

ON MODIFICATION THEOREMS

BY
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Abstract. Given a right continuous family F_t of complete σ -fields and a bounded right continuous family X_t of random variables, we show in this paper that it is possible to modify the conditional expectations $E(X_t|F_t)$ to be right continuous. When $X_t = X$, this reduces to a result of J. L. Doob.

A famous result of Doob states that any martingale has a right continuous with left limits modification. This is an important result and is very useful in providing modification theorems in the theory of Markov processes.

During a discussion with Professor T. Watanabe the following problem arose:

Suppose F_t is an increasing right continuous family of σ -fields and X_t is a right continuous with left limits stochastic process. Can we define a right continuous with left limits modification of $E(X_t|F_t)$?

We shall show that it is always possible to select such a modification. Note that whereas in the case of martingales we have available the upcrossing inequality of Doob (this is an indispensable tool in the standard proofs of the martingale modification theorem) there is no such tool in the general situation. The method developed here generalizes without changes to Banach space valued martingales. Thus a Banach space valued martingale has a right continuous with left limits modification. Our methods do not require the notion of separability. The only prerequisite to reading this note is the knowledge of the martingale convergence theorem and familiarity with the notion of stopping rules. In the beginning of §1 we develop what is needed about stopping rules in a more general setting than what is given in standard books. Then we proceed to prove the main theorem (Theorem 6).

In §2 we show how the modification theorem for super martingales can be deduced using Theorem 6. Lemma 9 gives a slight generalization of a theorem of Meyer without invoking the upcrossing inequality.

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1. Let F_n denote a sequence of σ -fields for $n=1, 2, \dots, \infty$; ∞ is included. An integral valued nonnegative function T is called a stopping rule relative to F_n iff for all n the event $(T=n) \in F_n$. If T is a stopping rule we denote by F_T the σ -fields of events A such that $A \cap (T=n) \in F_n$, $1 \leq n \leq \infty$. It is easy to verify that F_T is indeed a σ -field. We simply note the following:

1. T is F_T -measurable.

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2. If X_n is a sequence of random variables with X_n F_n -measurable for $1 \leq n \leq \infty$ then X_T is F_T -measurable.

3. Suppose the σ -fields F_n increase, i.e. $F_n \subset F_{n+1}$. Then if T, S are stopping rules and $T \leq S$ we have $F_T \subset F_S$. If on the other hand the σ -fields decrease and T, S are stopping rules with $T \leq S$ then $F_T \supset F_S$.

The following lemma has an easy proof.

LEMMA 1 (OPTIONAL SAMPLING LEMMA). *Let X_n be random variables and F_n σ -fields for $1 \leq n \leq \infty$. We will assume that all expectations occurring exist. Let T be a stopping rule. If $Y_n = E(X_n | F_n)$, $1 \leq n \leq \infty$, then $Y_T = E(X_T | F_T)$ almost surely.*

The proof is easy. In fact Y_T is F_T -measurable. Hence the result follows from the equalities

$$\int_{A \cap (T=n)} X_T = \int_{A \cap (T=n)} X_n = \int_{A \cap (T=n)} Y_n = \int_{A \cap (T=n)} Y_T$$

and now sum over n ; A denotes an arbitrary element of F_T .

REMARK. If for all n , $X_n = X$ and the σ -fields F_n increase or decrease, Lemma 1 is the optional sampling theorem for martingales. The general optional sampling theorem for super martingales then follows at once via the Doob decomposition [1, p. 104].

LEMMA 2. *Let F_n be an increasing sequence of σ -fields; $1 \leq n \leq \infty$. Let X_n , $1 \leq n \leq \infty$, be a sequence of random variables such that*

(1) $X_n \rightarrow X_\infty$ almost surely,

(2) $|X_n| \leq \varphi$ with $E(\varphi) < \infty$.

Then $|E(X_n | F_n) - E(X_\infty | F_n)| \rightarrow 0$ almost surely.

