)\, r^{\lambda},$$

where the constant is best possible.

Now suppose that h is a nonconstant subharmonic function in $\boldsymbol{C}$ and let

$$\rho = \limsup_{r\to\infty}\frac{\log M(r)}{\log r},\qquad k = \liminf_{r\to\infty}\frac{\log M(r)}{\log r}.$$

ρ and k are called the order and lower order of h respectively.

Let E be a Lebesgue measurable set of nonnegative real numbers. Given r, R, $0\leqq r<R<\infty$, let

$$E(r, R) = En[r, R],\qquad M_lE(r, R) = \int_{E(r,R)}\frac{dt}{t},$$

$$\bar{D}_lE = \limsup_{R\to\infty}\frac{m_lE(1, R)}{\log R},\qquad \underline{D}_lE = \liminf_{R\to\infty}\frac{m_lE(1, R)}{\log R}.$$

$\bar{D}_lE$ and $\underline{D}_lE$ are called the upper and lower logarithmic densities of E respectively.

Using this notation we have, as a first application of Theorem 1,

THEOREM 2. *Let h be a nonconstant subharmonic function in* $\boldsymbol{C}$. *If* $\rho<\lambda<1$, *then*

$$\underline{D}_l\{m(r) > \cos\pi\lambda\, M(r)\} \geqq 1-\rho/\lambda.$$

We mention that Barry [2] proved Theorem 2 using a different method than ours. Also Besicovitch [3], Huber [8], and Kjellberg [9] proved somewhat weaker theorems than Theorem 2.

As a second application of Theorem 1, we shall prove

THEOREM 3. *Let h be a nonconstant subharmonic function in* $\boldsymbol{C}$. *If* $\varepsilon>0$ *and* $k<\lambda<1$, *then*

$$\bar{D}_l\{m(r)-\cos\pi\lambda\, M(r) > r^{k-\varepsilon}\} \geqq 1-k/\lambda.$$

We remark that Kjellberg (see [9, pp. 19–21]) proved Theorem 3 for $\lambda=\frac{1}{2}$. He deduced this theorem from Beurling's Theorem (Theorem 1 with $\lambda=\frac{1}{2}$). In a similar way we shall deduce Theorems 2 and 3 from Theorem 1. We also remark that the lower bounds in Theorems 2 and 3 are best possible, as we shall show in a forthcoming paper.

Finally, we prove

THEOREM 4. *Let h be a nonconstant subharmonic function in* $\boldsymbol{C}$. *If*

$$m_l\{r > 1 : m(r) > \cos\pi\lambda\, M(r)\} < \infty,$$

then $\lim_{r\to\infty} M(r)/r^{\lambda}=a$. *Here* $0<a\leqq+\infty$.

We note that Kjellberg [11] proved Theorem 4 when $\{m(r) > \cos \pi\lambda\, M(r)\}$ is bounded.

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2. **Notation.** We introduce here some notation which will be used throughout the proof of Theorem 1. Let $\Delta = \{|z| < 1\}$, $\Omega = \Delta - \Gamma$, $G = \Delta - \{\Gamma \cup \Gamma^+\}$, $H = \{|z| < 1\} \cap \{\operatorname{Re} z > 0\}$. Given v subharmonic in an open set O, we write, as in Beurling [4, p. 22, b], $v^{\to}(\zeta) = \limsup_{z \to \zeta} v(z)$ when $\zeta \in \partial O$. Moreover, if $\lim_{z \to \zeta} v(z)$ exists, we shall denote this limit by $v^{\leftrightarrow}(\zeta)$.

3. **Proof of Theorem 1.** The proof of Theorem 1 is long. In §§3–5 we construct U and show that U satisfies (i)–(vi). These sections are motivated by the work of Hellsten, Kjellberg, and Norstad [7]. In §§6–9 we prove (vii) in the special case $\Gamma = [-r_1', -r_1]$ and $r = 0$. Finally, in §10 we reduce the general case to the case $\Gamma = [-r_n', (-r_1 r_2 \cdots r_n)/(r_1' r_2' \cdots r_{n-1}')]$ and $r = 0$. This section is based upon a method of Beurling [4, pp. 60–62].

Let u be subharmonic in Δ and suppose that u satisfies (1.1) and (1.2). It is well known (see Tsuji [13, IV.10]) that u can be written in the form

$$u = u_1 + u_2 \tag{3.1}$$

where

$$u_1(z) = \int_{|\zeta|<1} \log \left| \frac{z-\zeta}{1-\bar{\zeta} z} \right| d\mu(\zeta), \qquad z \in \Delta,$$

$$u_2(z) = 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\beta(\theta), \qquad z \in \Delta.$$

Here μ is the Riesz mass associated with u. β is a nondecreasing function on $[-\pi, \pi]$ which is continuous on the left and satisfies $\beta(-\pi) = 0$. $-u_1$ is a Green's potential and u_2 is the least harmonic majorant of u in Δ.

Let

$$u_1^*(z) = \int_{|\zeta|<1} \log \left| \frac{z+|\zeta|}{1+z|\zeta|} \right| d\mu(\zeta), \qquad z \in \Delta,$$

$$u_2^*(z) = 1 - \frac{1}{2\pi} \operatorname{Re} \frac{1-z}{1+z} \int_{-\pi}^{\pi} d\beta(\theta), \qquad z \in \Delta,$$

$$u^* = u_1^* + u_2^*. \tag{3.2}$$

Thus u^* is subharmonic in Δ, harmonic in $\Delta - (-1, 0]$, and $u^* \leqq 1$. Moreover, using the inequalities

$$\frac{|z|-|\zeta|}{1-|z|\,|\zeta|} \leqq \left| \frac{z-\zeta}{1-\bar{\zeta} z} \right| \leqq \frac{|z|+|\zeta|}{1+|\zeta|\,|z|}, \tag{3.3}$$

$$\frac{1-|z|}{1+|z|} \leqq \operatorname{Re} \frac{e^{i\theta}+z}{e^{i\theta}-z} \leqq \frac{1+|z|}{1-|z|}, \tag{3.4}$$

it is easily verified (see Hellsten, Kjellberg, and Norstad [7]) that

(3.5) $u^*(-|z|) \leqq m(|z|) \leqq M(|z|) \leqq u^*(|z|)$, $z \in \Delta$,

(3.6) $u^*(-|z|) \leqq u^*(z) \leqq u^*(|z|)$, $z \in \Delta$,

(3.7) $u^*(-|z|) + u^*(|z|) \leqq u(-z) + u(z)$, $z \in \Delta$,

(3.8) $u^*(-|z|) - \cos \pi\lambda\, u^*(|z|) \leqq m(|z|) - \cos \pi\lambda\, M(|z|)$, $z \in \Delta$.

From (3.6) and (3.8) it follows that u^* also satisfies (1.1) and (1.2). In view of (3.5) we see that it suffices to prove (v) for u^*.

4. **Existence of** U. Now we shall construct U. For this purpose let u_1 be the bounded continuous function on $\bar{\Delta} - \{-1\}$ which is harmonic in Ω and satisfies $u_1(-r) = 0$, $r \in \Gamma^+ - \{1\}$; $u_1(e^{i\theta}) = 1$, $|\theta| < \pi$.

Proceeding by induction, we let u_{n+1}, n a positive integer, be the bounded continuous function on $\bar{\Delta} - \{-1\}$ which is harmonic in Ω and satisfies $u_{n+1}(-r) = \cos \pi\lambda\, u_n(r)$, $r \in \Gamma^+ - \{1\}$; $u_{n+1}(e^{i\theta}) = 1$, $|\theta| < \pi$. For the existence and uniqueness of u_{n+1} see Heins [5, Theorems 12.3 and 12.4].

