

ADDENDUM TO "ON A PROBLEM OF TURÁN ABOUT POLYNOMIALS WITH CURVED MAJORANTS"

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Let π_n denote the class of polynomials $p_n(x) = \sum_{k=0}^n a_k x^k$ of degree n which satisfy $|p_n(x)| \leq (1-x^2)^{1/2}$ for $-1 < x < 1$. Given any x_0 in $[-1, 1]$, how large can $|p_n^{(k)}(x_0)|$ (the k th derivative at x_0) be if $p_n(x)$ belongs to the class π_n ? In case $x_0 = 0$ the problem is equivalent to the problem of estimating $|a_k|$ if $p_n(x) = \sum_{k=0}^n a_k x^k \in \pi_n$. It has been shown [3] that if $p_n(x) = \sum_{k=0}^n a_k x^k \in \pi_n$ then

$$|a_1| \leq n-1, \quad |a_2| \leq \{(n-1)^2 + 1\}/2.$$

Here we prove the following theorem which gives a sharp estimate for each of the coefficients.

THEOREM. *If $p_n(x) = \sum_{k=0}^n a_k x^k$ is a polynomial of degree n such that $|p_n(x)| \leq (1-x^2)^{1/2}$ for $-1 < x < 1$ and $U_n(x)$ denotes the n th Chebyshev polynomial of the second kind, then, according as $n-k$ is even or odd, $|a_k|$ is bounded above by the absolute value of the coefficient of x^k in $e^{i\gamma}(1-x^2)U_{n-2}(x)$ or $e^{i\gamma}(1-x^2)U_{n-3}(x)$, respectively.*

The idea of proof comes from a paper of O. D. Kellogg [2].

Proof. Without loss of generality we may suppose $p_n(x)$ to be real for real x .

Taking first the case in which k, n are both even, we consider the polynomial

$$\frac{1}{2}\{p_n(x) + p_n(-x)\} = a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{n-2} x^{n-2} + a_n x^n$$

and compare it with the polynomial

$$\begin{aligned} (1-x^2)U_{n-2}(x) &= -2^{n-2}x^n + \sum_{j=1}^{n/2-1} (-1)^{j-1} \left\{ \frac{(n-1-j)!}{(j-1)!(n-2j)!} + \frac{1}{4} \frac{(n-2-j)!}{j!(n-2-2j)!} \right\} \\ &\quad \times (2x)^{n-2j} + (-1)^{n/2-1} \\ &= \sum_{k=0}^n A_k x^k \quad (\text{say}). \end{aligned}$$

If $-1 < \lambda < 1$, the difference

$$D(x, \lambda) = (1-x^2)U_{n-2}(x) - \lambda\{a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{n-2} x^{n-2} + a_n x^n\}$$

is positive at all points of the interval $(-1, 1)$ where $(1-x^2)U_{n-2}(x) = (1-x^2)^{1/2}$ and negative where $(1-x^2)U_{n-2}(x) = -(1-x^2)^{1/2}$. It is readily seen that there are

a total of $n-1$ points on the interval $(-1, 1)$ where $(1-x^2)U_{n-2}(x)$ is alternately equal to $(1-x^2)^{1/2}$, $-(1-x^2)^{1/2}$. Hence the polynomial $D(x, \lambda)$ has at least $n-2$ real zeros in $(-1, 1)$. Since it also vanishes at $-1, +1$ we conclude that all the zeros of $D(x, \lambda)$ are real and distinct. All the odd coefficients of $D(x, \lambda)$ being zero, none of the even coefficients can vanish; for then by Descartes' rule of signs, the zeros of $D(x, \lambda)$ could not all be real. Thus, for every even $k \leq n$ and for $-1 < \lambda < 1$, $|A_k - \lambda a_k| \neq 0$. This is possible only if $|a_k| \leq |A_k|$ for $k=0, 2, 4, \dots, n$.

In case k, n are both odd we apply the above reasoning to the polynomial

$$(1-x^2)U_{n-2}(x) - (\lambda/2)\{p(x) - p(-x)\}$$

(whose even coefficients are all zero) in order to get the desired conclusion.

If k is even and n is odd we consider

$$(1-x^2)U_{n-3}(x) - (\lambda/2)\{p(x) + p(-x)\}$$

whereas if k is odd and n is even we argue with

$$(1-x^2)U_{n-3}(x) - (\lambda/2)\{p(x) - p(-x)\}.$$

One can in fact show that only those polynomials given at the end of the statement of the theorem are extremal for the k th coefficient when $n-k$ is even or odd respectively.

We also observe that Theorem 2 of our paper [3] is an immediate consequence of a theorem of Levin which appears as Theorem 11.7.2 in [1]. If we set $f(z) = p_n(\cos z)$, $\omega(z) = e^{i(n-1)z} \sin z$ the conditions of Levin's theorem are satisfied with $\tau = n$, $\sigma = n$. Since differentiation is a B -operator we have

$$|(d/dx)p_n(\cos x)| \leq |(d/dx)\{e^{i(n-1)x} \sin x\}|, \quad -\infty < x < \infty,$$

which readily gives the desired result.

We take this opportunity to make it clear that z^* appearing on p. 448 of [3] is real and the polynomials $p_n(z)$ in Theorems C and 4 of that paper are supposed to have real coefficients.

REFERENCES

1. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954. MR 16, 914.
2. O. D. Kellogg, *On bounded polynomials in several variables*, Math. Z. 27 (1927), 55-64.
3. Q. I. Rahman, *On a problem of Turán about polynomials with curved majorants*, Trans. Amer. Math. Soc. 163 (1972), 447-455.

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