Proof. We may assume that $X_\infty = 0$. Put $Y_n = E(X_n | F_n)$. We shall show that $Y_n \rightarrow 0$ a.s. Let $\varepsilon > 0$. Define the stopping rules T_k by

$$\begin{aligned} T_1 &= \inf(n : |Y_n| \geq \varepsilon), & T_{k+1} &= \inf(n : n \geq T_k + 1, |Y_n| \geq \varepsilon), \\ &= \infty \text{ if there is no such } n; & &= \infty \text{ if there is no such } n. \end{aligned}$$

Then $1 + T_k \leq T_{k+1}$, $T_k < \infty$ implies $|Y_{T_k}| \geq \varepsilon$, and if $T_k = \infty$ then $|Y_{T_k}| = 0$. We have

$$\varepsilon P(T_k < \infty) \leq E(|Y_{T_k}|) \leq E(|X_{T_k}|)$$

because $Y_{T_k} = E(X_{T_k} | F_{T_k})$. Since $T_k \geq k$ by definition (note that $T_1 \geq 1$), $T_k \rightarrow \infty$, and hence $X_{T_k} \rightarrow 0$. Therefore $E(|X_{T_k}|)$ and hence $P(T_k < \infty)$ tend to zero as $k \rightarrow \infty$. Also $(\limsup |Y_n| \geq \varepsilon) = (T_k < \infty \text{ for all } k) = \bigcap_k (T_k < \infty)$. Hence $P(\limsup |Y_n| \geq \varepsilon) \leq \liminf P(T_k < \infty) = 0$. Q.E.D.

A version of Lemma 2 for decreasing σ -fields also holds. We have

LEMMA 3. *Let F_n be a decreasing sequence of σ -fields. Let X_n , $1 \leq n \leq \infty$, be a sequence of random variables with*

(1) $X_n \rightarrow X_\infty$ almost surely,

(2) $|X_n| \leq \varphi$ with $E(\varphi) < \infty$.

Then $|E(X_n | F_n) - E(X_\infty | F_n)| \rightarrow 0$ almost surely.

Proof. Assume that $X_\infty = 0$ and put $Y_n = E(X_n | F_n)$. Let $\varepsilon > 0$ and define the stopping rules T_n by

$$T_n = \sup (k : k \leq n \text{ and } |Y_k| \geq \varepsilon), \\ = 1 \text{ if there is no such } k.$$

Clearly $T_n \leq n$ and $T_n \leq T_{n+1}$. If $S = \lim_n T_n$ then S is F_{T_n} -measurable for all n . The event $(S = \infty)$ is contained in $\bigcup_n (T_n > 1)$ and so $(S = \infty) = \bigcup_n (S = \infty) \cap (T_n > 1)$. Since T_n increases the events $(T_n > 1)$ are nondecreasing. It follows that $P(S = \infty) = \lim P(S = \infty, T_n > 1)$. Now $T_n > 1$ implies $|Y_{T_n}| \geq \varepsilon$. We have

$$E(|Y_{T_n}| : S = \infty, T_n > 1) \leq E(|X_{T_n}| : S = \infty, T_n > 1) \leq E(|X_{T_n}| : S = \infty).$$

We deduce that

$$\varepsilon P(S = \infty, T_n > 1) \leq E(|X_{T_n}| : S = \infty).$$

Let $n \rightarrow \infty$ to get $P(S = \infty) = 0$. Finally note that $(S = \infty) = (\limsup |Y_n| \geq \varepsilon)$.

REMARKS. 1. Martingale convergence theorem implies

$$E(X_\infty | F_n) \rightarrow E(X_\infty | F_\infty) \text{ almost surely.}$$

Thus Lemma 3 shows that $E(X_n | F_n) \rightarrow E(X_\infty | F_\infty)$ almost surely. This will be used below.

2. The absence of an upcrossing inequality makes the following theorem interesting.

We have the following modification theorem.

THEOREM 4. *Let X_t be right continuous and uniformly bounded. Suppose F_t is an increasing and right continuous family of σ -fields. Then there exists a right continuous modification of the stochastic process $E(X_t | F_t)$.*

Proof. Assume X_t is right continuous. Let T be any stopping rule relative to F_t . Put $Y_t = E(X_t | F_t)$. We shall show below that the limit as $r \downarrow T$, r running over the dyadic rationals of Y_r exists and equals $E(X_T | F_T)$. For this purpose we may assume that $E(X_T | F_T) = 0$. Let $\varepsilon > 0$ be given. Define a decreasing sequence T_n of stopping rules by

$$T_n = \inf (i/2^n : i \geq [2^n T] + 1 \text{ and } Y_{i/2^n} \geq \varepsilon), \\ = \infty \text{ if there is no such } i,$$

where $[x]$ denotes the integral part of x . If $T_n < \infty$ then $Y_{T_n} \geq \varepsilon$. Let $S = \lim T_n$. The event $(S = T)$ is identical with the event $(\limsup_{r \downarrow T, r \in T} Y_r \geq \varepsilon)$.