We shall prove that $u_n|_\Delta \to U$. For this purpose we note that

$$|u_2 - u_1|(-r) = |\cos \pi\lambda| u_1(r) \leqq |\cos \pi\lambda|, \qquad r \in \Gamma^+ - \{1\},$$

and $|u_2 - u_1|(e^{i\theta}) = 0$, $|\theta| < \pi$. Since $u_2 - u_1$ is harmonic in Ω and continuous on $\bar{\Delta} - \{-1\}$, it follows from the Phragmén-Lindelöf Maximum Principle (see Heins [5, p. 76]) that $|u_2 - u_1| \leqq |\cos \pi\lambda|$.

Let k be a positive integer. Then if

$$|u_{k+1} - u_k| \leqq |\cos \pi\lambda|^k, \tag{4.1}$$

we have

$$|u_{k+2} - u_{k+1}|(-r) = |\cos \pi\lambda|\, |u_{k+1} - u_k|(r) \leqq |\cos \pi\lambda|^{k+1}$$

when $r \in \Gamma^+ - \{1\}$, and

$$|u_{k+2} - u_{k+1}|(e^{i\theta}) = 0, \qquad |\theta| < \pi.$$

Since $u_{k+2} - u_{k+1}$ is harmonic in Ω, it follows that $|u_{k+2} - u_{k+1}| \leqq |\cos \pi\lambda|^{k+1}$. Hence by induction (4.1) is true for each positive integer k.

If m, n are positive integers and $m > n$, then by (4.1) we see that

$$\begin{aligned} |u_m - u_n| &= \left| \sum_{k=n}^{m-1} (u_{k+1} - u_k) \right| \\ &\leqq \sum_{k=n}^{m-1} |u_{k+1} - u_k| \leqq \sum_{k=n}^{m-1} |\cos \pi\lambda|^k \leqq \frac{|\cos \pi\lambda|^n}{1 - |\cos \pi\lambda|}. \end{aligned} \tag{4.2}$$

From (4.2) we see that $(u_n)_1^\infty$ tends uniformly to a continuous function V on $\bar{\Delta} - \{-1\}$, which is harmonic in Ω. Moreover,

$$V(-r) = \lim_{n \to \infty} u_n(-r) = \cos \pi\lambda \lim_{n \to \infty} u_{n-1}(r) = \cos \pi\lambda\, V(r)$$

when $r \in \Gamma^+ - \{1\}$ and $V(e^{i\theta}) = 1$, $|\theta| < \pi$. Let $U = V|\Delta$. Then from our previous work we conclude that U satisfies the stipulated properties (ii)–(iv) and that U is bounded and continuous in Δ.

5. **A maximum principle.** We now prove (i), (v), and (vi). Theorem 1 is thereby established save for (vii) which will be treated in §§6–10. We shall want the following lemma.

LEMMA 1. *Let h be subharmonic and bounded above on Ω. Let E be a finite subset (possibly empty) of Γ. For given $\sigma \in (0, 1)$ let h satisfy*

(a) $$h^{\rightarrow}(\zeta) \leqq \cos \pi\sigma\, h(|\zeta|), \qquad \zeta \in \Gamma - (E \cup \{0, -1\}),$$

and either the condition

(b) $$h^{\rightarrow}(e^{i\theta}) \leqq 0, \qquad |\theta| < \pi,$$

or alternatively, the three conditions (c), (d), (e):

(c) $$\limsup_{z \to e^{i\theta}} [h(z) + h(-z)] \leqq 0, \qquad 0 < |\theta| < \pi,$$

(d) $$h(z) = h(\bar{z}), \qquad z \in \Omega,$$

(e) $$h^{\rightarrow}(e^{i\theta}) \leqq 0, \qquad |\theta| \leqq \pi/2;$$

then $h|_H \leqq 0$.

Proof. Let $p(z) = \max\{h(z), h(\bar{z})\}$, $z \in \Omega$, and suppose on the one hand that (a) and (b) are fulfilled. Then p is bounded above and subharmonic on Ω. Moreover, since h satisfies (a) and (b), we have

$$p^{\rightarrow}(\zeta) = h^{\rightarrow}(\zeta) \leqq \cos \pi\sigma\, h(|\zeta|) = \cos \pi\sigma\, p(|\zeta|),$$

when $\zeta \in \Gamma - (E \cup \{0, -1\})$, *and*

$$p^{\rightarrow}(e^{i\theta}) = \max\{h^{\rightarrow}(e^{i\theta}), h^{\rightarrow}(e^{-i\theta})\} \leqq 0, \qquad |\theta| < \pi.$$

On the other hand, if h satisfies (a) and (c)–(e), then by (d) we have $h = p$. Hence in either case p satisfies (a) and (b) or (c), (d), (e).

If p satisfies (b), then p satisfies (c) and (e). Therefore, we assume, as we may, that p satisfies (a) and (c)–(e).

Let $q(z) = p(z) + p(-z)$, $z \in G$; $A = \max\{0, \sup_{r \in (0,1)} p(r)\}$. By (a) we see that

$$q^{\rightarrow}(\zeta) \leqq (1 + \cos \pi\sigma) p(|\zeta|) \leqq (1 + \cos \pi\sigma) A$$

when $\zeta \in \Gamma \cup \Gamma^{+} - (E \cup E^{+} \cup \{0, \pm 1\})$. Here $E^{+} = \{r : -r \in E\}$. Then by (c) and the Phragmén-Lindelöf Maximum Principle, it follows that $q \leqq (1 + \cos \pi\sigma) A$. Since p satisfies (d), we have

$$q(it) = p(it) + p(-it) = 2p(it) \leqq (1 + \cos \pi\sigma) A$$

when $t \in (-1, 1) - \{0\}$, and so

$$p(it) \leqq (1 + \cos \pi\sigma) A/2, \qquad t \in (-1, 1) - \{0\}.$$

Using this inequality and (e), we find that

$$p|_H \leqq (1+\cos \pi\sigma)A/2. \tag{5.1}$$

Hence, $\sup_{r\in(0,1)} p(r) \leqq (1+\cos \pi\sigma)A/2$. Since $0<(1+\cos \pi\sigma)/2<1$ and $A \geqq 0$, it follows that $A=0$. Putting $A=0$ in (5.1) we see that Lemma 1 is true.

We now prove (v). For this purpose let u^* be as in (3.2). Then by properties (iii) and (iv) of U, (3.8), and the fact that u^*-U is upper semicontinuous in Δ, we have

$$[u^*-U]^{\to}|_\Omega(-r) \leqq \cos \pi\lambda\, [u^*-U](r)$$

when $r \in \Gamma^+ - \{0, 1\}$, and

$$[u^*-U]^{\to}|_\Omega(e^{i\theta}) \leqq 0, \qquad |\theta| < \pi.$$

Letting $h=[u^*-U]|_\Omega$ and $\sigma=\lambda$ in Lemma 1, we obtain that $u^*|_H \leqq U|_H$. Hence, $u^*(r) \leqq U(r)$, $r \in (0, 1)$. Since $u^*(r)=M(r, u^*)$, it follows that $u^*(0) \leqq U(0)$. We conclude by (3.5) that $M(r, u) \leqq u^*(r) \leqq U(r)$, $r \in [0, 1)$.

To prove (i) we observe from (v) with $u=0$ that $0 \leqq U(r)$ when $0 \leqq r < 1$. Then from the boundary values of $U(z)-\cos \pi\lambda\, U(-z)$ in G, we find that

$$U(z) \geqq \cos \pi\lambda\, U(-z), \qquad z \in G.$$

Using this inequality, we obtain for $r \in \Gamma^+ - \{0, 1\}$ and $s>0$ small that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} U(-r+se^{i\theta})\, d\theta &\geqq \frac{\cos \pi\lambda}{2\pi}\int_0^{2\pi} U(r-se^{i\theta})\, d\theta \\ &= \cos \pi\lambda\, U(r) = U(-r). \end{aligned}$$

Here we have used the fact that U is harmonic in $\{|z-r|<s\}$ when $s>0$ is small. Moreover, if $0 \in \Gamma$ and $t \in (0, 1)$, then

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} U(te^{i\theta})\, d\theta &\geqq \frac{\cos \pi\lambda}{2\pi}\int_0^{2\pi} U(-te^{i\theta})\, d\theta \\ &= \frac{\cos \pi\lambda}{2\pi}\int_0^{2\pi} U(te^{i\theta})\, d\theta \end{aligned}$$

and so since by (iv), $U(0)=0$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} U(te^{i\theta})\, d\theta \geqq 0 = U(0).$$

Hence (i) is true.

Finally, we prove (vi) (unicity). Suppose that g also satisfies (i)–(v). Then $U-g$ and $g-U$ satisfy the hypotheses of Lemma 1 and hence $g|_H=U|_H$. It follows from this equality and the Identity Theorem for analytic functions that $U \equiv g$. Hence (vi) is true.

6. **A lemma.** In §§6–10 our object is to prove (vii). To avoid confusion it will be necessary to indicate the dependence of U upon the set Γ. Therefore, we shall often write $U(\cdot, \Gamma)$ instead of U.

We shall want the following lemma.

LEMMA 2. *If $r_1 \in (0, 1)$ and $\Gamma = [-1, -r_1]$, then $U(0) \leqq C_1(\lambda) r_1^\lambda$. Here $C_1(\lambda)$ is a positive constant which depends only on λ.*

Proof. First assume that $\lambda \in [\frac{1}{2}, 1)$. Let $\Gamma_1 = [-1, 0]$. We write $U(z) = U(z, \Gamma)$, $U_1(z) = U(z, \Gamma_1)$ when $z \in \Delta$. We consider the function $f(z) = (z + r_1)/(1 + r_1 z)$, $z \in \Delta$. f maps the disk Δ univalently onto Δ and the slit $(-1, -r_1]$ onto $(-1, 0]$. From this remark we see that $U_1 \circ f$ is harmonic in $\Omega = \Delta - \Gamma$.

We note that $U_1(r) = M(r, U_1)$, $r \in [0, 1)$. Indeed, if U_1^* is the subharmonic function in Δ associated with U_1 (see display (3.2)), then by (3.5), (3.8), and (v) of Theorem 1 we have $M(r, U_1) \leqq U_1^*(r) \leqq U_1(r)$, $r \in [0, 1)$. Since $U_1(r) \leqq M(r, U_1)$, it follows that $M(r, U_1) = U_1(r)$ when $r \in [0, 1)$. We also note that $\cos \pi\lambda \leqq 0$ when $\lambda \in [\frac{1}{2}, 1)$.

Using these facts, (iv) of Theorem 1, and (3.3) we obtain for $r \in [r_1, 1)$ that

$$\begin{aligned} U_1 \circ f(-r) &= U_1\left(\frac{-r+r_1}{1-rr_1}\right) = \cos \pi\lambda\, U_1\left(\frac{r-r_1}{1-rr_1}\right) \\ &\geqq \cos \pi\lambda\, U_1\left(\frac{r+r_1}{1+rr_1}\right) = \cos \pi\lambda\, U_1 \circ f(r). \end{aligned}$$

Then by Lemma 1 with $h = [U - U_1 \circ f]|_\Omega$ and $\sigma = \lambda$ we have $U|_H \leqq U_1 \circ f|_H$. In particular, $U(r) \leqq U_1 \circ f(r)$, $r \in (0, 1)$. Letting $r \to 0$ in the above inequality and using Theorem A we obtain

$$U(0) \leqq U_1 \circ f(0) = U_1(r_1) \leqq (2/\pi\lambda) \tan (\pi\lambda/2)\, r_1^\lambda.$$

Hence if $C_1(\lambda) \geqq (2/\pi\lambda) \tan (\pi\lambda/2)$, then Lemma 2 is true when $\lambda \in [\frac{1}{2}, 1)$.

The proof of Lemma 2 for $0 < \lambda < \frac{1}{2}$ is more difficult since conformal mapping does not preserve the $\cos \pi\lambda$ inequality. Here the idea is to control U with the aid of a function W subharmonic in $\boldsymbol{C}$. For this purpose we recall certain standard facts about functions subharmonic in $\boldsymbol{C}$.

7. **Subharmonic functions in $\boldsymbol{C}$.** Let v be subharmonic in $\boldsymbol{C}$, harmonic at 0, and of order < 1. Then v can be represented as (see Heins [6])

$$v(z) = v(0) + \int_{|\zeta| < \infty} \log \left| 1 - \frac{z}{\zeta} \right| d\gamma(\zeta), \qquad z \in \boldsymbol{C}. \tag{7.1}$$

Here γ is the Riesz mass associated with v. Let

$$\tilde{v}(z) = v(0) + \int_{|\zeta| < \infty} \log \left| 1 + \frac{z}{|\zeta|} \right| d\gamma(\zeta), \qquad z \in \boldsymbol{C}. \tag{7.2}$$

It is easily verified (see Anderson [1, (8.3)]) that

$$\tilde{v}(-r) \leqq m(r, v) \leqq M(r, v) \leqq \tilde{v}(r), \qquad 0 < r < \infty, \tag{7.3}$$

$$\tilde{v}(-r) - \cos \pi\sigma\, \tilde{v}(r) \leqq m(r, v) - \cos \pi\sigma\, M(r, v), \qquad 0 < r < \infty. \tag{7.4}$$

Here $\sigma \in (0, 1)$.

Next we state two theorems.

THEOREM B. *Let v be subharmonic in C, and suppose that $0<\sigma<1$. If $a=\liminf_{r\to\infty} M(r)/r^\sigma<\infty$ and*

$$\limsup_{r_1,r_2\to\infty}\int_{r_1}^{r_2}[m(r)-\cos\pi\sigma\, M(r)]\frac{dr}{r^{1+\sigma}}\leqq 0,$$

then $M(r)\sim ar^\sigma$ $(r\to\infty)$. Moreover, $\tilde{v}(r)\sim ar^\sigma$ $(r\to\infty)$.

Proof. (See Anderson [1].)

THEOREM C. *Let v be subharmonic in C, harmonic at zero and of order <1. Suppose that $v(0)=0$. Then if $\sigma\in(0,1)$ we have*

$$\int_{r_1}^{r_2}[\tilde{v}(-r)-\cos\pi\sigma\,\tilde{v}(r)]\frac{dr}{r^{1+\sigma}}\geqq A(\sigma)\frac{\tilde{v}(r_1)}{r_1^\sigma}-B(\sigma)\frac{\tilde{v}(r_2)}{r_2^\sigma}.$$

Here $A(\sigma)=(1-\sin(\pi|\frac{1}{2}-\sigma|))/(\frac{1}{2}-|\frac{1}{2}-\sigma|)$, $0<r_1<r_2<\infty$, and $0<B(\sigma)<10$.

Proof. (See Kjellberg [10, p. 193, (23)].)

Anderson and Kjellberg prove Theorems B and C only for subharmonic functions of the form $v=\log|f|$, where f is entire. However, the proofs are exactly the same in the general case.