Since $\bigcap_n F_{T_n} = F_S$ [1, T42, p. 67], Lemma 3 implies that $\lim Y_{T_n}$ exists and equals $E(X_S | F_S)$, almost surely. Thus

$$\varepsilon P(S = T) \leq E(\lim Y_{T_n} : (T = S)) = E(E(X_S | F_S) : (T = S)).$$

The event $(S=T) \in F_T \subset F_S$. Hence

$$\begin{aligned} E(E(X_S|F_S) : S=T) &= E(X_S : S=T) \\ &= E(X_T : S=T) = E(E(X_T|F_T) : S=T) \\ &= 0 \quad \text{because } E(X_T|F_T) = 0 \text{ by assumption.} \end{aligned}$$

Thus $P(\limsup_{r \downarrow T; r \downarrow T} Y_r \geq \varepsilon) = 0$ for all $\varepsilon > 0$, i.e. $\limsup_{r \downarrow T; r \downarrow T} Y_r = 0$. A similar reasoning shows that $\liminf_{r \downarrow T; r \downarrow T} Y_r = 0$ almost surely. Thus with probability one $\lim_{r \downarrow T} Y_r$ exists and equals $E(X_T|F_T)$. Define the processes U_t and V_t by

$$\begin{aligned} U_t &= \limsup_{r \downarrow t} Y_r, \\ V_t &= \liminf_{r \downarrow t} Y_r, \end{aligned} \quad r \text{ diadic rational.}$$

U_t and V_t are progressively measurable [1, D48, p. 70]. We also have

$$U_t \geq \limsup_{s \downarrow t} U_s, \quad V_t \leq \liminf_{s \downarrow t} V_s.$$

Fix $\varepsilon > 0$ and let the stopping rule T be defined by

$$\begin{aligned} T &= \inf(t : U_t - V_t \geq \varepsilon), \\ &= \infty \quad \text{if there is no such } t. \end{aligned}$$

If $T < \infty$ from what we said above $U_T - V_T \geq \varepsilon$, i.e. that for the stopping rule T , $\limsup_{r \downarrow T} Y_r \geq \varepsilon + \liminf_{r \downarrow T} Y_r$. We have shown this cannot occur with positive probability. Thus except for a null set $\lim_{r \downarrow t} Y_r$ exists for all t . This limit then is the right continuous modification of Y_t we were looking for. Q.E.D.

Now a natural question is whether we can require Y_t to have left limits provided X_t has left limits. The answer is yes. We need

LEMMA 5. *Let X_t and F_t be as in Theorem 4, and let Y_t be a right continuous modification of $E(X_t|F_t)$. Then for any stopping rule T we have $Y_T = E(X_T|F_T)$.*

Proof. The proof is simple. If the stop rules T_n are defined by $T_n = ([2^n T] + 1)/2^n$ where $[x]$ denotes the integral part of x , then $T_n \downarrow T$. By right continuity $Y_{T_n} \rightarrow Y_T$. Lemma 1 implies that $Y_{T_n} = E(X_{T_n}|F_{T_n})$. Since $X_{T_n} \rightarrow X_T$ we may use Lemma 3 to complete proof.

Now we have

THEOREM 6. *Let X_t be right continuous, have left limits and be uniformly bounded. If F_t is an increasing right continuous family of σ -fields then we can choose a right continuous with left limits modification of $E(X_t|F_t)$.*

Proof. Let Y_t be a right continuous modification of $E(X_t|F_t)$; this is guaranteed by Theorem 4. We will show that Y_t has left limits almost surely. Let $\varepsilon > 0$ and define stopping rules T_n as follows:

$$\begin{aligned} T_1 &= \inf(t : |Y_t - Y_0| \geq \varepsilon), \\ &= \infty \quad \text{if there is no such } t, \end{aligned}$$

and, in general,

$$\begin{aligned} T_{n+1} &= \inf(t : t \geq T_n, |Y_t - Y_{T_n}| \geq \varepsilon), \\ &= \infty \quad \text{if there is no such } t. \end{aligned}$$

If $T_{n+1} < \infty$, by right continuity of Y_t we have $|Y_{T_{n+1}} - Y_{T_n}| \geq \varepsilon$. By Lemma 5 we have $Y_{T_n} = E(X_{T_n} | F_{T_n})$. Since the sequence T_n increases and X_t has left limits X_{T_n} converges on the set $(\lim T_n < \infty)$. We have

$$\begin{aligned} \varepsilon P(\lim T_n < \infty) &\leq \varepsilon P(T_{n+1} < \infty) \\ &\leq E(|Y_{T_{n+1}} - Y_{T_n}| : T_{n+1} < \infty) \leq E(|X_{T_{n+1}} - X_{T_n}| : T_{n+1} < \infty). \end{aligned}$$

Letting $n \rightarrow \infty$ we conclude that

$$P(\lim T_n < \infty) = 0.$$

Thus almost surely for each $\varepsilon > 0$, there exists a sequence T_n of stopping rules with $T_n < T_{n+1}$, $T_n \uparrow \infty$ such that the oscillation in the intervals $[T_n, T_{n+1})$ is less than 2ε . This means that Y_t has left limits almost surely. That finishes the proof.