Using Theorems B and C we prove

LEMMA 3. *Let v be subharmonic in C, harmonic at zero, and of order <1. Let $0<R_0<\infty$ and suppose that*

(a) $v(-r)\leqq\cos\pi\sigma\, v(r)$, $r\in[R_0,\infty)$,

(b) $v(r)\geqq r^\sigma$, $r\geqq 0$,

(c) $\lim_{r\to\infty} M(r,v)/r^\sigma=1$.

Then if $\sigma\in(0,\frac{1}{2})$ we have $M(r,v)\leqq C_0(\sigma)r^\sigma$, $r\in[R_0,\infty)$. Here $C_0(\sigma)=B(\sigma)/A(\sigma)$ and $A(\sigma)$ and $B(\sigma)$ are defined as in Theorem C.

Proof. Clearly since $\cos\pi\sigma>0$, we have

$$m(r,v)-\cos\pi\sigma\, M(r,v)\leqq v(-r)-\cos\pi\sigma\, v(r). \tag{7.5}$$

Then by (a), (c), and Theorem B it follows that

$$\lim_{r\to\infty}\frac{\tilde{v}(r)}{r^\sigma}=1. \tag{7.6}$$

Let $h=\tilde{v}-v(0)$. By Theorem C we have

$$\int_{r_1}^{r_2}\frac{[h(-r)-\cos\pi\sigma\, h(r)]}{r^{1+\sigma}}dr\geqq A(\sigma)\frac{h(r_1)}{r_1^\sigma}-B(\sigma)\frac{h(r_2)}{r_2^\sigma},\qquad 0<r_1<r_2<\infty.$$

Then by (a), (7.5), and (7.4) we have

$$\int_{(0,R_0)\cap(r_1,r_2)} \frac{[\tilde{v}(-r)-\cos \pi\sigma\, \tilde{v}(r)]}{r^{1+\sigma}}\,dr+(\cos \pi\sigma-1)v(0)\int_{r_1}^{r_2}\frac{dr}{r^{1+\sigma}} \geqq \frac{1-\cos \pi\sigma}{\sigma}\left(\frac{\tilde{v}(r_1)}{r_1^\sigma}-\frac{v(0)}{r_1^\sigma}\right)-B(\sigma)\left(\frac{\tilde{v}(r_2)}{r_2^\sigma}-\frac{v(0)}{r_2^\sigma}\right). \tag{7.7}$$

In the above inequality we have inserted the exact value of $A(\sigma)$.

From (b), (7.3), and (7.6) we see that $\tilde{v}(r)/r^\sigma$, $R_0 \leqq r<\infty$, attains its maximum at a point $r_0 \in [R_0, \infty)$. In (7.7) we let $r_1=r_0$ and $r_2 \to \infty$. We obtain, the first term now being 0,

$$\frac{(\cos \pi\sigma-1)}{\sigma}\frac{v(0)}{r_0^\sigma} = (\cos \pi\sigma-1)v(0)\int_{r_0}^{\infty}\frac{dr}{r^{1+\sigma}} \geqq \frac{1-\cos \pi\sigma}{\sigma}\left(\frac{\tilde{v}(r_0)}{r_0^\sigma}-\frac{v(0)}{r_0^\sigma}\right)-B(\sigma).$$

Thus, $(\tilde{v}(r_0)/r_0^\sigma)(1-\cos \pi\sigma)/\sigma \leqq B(\sigma)$. It follows from this inequality and (7.3) that

$$\frac{M(r, v)}{r^\sigma} \leqq \frac{\sigma B(\sigma)}{1-\cos \pi\sigma} = \frac{B(\sigma)}{A(\sigma)} = C_0(\sigma), \qquad R_0 \leqq r < \infty.$$

This completes the proof of Lemma 3.

Finally in this section we consider the function s defined in $\boldsymbol{C}$ by $s(z)=\operatorname{Re}(z^\lambda)$, $|\arg z| \leqq \pi$. Then s is subharmonic in $\boldsymbol{C}$ and harmonic in $\boldsymbol{C}-(-\infty, 0]$. Moreover,

(7.8) $s(z) \geqq \cos \pi\lambda\, s(-z)$, $z \in \boldsymbol{C}$,

(7.9) $s(-|z|)+s(|z|) \leqq s(z)+s(-z)$, $z \in \boldsymbol{C}$,

(7.10) $s(z) \geqq s(i|z|)$, $\operatorname{Re} z \geqq 0$.

(7.8) and (7.9) can be verified by direct calculation or by using the Phragmén-Lindelöf Theorem (see Heins [5, p. 111]) in the upper and lower half planes. (7.10) is obvious.

8. **Proof of Lemma 2 for** $0<\lambda<\frac{1}{2}$. We now begin the proof of Lemma 2 for $0<\lambda<\frac{1}{2}$. Let W be the bounded continuous function in Δ defined by

(A) W is harmonic in Ω,

(B) $W(-r)=\cos \pi\lambda\, W(r)$, $r \in \Gamma^+-\{1\}$,

(C) $W^{\leftrightarrow}(e^{i\theta})=s(e^{i\theta})$, $|\theta| \leqq \pi$.

Here s is as in §7. The existence and uniqueness of W follow as in §§4–5.

We extend W continuously to $\boldsymbol{C}$ by defining $W(z)=s(z)$, $|z| \geqq 1$. We assert that W is subharmonic in $\boldsymbol{C}$. To verify this assertion we first observe from the boundary values of $W|_\Omega$ and (7.8) that $W(z) \geqq \cos \pi\lambda\, W(-z)$, $z \in \Omega$. Then arguing as in §5 we find that W is subharmonic in Δ. Moreover, applying Lemma 1 with $h=[s-W]|_\Omega$ and $\sigma=\lambda$, we see that $s|_H \leqq W|_H$. Using this fact and comparing the boundary values of both functions in Ω, we obtain $s|_\Delta \leqq W|_\Delta$, and thereupon that $s \leqq W$. Since s is subharmonic in $\boldsymbol{C}$ and $s(z)=W(z)$, $|z| \geqq 1$, it follows upon examining circumferential means that W is subharmonic in $\{|z| \geqq 1\}$. Hence our assertion is true.

Now let U be as in Lemma 2. We proceed to control U with the aid of W. We let

$$\begin{aligned} T(z) &= U(z)-2W(z)/(1+\cos \pi\lambda), \qquad && z \in \Delta, \\ S(z) &= T(z)+T(-z), && z \in \Delta. \end{aligned}$$

We shall show that $T|_\Omega$ satisfies (a) and (c)–(e) in Lemma 1 with $\sigma=\lambda$. Clearly $T|_\Omega$ satisfies (a). To show (d), we first observe that $U(z)=U(\bar{z})$, $z \in \Delta$, since both functions have the same boundary values in Ω. Similarly, $W(z)=W(\bar{z})$, $z \in \Delta$. Hence T satisfies (d) of Lemma 1.

If $0<|\theta|<\pi$, then by (7.9) we see that

$$\begin{aligned} S^{\leftrightarrow}(e^{i\theta}) &= 2-2[W(e^{i\theta})+W(-e^{i\theta})]/(1+\cos \pi\lambda) \\ &\leqq 2-2[W(-1)+W(1)]/(1+\cos \pi\lambda) = 0. \end{aligned}$$

Moreover, if $|\theta| \leqq \pi/2$, then by (7.10) we find that

$$\begin{aligned} T^{\leftrightarrow}(e^{i\theta}) &= 1-\frac{2W(e^{i\theta})}{1+\cos \pi\lambda} \leqq 1-\frac{2\cos(\pi\lambda/2)}{1+\cos \pi\lambda} \\ &= 1-\sec(\pi\lambda/2) \leqq 0. \end{aligned}$$

Therefore T satisfies (c) and (e).

Let $h=T|_\Omega$ and $\sigma=\lambda$. Then from our previous work and Lemma 1 we have $U(z) \leqq 2W(z)/(1+\cos \pi\lambda)$, $z \in H$. Applying Lemma 3 with $v=W$, $\sigma=\lambda$, and $R_0=r_1$, we obtain

$$U(0) \leqq \frac{2}{1+\cos \pi\lambda} W(0) \leqq \frac{2C_0(\lambda)}{1+\cos \pi\lambda} r_1^\lambda.$$

Hence Lemma 2 is true for $0<\lambda<\frac{1}{2}$. Since we have already considered the case $\frac{1}{2} \leqq \lambda < 1$, the proof of Lemma 2 is complete.

9. **Proof of (vii) in a special case.** Using Lemma 2 we shall prove the following lemma.

LEMMA 4. *If* $\Gamma=[-r_1', -r_1]$, *then* $U(0) \leqq C_1(\lambda)(r_1/r_1')^\lambda$.

Proof. If $r_1=0$, then by (iv) of Theorem 1 we have $U(0)=0$, and so Lemma 4 is trivially true. If $r_1>0$, let

$$\Gamma_2 = \left[-1, -\frac{r_1}{r_1'}\right], \qquad U_2 = U(\cdot, \Gamma_2), \qquad U = U(\cdot, \Gamma),$$
$$v(z) = U(r_1'z)-U_2(z), \qquad z \in \Delta.$$

We note that v is harmonic in $\Omega_2=\Delta-\Gamma_2$, and

$$\begin{aligned} v^{\leftrightarrow}(-r) &= \cos \pi\lambda\, v(r), \qquad && r \in \Gamma_2^+-\{1\}, \\ v^{\leftrightarrow}(e^{i\theta}) &\leqq 0, && |\theta| < \pi. \end{aligned}$$

Then by Lemma 1 with $h=v$ and $\sigma=\lambda$ we have $v|_H \leqq 0$. Letting $z \to 0$ in H, we get $U(0) \leqq U_2(0)$. From this inequality and Lemma 2 we conclude that Lemma 4 is true.

10. **Proof of (vii) in the general case.** Next we shall show that (vii) of Theorem 1 is true when $r=0$. Our method of proof is a modification of a method of Beurling [4, pp. 60–62].

We recall that $\Gamma = \bigcup_{i=1}^{n} [-r_i', -r_i]$. We assume, as we may, that the closed segments $[-r_i', -r_i]$ are disjoint. If $n=1$ and $C(\lambda) \geqq C_1(\lambda)$, then by Lemma 4 it follows that (vii) is true when $r=0$. If $r_1=0$, then by (iv), $U(0)=0$. Hence in this case (vii) is trivially true at $r=0$. There remains the situation: $r_1>0$ and $n>1$. In this case let α be the Riesz mass associated with U (see display (3.1)). Since U is harmonic in Ω, it follows by a well-known representation theorem of F. Riesz [12] that the support of α is contained in Γ. Using this fact and (3.1), we see that

$$U(z) = 1+\int_{\Gamma^+} \log\left|\frac{z+t}{1+tz}\right| d\alpha(-t), \qquad z \in \Delta. \tag{10.1}$$

Let

$$\Gamma_1 = \left[-r_2, -\frac{r_1 r_2}{r_1'}\right] \cup \left(\bigcup_{i=2}^{n} [-r_i', -r_i]\right),$$

$$q_1(z) = \int_{r_1}^{r_1'} \log\left|\frac{z+t}{1+zt}\right| d\alpha(-t), \qquad z \in \Delta,$$

$$q_2(z) = \sum_{i=2}^{n} \int_{r_i}^{r_i'} \log\left|\frac{z+t}{1+zt}\right| d\alpha(-t), \qquad z \in \Delta,$$

$$F(z) = 1+q_1\left(\frac{r_1'}{r_2} z\right)+q_2(z), \qquad z \in \Delta.$$

Clearly F is subharmonic in Δ and harmonic in $\Delta - \Gamma_1$.

We note that if $r, t \in [0, 1)$, then

$$\frac{d}{dr} \log\left(\frac{r+t}{1+rt}\right) = \frac{1}{r+t} - \frac{t}{1+rt} = \frac{1-t^2}{(1+rt)(r+t)}. \tag{10.2}$$

Hence for fixed t, $\log((r+t)/(1+rt))$ is nondecreasing on $[0, 1)$.

We shall prove that

$$F(-r) \leqq \cos \pi\lambda\, F(r), \qquad r \in \Gamma_1^+ - \{1, r_2\}. \tag{10.3}$$

To this end, for fixed r, $r_1 r_2/r_1' \leqq r < r_2$, and with $\beta = r_1'/r_2$, let

$$\begin{aligned} \xi_1(\delta) &= 1+q_1(-\beta r)+q_2(-\delta r), \\ \xi_2(\delta) &= 1+q_1(\beta r)+q_2(\delta r), \qquad \beta \leqq \delta \leqq 1. \end{aligned}$$

From Lemma 1 with $h=-U$ and the Maximum Principle we see that $0<U(s)$, $s \in (0, 1)$. Then by (10.2) we have

$$0 < U(r_1) \leqq U(\beta r) = \xi_2(\beta) \leqq \xi_2(\delta), \qquad \delta \in [\beta, 1]. \tag{10.4}$$

We now introduce $A(\delta) = \xi_1(\delta)/\xi_2(\delta)$, $\delta \in [\beta, 1]$. If $\xi_1(1) \leqq -\xi_2(1)$, then

$$A(1) = F(-r)/F(r) \leqq -1 < \cos \pi\lambda.$$

Therefore, we assume that $\xi_1(1) > -\xi_2(1)$. We observe that q_2 is harmonic in $\{|z| < r_2\}$. Since $\delta r < r_2$, $\beta \leqq \delta \leqq 1$, it follows that ξ_1 and ξ_2 are continuous and differentiable on $[\beta, 1]$. Then with the aid of (10.4) we see that A is continuous and differentiable on $[\beta, 1]$.

Let $a = \inf\{\delta : A > -1 \text{ on } (\delta, 1]\}$. We shall show that A is nonincreasing on $[a, 1]$. Given $\delta \in [a, 1]$, we have

$$\begin{aligned} \xi_1'(\delta) &= \frac{d}{d\delta} \int_{\Gamma^+ - [r_1, r_1']} \log \left| \frac{\delta r - t}{1 - \delta r t} \right| d\alpha(-t) \\ &= r \int_{\Gamma^+ - [r_1, r_1']} \left(\frac{1}{\delta r - t} + \frac{t}{1 - \delta r t} \right) d\alpha(-t) \\ &= r \int_{\Gamma^+ - [r_1, r_1']} \frac{(1 - t^2)\, d\alpha(-t)}{(\delta r - t)(1 - \delta r t)}. \end{aligned}$$

Here we have differentiated under the integral sign. It is permissible since $\delta r < r_2 \leqq t$ when $\beta \leqq \delta \leqq 1$. Likewise,

$$\xi_2'(\delta) = r \int_{\Gamma^+ - [r_1', r_1]} \frac{(1 - t^2)\, d\alpha(-t)}{(\delta r + t)(1 + \delta r t)}.$$

Thus,

$$\begin{aligned} [\xi_2(\delta)]^2 A'(\delta) &= r\xi_2(\delta) \int_{\Gamma^+ - [r_1, r_1']} \frac{(1 - t^2)\, d\alpha(-t)}{(\delta r - t)(1 - \delta r t)} - r\xi_1(\delta) \int_{\Gamma^+ - [r_1, r_1']} \frac{(1 - t^2)\, d\alpha(-t)}{(\delta r + t)(1 + \delta r t)} \\ &\leqq r\xi_2(\delta) \int_{\Gamma^+ - [r_1, r_1']} (1 - t^2) \left[\frac{1}{(\delta r + t)(1 + \delta r t)} + \frac{1}{(\delta r - t)(1 - \delta r t)} \right] d\alpha(-t) \\ &= r\xi_2(\delta) \int_{\Gamma^+ - [r_1, r_1']} \frac{2\delta r (1 - t^4)\, d\alpha(-t)}{(\delta^2 r^2 - t^2)(1 - \delta^2 r^2 t^2)} \leqq 0. \end{aligned}$$

Hence A is nonincreasing on $[a, 1]$. It follows that $a = \beta$ and

$$\cos \pi\lambda = U(-\beta r)/U(\beta r) = A(\beta) \geqq A(1) = F(-r)/F(r).$$

Thus (10.3) is true for $r_1 r_2 / r_1' \leqq r < r_2$.