2. In this section we develop some results that contain the modification theorem on super martingales. We assume that all processes considered are progressively measurable relative to a fixed increasing right continuous family F_t of σ -fields. For simplicity we further assume that all processes are uniformly bounded. This restriction can easily be removed. If for the process $X(t)$, $X(\infty)$ is not defined, simply define it to be zero.

THEOREM 7. *Let $X(t)$ be a stochastic process with the following property:*

If T_n is a decreasing sequence of discrete valued stopping rules then $\lim X(T_n)$ exists almost surely.

Then except for a null set the limit $\lim_{r \downarrow t} X_r$, r running over diadic rationals, exists for all t .

Proof. We will only indicate the proof since it is similar to the proof of Theorem 4. Let T be any stopping rule. We will show that $\lim_{r \downarrow T} X(r)$, r running over diadic rationals, exists almost surely. If this were not the case there would exist numbers $b > a$ with

$$\limsup_{r \downarrow T} X(r) > b > a > \liminf_{r \downarrow T} X(r).$$

Define the stopping rules T_n and S_n as follows:

$$\begin{aligned} T_n &= \inf(i/2^n : i \geq [2^n T] + 1 \text{ and } X(i/2^n) > b), \\ &= \infty \quad \text{if there is no such } i; \\ S_n &= \inf(j/2^n : j \geq [2^n T] + 1 \text{ and } X(j/2^n) < a), \\ &= \infty \quad \text{if there is no such } j. \end{aligned}$$

Clearly T_n and S_n decrease. The set where $\limsup_{r \downarrow T} X(r) > b > a > \liminf_{r \downarrow T} X(r)$ is identical with $\lim T_n = \lim S_n = T$. Call this set A . Choose small positive ε_n with

$\sum_n \varepsilon_n < P(A)$. Define a decreasing sequence of discrete valued stopping rules U_n as follows: $U_1 = T_1$. There exists an integer n_1 with

$$P(A, T_1 \geq S_{n_1}) < \varepsilon_1, \quad \text{since } S_n \downarrow T \text{ on } A.$$

Put $U_2 = U_1 \wedge S_{n_1}$. There exists an n_2 with $P(A, U_2 \geq T_{n_2}) < \varepsilon_2$. Put $U_3 = U_2 \wedge T_{n_2}$. And so on. We thus get a decreasing sequence U_n of stopping rules such that at least on a subset of A of positive measure, $X(U_{2n+1}) > b$ and $X(U_{2n}) < a$, contradicting the assumption that $X(U_n)$ converges almost surely. Now define, as in Theorem 4,

$$Y_1(t) = \limsup_{r \downarrow t} X(r), \quad Y_2(t) = \liminf_{r \downarrow t} X(r).$$

If $P(Y_1(t) \neq Y_2(t) \text{ for some } t) > 0$, then there exist a stopping rule T and $\varepsilon > 0$ with $Y_1(T) > Y_2(T) + \varepsilon$ and this contradicts what we have already proved. This completes the proof.

REMARK. In Theorem 7 if $X(t)$ is a super martingale and we put $Y(t) = \lim_{r \downarrow t} X(r)$ then $Y(t)$ is right continuous. $Y(t)$ is F_t -measurable since F_t is right continuous. This then implies the existence of right continuous modifications of measurable super martingales. The existence of left limits is no problem.

LEMMA 8. *If $X(t)$ is a right continuous stochastic process such that for all increasing sequences T_n of stopping times $\lim X(T_n)$ exists then $X(t)$ has left limits almost surely.*

Proof. This is implicit in the proof of Theorem 6. Indeed if $\varepsilon > 0$ and the stopping rules T_n are defined by

$$\begin{aligned} T_1 &= \inf(t : |X_t - X_0| \geq \varepsilon), \\ &= \infty \quad \text{if there is no such } t; \\ T_{n+1} &= \inf(t : t \geq T_n \text{ and } |X_t - X_{T_n}| \geq \varepsilon), \\ &= \infty \quad \text{if there is no such } t, \end{aligned}$$

the assumption that X_{T_n} converges implies that $T_n \uparrow \infty$ almost surely. It follows that $X(t)$ has left limits almost surely. Q.E.D.