If $r \in \Gamma^+ - [r_1, r_1']$ and $r \neq r_2$, we let

$$\begin{aligned} \eta_1(\delta) &= 1 + q_1(-\delta r) + q_2(-r), \\ \eta_2(\delta) &= 1 + q_1(\delta r) + q_2(r), \qquad \beta \leqq \delta \leqq 1. \end{aligned}$$

From (10.2) we see that

$$(10.5) \qquad 0 < U(r_1') \leqq \eta_2(\beta) \leqq \eta_2(\delta), \qquad \delta \in [\beta, 1].$$

Therefore we let $L(\delta) = \eta_1(\delta)/\eta_2(\delta)$, $\beta \leqq \delta \leqq 1$.

If $\eta_1(\beta) \leqq -\eta_2(\beta)$, then $L(\beta) = F(-r)/F(r) \leqq -1 < \cos \pi\lambda$. Hence we assume that $\eta_1(\beta) > -\eta_2(\beta)$. We observe that q_1 is harmonic in the annulus $\{r_1' < |z| < 1\}$. Since

$\delta r > r_1'$ when $\beta \leqq \delta \leqq 1$, it follows that η_1 and η_2 are continuous and differentiable on $[\beta, 1]$. Then by (10.5) we see that L is differentiable on $[\beta, 1]$. We shall show, using the differential calculus, that L is nondecreasing on $[\beta, 1]$. Indeed, if $\delta \in [\beta, 1]$, then

$$\begin{aligned}[\eta_2(\delta)]^2 L'(\delta) &= r\eta_2(\delta)\int_{r_1}^{r_1'} \frac{(1-t^2)\,d\alpha(-t)}{(\delta r - t)(1-\delta r t)} - r\eta_1(\delta)\int_{r_1}^{r_1'} \frac{(1-t^2)\,d\alpha(-t)}{(\delta r + t)(1+\delta r t)} \\ &\geqq r\eta_2(\delta)\int_{r_1}^{r_1'} (1-t^2)\left[\frac{1}{(\delta r - t)(1-\delta r t)} - \frac{1}{(\delta r + t)(1+\delta r t)}\right] d\alpha(-t) \\ &= r\eta_2(\delta)\int_{r_1}^{r_1'} \frac{2[(\delta r)^2+1]t(1-t^2)\,d\alpha(-t)}{(\delta^2 r^2 - t^2)(1-\delta^2 r^2 t^2)} \geqq 0.\end{aligned}$$

Again we have differentiated under the integral sign. It is permissible since $\delta r > r_1'$. Also, we have used the fact that $\eta_1(\delta) \leqq \eta_2(\delta)$ when $\beta \leqq \delta \leqq 1$. Hence L is nondecreasing on $[\beta, 1]$. It follows that

$$F(-r)/F(r) = L(\beta) \leqq L(1) = U(-r)/U(r) = \cos \pi\lambda.$$

From the above inequality we conclude that (10.3) is true. Since $F^{\rightarrow}(e^{i\theta}) \leqq 1$ when $|\theta| < \pi$, it now follows from Lemma 1 with $h = [F - U(\cdot, \Gamma_1)]|_{\Delta - \Gamma_1}$, $\sigma = \lambda$, and $E = \{-r_2\}$ that $F|_H \leqq U(\cdot, \Gamma_1)|_H$. Letting $z \to 0$ in H, we get

$$U(0, \Gamma) = F(0) \leqq U(0, \Gamma_1). \tag{10.6}$$

We now proceed by induction. Let k be a positive integer and suppose that

$$U(0, \Gamma) \leqq C_1(\lambda) \exp[-\lambda m_l \Gamma^+(0, 1)], \tag{10.7}$$

whenever $n \leqq k$. Here $C_1(\lambda)$ is defined as in Lemma 2 and n, as previously in §10, denotes the number of disjoint closed segments contained in Γ. Then if $n = k+1$ we have

$$\begin{aligned} U(0, \Gamma) \leqq U(0, \Gamma_1) &\leqq C_1(\lambda) \exp[-\lambda m_l \Gamma_1^+(0, 1)] \\ &= C_1(\lambda) \exp[-\lambda m_l \Gamma^+(0, 1)],\end{aligned}$$

thanks to (10.6) and the fact that Γ_1 is the union of k disjoint segments. Since $U(\cdot, \Gamma)$ satisfies (10.7) for $n = 1$, we conclude by induction that (10.7) is true whenever n is a positive integer. Hence, if $C(\lambda) \geqq C_1(\lambda)$, then $U(\cdot, \Gamma)$ satisfies (vii) for $r = 0$.

To prove (vii) for $r \in [0, 1)$ we argue as follows: Let U^* be the function associated with U (see display (3.2)). Then by (10.1) we see that $U = U^*$. Hence by (3.7) with $u = U$ we have

$$U(-|z|) + U(|z|) \leqq U(z) + U(-z), \qquad z \in \Delta. \tag{10.8}$$

Now suppose that $r_0 \in \Gamma^+ - \{1\}$. Let v be the subharmonic function on Δ defined by

$$\begin{aligned} v &= \text{least harmonic majorant of } U \text{ restricted to } \{|z| < r_0\}, \\ v &= U \quad \text{in } \Delta - \{|z| < r_0\}.\end{aligned}$$

Then

$$v(-r) = \cos \pi\lambda \, v(r), \qquad r \in \Gamma^+(r_0, 1)-\{1\},$$

and

$$v^{\leftrightarrow}(e^{i\theta}) = 1, \qquad |\theta| < \pi.$$

Hence by (v) of Theorem 1 and (10.7) we have

$$(10.9) \qquad v(0) \leqq U(0, \Gamma \cap [-1, -r_0]) \leqq C_1(\lambda) \exp[-\lambda m_l \Gamma^+(r_0, 1)].$$

Using (iv) of Theorem 1, (10.8), the fact that $v(0)=(1/2\pi)\int_0^{2\pi} U(r_0e^{i\theta}, \Gamma)\, d\theta$, and (10.9) we obtain

$$\begin{aligned}(1+\cos \pi\lambda)U(r_0, \Gamma) &= U(-r_0, \Gamma)+U(r_0, \Gamma)\\ &\leqq \frac{1}{2\pi}\int_0^{2\pi} [U(r_0e^{i\theta}, \Gamma)+U(-r_0e^{i\theta}, \Gamma)]\, d\theta = 2v(0)\\ &\leqq 2C_1(\lambda)\exp[-\lambda m_l \Gamma^+(r_0, 1)].\end{aligned}$$

Hence,

$$(10.10) \qquad U(r_0, \Gamma) \leqq \frac{2\max\{1, C_1(\lambda)\}}{1+\cos \pi\lambda}\exp[-\lambda m_l \Gamma^+(r_0, 1)].$$

Now consider the remaining case where $r_0 \in [0, 1)-\Gamma^+$. If $m_l\Gamma^+(r_0, 1)=0$, then (10.10) is trivially true since $U \leqq 1$. Therefore we assume that $m_l\Gamma^+(r_0, 1)>0$. Let $k=\min\{i : r_i > r_0\}$. In (10.10) we replace r_0 by r_k. Then by this inequality we have

$$\begin{aligned}U(r_0, \Gamma) \leqq U(r_k, \Gamma) &\leqq \frac{2\max\{C_1(\lambda), 1\}}{1+\cos \pi\lambda}\exp[-\lambda m_l\Gamma^+(r_k, 1)]\\ &= \frac{2\max\{C_1(\lambda), 1\}}{1+\cos \pi\lambda}\exp[-\lambda m_l\Gamma^+(r_0, 1)].\end{aligned}$$

Hence (10.10) is true when $r_0 \in [0, 1)$.