Now suppose that $X(t)$ is a super martingale. Consider the processes

$$X_n(t) = X((i+1)/2^n) \quad \text{if } i/2^n \leq t < (i+1)/2^n.$$

$X_n(t)$ are right continuous. Then there exists a right continuous modification $Y_n(t)$ of the process $E(X_n(t) | F_t)$. Since $X(t)$ is a super martingale $Y_n(t)$ is nondecreasing, i.e. $Y_n(t) \leq Y_{n+1}(t)$. Put $Y(t) = \lim_{n \rightarrow \infty} Y_n(t)$. $Y(t)$ is clearly a measurable modification of $X(t)$. (Of course we assume that $E(X(t))$ is a right continuous function of t .) Theorem 7 and Lemma 8 say that $Y(t)$ has a right continuous with left limits modification. Since, for each n , $Y_n(t)$ is a super martingale, a result of Meyer [1, T, p. 99] implies that $Y(t)$ is right continuous and has left limits. The following is a slight generalisation of the above mentioned result of Meyer.

LEMMA 9. Let $X(t)$ be a super martingale with the following properties:

1. $X(t)$ is progressively measurable and is lower semicontinuous from the right, i.e. $X(t) \leq \liminf_{s \downarrow t} X(s)$.

2. For stopping times T and S with $T \leq S$ we have $E(X(T)) \geq E(X(S))$. Then $X(t)$ is right continuous and has left limits almost surely.

Proof. If T_n are decreasing, discrete valued stopping rules, $X(T_n)$ converges by the super martingale convergence theorem. By Theorem 7, except for a null set, $Y_t = \lim_{r \downarrow t} X(r)$ exists for all t . Lower semicontinuity on the right implies that $Y(t) \geq X(t)$ for all t .

For each stopping time T we get

$$Y(T) = \lim_{r \downarrow T} X(r) = \lim X\left(\frac{[2^n T] + 1}{2^n}\right)$$

so that

$$E(Y(T)) = \lim E\left(X\left(\frac{[2^n T] + 1}{2^n}\right)\right) \leq E(X(T))$$

because $([2^n T] + 1)/2^n$ being a stopping rule $> T$ we have

$$E\left(X\left(\frac{[2^n T] + 1}{2^n}\right)\right) \leq E(X(T)).$$

Thus for every stopping rule T , $X(T) = Y(T)$ almost surely. If $\varepsilon > 0$ and

$$\begin{aligned} T &= \inf(t : Y_t - X_t \geq \varepsilon), \\ &= \infty \quad \text{if there is no such } t, \end{aligned}$$

then by right continuity of Y_t and lower semicontinuity of X_t from the right we must have $Y_T - X_T \geq \varepsilon$ on the set $(T < \infty)$. But this contradicts $Y_T = X_T$ for all stopping rules. Thus

$$P(Y_t = X_t \text{ for all } t) = 1,$$

i.e. X_t is almost surely right continuous and by Lemma 8 has left limits. That completes the proof.

REMARK. Let us indicate to what extent the results of §§1 and 2 generalise to Banach space valued variables. The conditional expectation of a variable X relative to a sub- σ -field G will be defined as in [2, p. 22]. The proofs of Lemmas 1, 2 and 3 are generalised verbatim. Theorem 4 needs slight changes. Assume as in Theorem 4 that $E(X_T | F_T) = 0$. Define

$$\begin{aligned} T_n &= \inf(i/2^n : i \geq [2^n T] + 1 \text{ and } |Y_{i/2^n}| \geq \varepsilon) \\ &= \infty \quad \text{if there is no such } i. \end{aligned}$$

Put $S_n = ([2^n T] + 1)/2^n$. If

$$P\left(\limsup_{r \downarrow T} |Y_r| \geq \varepsilon\right) > 0$$

we can always choose indices k_n such that $P(T_n < S_{k_n} \text{ for all } n) > 0$. Define $U_n = S_{k_n} \wedge T_n$. $U_n \downarrow T$ so that Y_{U_n} tends to $E(X_T/F_T) = 0$. This is not possible on the set $(U_n < S_{k_n} \text{ for all } n)$. Now let

$$Z_t = \limsup_{r,s > t; r,s \downarrow t} |Y_r - Y_s|.$$

Z_t is progressively measurable and $Z_T = 0$ for all stopping rules T . It follows that $P(Z_t > 0 \text{ for some } t) = 0$.

Lemma 5 and Theorem 6 do not need any changes. The changes needed in the proof of Theorem 7 parallel those that we have already indicated in the case of Theorem 4.

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