Let $C(\lambda)=2\max\{C_1(\lambda), 1\}/(1+\cos \pi\lambda)$. Then by the preceding argument we see that (vii) is true. This completes the proof of Theorem 1.

11. **Remark.** From our previous work and Theorem C we see that

$$C(\lambda) = \frac{4}{(1+\cos \pi\lambda)^2}\frac{B(\lambda)}{A(\lambda)} \leqq 20\lambda \csc^2(\pi\lambda)\sec^2\left(\frac{\pi\lambda}{2}\right)$$

when $0<\lambda<\frac{1}{2}$ and

$$C(\lambda) \leqq \frac{2}{\pi\lambda}\sec^2\left(\frac{\pi\lambda}{2}\right)\tan\left(\frac{\pi\lambda}{2}\right),$$

when $\frac{1}{2}\leqq\lambda<1$. We note that $C(\frac{1}{2})=8/\pi$. Actually Beurling [4, p. 64] showed that (vii) is true for $\lambda=\frac{1}{2}$ when $C(\frac{1}{2})=(4/\pi)\sqrt{2}$.

12. **Proof of Theorem 2.** Let h be as in Theorem 2. We note that $m(r) - \cos \pi\lambda\, M(r)$ is upper semicontinuous on $(0, \infty)$, as is easily verified. Hence, if $\lambda \in (0, 1)$, then

$$F_\lambda = \{r > 0 : m(r) - \cos \pi\lambda\, M(r) < 0\}$$

is open.

Choose $r_0 > 0$ such that $M(r_0) > 0$. This choice is possible since h is nonconstant. If $R > r_0$, let

$$u(z) = h(Rz)/M(R), \qquad z \in \Delta,$$
$$T = \{r : rR \in F_\lambda\} \cap (r_0/R, 1).$$

Then

$$m(r, u) < \cos \pi\lambda M(r, u), \qquad r \in T,$$

and $u^{\rightarrow}(e^{i\theta}) \leqq 1$, $|\theta| \leqq \pi$.

Now T is open and so $T = \bigcup_{n=1}^{\infty} (s_n, t_n)$, where $r_0/R \leqq s_n < t_n \leqq 1$. Here we allow repetition of intervals. Let

$$T_j = \bigcup_{n=1}^{j} \left[s_n + \frac{(t_n - s_n)}{3j},\ t_n - \frac{(t_n - s_n)}{3j}\right]$$

for j a positive integer. Then by Theorem 1 we have

$$\frac{M(r_0, h)}{M(R, h)} = M(r_0/R, u) \leqq C(\lambda) \exp(-\lambda m_l T_j).$$

Since $\lim_{j\to\infty} m_l T_j = m_l T$, we obtain, as $j \to \infty$, that

$$M(r_0, h)/M(R, h) \leqq C(\lambda) \exp(-\lambda m_l T). \tag{12.1}$$

Using (12.1) we find that

$$\frac{M_l F_\lambda(r_0, R)}{\log R} = \frac{m_l T}{\log R} \leqq \frac{\log M(R, h) + \log C(\lambda) - \log M(r_0, h)}{\lambda \log R}. \tag{12.2}$$

Letting $R \to +\infty$ through suitably chosen sequences, we get

$$\bar{D}_l F_\lambda \leqq \rho/\lambda, \qquad \underline{D}_l F_\lambda \leqq k/\lambda.$$

Let $G_\lambda = (0, +\infty) - F_\lambda$. Then the above inequalities imply that

$$\underline{D}_l G_\lambda \geqq 1 - \rho/\lambda, \tag{12.3}$$

$$\bar{D}_l G_\lambda \geqq 1 - k/\lambda. \tag{12.4}$$

Let $\alpha \in (0, \infty)$ and put $h_1 = h - \alpha$. In (12.3) and (12.4) we replace h by h_1. Then

$$\begin{aligned} &\underline{D}_l\{r : m(r, h) - \cos \pi\lambda\, M(r, h) > 0\} \\ &\quad \geqq \underline{D}_l\{r : m(r, h) - \cos \pi\lambda\, M(r, h) \geqq (1 - \cos \pi\lambda)\alpha\} \\ &\quad = \underline{D}_l\{r : m(r, h_1) - \cos \pi\lambda\, M(r, h_1) \geqq 0\} \\ &\quad \geqq 1 - \rho/\lambda. \end{aligned} \tag{12.5}$$

Likewise,

$$\overline{D}_l\{r : m(r, h) - \cos \pi\lambda\, M(r, h) \geqq (1-\cos \pi\lambda)\alpha\} \geqq 1-k/\lambda. \tag{12.6}$$

From (12.5) we conclude that Theorem 2 is true.

13. **Proof of Theorem 3.** We now prove Theorem 3. We proceed as in Kjellberg [9, pp. 19–21]. If $\varepsilon > 0$ and $k-\varepsilon \leqq 0$, then Theorem 3 follows from (12.6) with α taken as $(1-\cos \pi\lambda)^{-1}$. If $k-\varepsilon > 0$, we choose $r_1 \geqq 2$ such that

$$M(r, h) > r^{(1+\varepsilon)(k-\varepsilon)}, \qquad r \geqq r_1. \tag{13.1}$$

This choice is possible since $(1+\varepsilon)(k-\varepsilon) = k+(k-1)\varepsilon-\varepsilon^2 < k$, and h has lower order $k<1$.

Let

$$\begin{aligned} L &= \{r : m(r, h) < \cos \pi\lambda\, M(r, h) + (1-\cos \pi\lambda) r^{k-\varepsilon}\}, \\ r_n &= r_1^{(1+\delta)^{n-1}}, \\ v_n(z) &= h(z) - r_n^{k-\varepsilon}, \qquad z \in \{|z| < r_n\}, \\ L_n &= \{r : m(r, v_n) < \cos \pi\lambda\, M(r, v_n)\}, \end{aligned}$$

where n is a positive integer and $\delta \in (0, \varepsilon/2)$.

We observe that

$$\begin{aligned} M(r_n, v_{n+1}) &= M(r_n, h) - r_{n+1}^{k-\varepsilon} \\ &= M(r_n, h) - r_n^{(1+\delta)(k-\varepsilon)} > r_n^{(1+\varepsilon)(k-\varepsilon)} - r_n^{(1+\varepsilon/2)(k-\varepsilon)} > 0. \end{aligned} \tag{13.2}$$

Then as in (12.2) we find that

$$\lambda m_l L_{n+1}(r_n, r_{n+1}) \leqq \log M(r_{n+1}, v_{n+1}) + \log C(\lambda) - \log M(r_n, v_{n+1}). \tag{13.3}$$

Now if $r \in L(r_n, r_{n+1})$, then

$$\begin{aligned} m(r, h) &< \cos \pi\lambda\, M(r, h) + (1-\cos \pi\lambda) r^{k-\varepsilon} \\ &\leqq \cos \pi\lambda\, M(r, h) + (1-\cos \pi\lambda) r_{n+1}^{k-\varepsilon}, \end{aligned}$$

and so

$$\begin{aligned} m(r, v_{n+1}) &= m(r, h) - r_{n+1}^{k-\varepsilon} \\ &< \cos \pi\lambda\, (M(r, h) - r_{n+1}^{k-\varepsilon}) = \cos \pi\lambda\, M(r, v_{n+1}). \end{aligned}$$

Hence, $L(r_n, r_{n+1}) \subset L_{n+1}(r_n, r_{n+1})$.

Using the above inclusion and summing both sides of (13.3), we obtain

$$\begin{aligned} \lambda m_l L(r_1, r_{n+1}) \leqq{}& \log M(r_{n+1}, h) + n \log C(\lambda) \\ &+ \sum_{i=2}^{n} \log \left(\frac{M(r_i, h) - r_i^{k+\varepsilon}}{M(r_i, h) - r_{i+1}^{k-\varepsilon}} \right) - \log (M(r_1, h) - r_2^{k-\varepsilon}). \end{aligned} \tag{13.4}$$

We observe that

$$\log M(r_{n+1}, h) \geqq \log r_{n+1}^{(1+\varepsilon)(k-\varepsilon)} \geqq (1+\delta)^{n+1}(k-\varepsilon) \log r_1,$$

thanks to (13.1). Moreover, by (13.2) and the fact that $\log(1-x)$ is bounded on $[0, \frac{1}{2}]$, we see that

$$\sum_{i=2}^{n} \log\left(\frac{M(r_i, h)-r_i^{k-\varepsilon}}{M(r_i, h)-r_{i+1}^{k-\varepsilon}}\right) = \sum_{i=2}^{n} \log\left(\frac{1-r_i^{k-\varepsilon}/M(r_i, h)}{1-r_{i+1}^{k-\varepsilon}/M(r_i, h)}\right)$$
$$= O(n) = o[\log M(r_{n+1}, h)],$$

as $n \to +\infty$. Then by (13.4) we have

$$\lambda m_l L(r_1, r_{n+1}) \leqq [1+o(1)] \log M(r_{n+1}, h)$$

as $n \to +\infty$. Dividing by $\lambda \log r_{n+1}$ and letting $n \to \infty$, we obtain

$$\underline{D}_l L \leqq \liminf_{n\to\infty} \frac{m_l L(r_1, r_{n+1})}{\log r_{n+1}} \leqq \liminf_{n\to\infty} \frac{\log M(r_{n+1}, h)}{\lambda \log r_{n+1}}. \tag{13.5}$$

Now if $s \in [r_n, r_{n+1})$, then

$$\frac{\log M(s, h)}{\log s} \geqq \frac{\log M(r_n, h)}{\log r_{n+1}} = \frac{\log M(r_n, h)}{\log r_n^{1+\delta}} = \frac{\log M(r_n, h)}{(1+\delta)\log r_n}.$$

Hence,

$$\liminf_{n\to\infty} \frac{\log M(r_n, h)}{\log r_n} \leqq (1+\delta)k.$$

Using this inequality and (13.5) we find that $\underline{D}_l L \leqq (1+\delta)k/\lambda$. Letting $\delta \to 0$, we get $\underline{D}_l L \leqq k/\lambda$. Therefore,

$$\bar{D}_l\{r : m(r, h) - \cos \pi\lambda\, M(r, h) \geqq (1-\cos \pi\lambda) r^{k-\varepsilon}\} \geqq 1-k/\lambda.$$

Since $\varepsilon > 0$ is arbitrary, it follows that Theorem 3 is true.

14. **Proof of Theorem 4.** Let $h_1 = h-1$ and observe that

$$m_l\{r : m(r, h_1) - \cos \pi\lambda\, M(r, h_1) \geqq 0\} = m_l\{r : m(r, h) - \cos \pi\lambda\, M(r, h) \geqq 1-\cos \pi\lambda\}$$
$$\leqq m_l\{r : m(r, h) - \cos \pi\lambda\, M(r, h) > 0\} < \infty.$$

Hence, we assume, as we may, that

$$m_l G_\lambda = m_l\{r : m(r, h) - \cos \pi\lambda\, M(r, h) \geqq 0\} < \infty,$$

since otherwise we consider h_1.

Choose $r > 1$ such that $M(r) > 0$. If $R > r$, then by (12.1) with $r = r_0$ we have

$$\begin{aligned} M(r) &\leqq C(\lambda) M(R) \exp[-\lambda m_l F_\lambda(r, R)] \\ &= C(\lambda) M(R) \exp[-\lambda \log(R/r) + \lambda m_l G_\lambda(r, R)] \\ &= C(\lambda) M(R)(r/R)^\lambda \exp[\lambda m_l G_\lambda(r, R)] \\ &\leqq C(\lambda) M(R)(r/R)^\lambda \exp[\lambda m_l G_\lambda(1, +\infty)] < +\infty. \end{aligned}$$

Let

$$B(\lambda) = C(\lambda) \exp[\lambda m_l G_\lambda(1, +\infty)].$$

Then from the above inequality we deduce that

$$M(r)/r^{\lambda} \leqq B(\lambda)M(R)/R^{\lambda}. \tag{14.1}$$

If $\liminf_{R\to\infty} M(R)/R^{\lambda}=0$, then by (14.1) we have

$$\frac{M(r)}{r^{\lambda}} \leqq B(\lambda) \liminf_{R\to\infty} \frac{M(R)}{R^{\lambda}} = 0.$$

This inequality contradicts the assumption that h is nonconstant. Hence

$$\liminf_{R\to\infty} \frac{M(R)}{R^{\lambda}} = b > 0. \tag{14.2}$$

If $\limsup_{r\to\infty} M(r)/r^{\lambda}=+\infty$, then by (14.1) we see that

$$+\infty = \limsup_{r\to\infty} \frac{M(r)}{r^{\lambda}} \leqq B(\lambda) \liminf_{R\to\infty} \frac{M(R)}{R^{\lambda}}.$$

Thus, $\lim_{r\to\infty} M(r)/r^{\lambda}=+\infty$.

If $\limsup_{R\to\infty} M(R)/R^{\lambda}<+\infty$, then

$$\begin{aligned} \int_1^{\infty} \frac{[m(r)-\cos\pi\lambda\, M(r)]^+}{r^{1+\lambda}}\,dr &= \int_{G_\lambda(1,+\infty)} \frac{[m(r)-\cos\pi\lambda\, M(r)]}{r^{1+\lambda}}\,dr \\ &\leqq (1-\cos\pi\lambda)\int_{G_\lambda(1,+\infty)} \frac{M(r)}{r^{1+\lambda}}\,dr = (1-\cos\pi\lambda)\int_{G_\lambda(1,+\infty)} \frac{O(r^{\lambda})}{r^{1+\lambda}}\,dr < +\infty. \end{aligned} \tag{14.3}$$

Here $a^+=\max\{a, 0\}$. From (14.3) we deduce that

$$\limsup_{r_1,r_2\to+\infty} \int_{r_1}^{r_2} \frac{[m(r)-\cos\pi\lambda\, M(r)]}{r^{1+\lambda}}\,dr \leqq 0.$$

Then by Anderson's Theorem (Theorem B) and (14.2) we conclude that $\lim_{r\to\infty} M(r)/r^{\lambda}=b$, where $0<b<+\infty$.

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