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KNOTS WHOSE BRANCHED CYCLIC COVERINGS HAVE PERIODIC HOMOLOGY

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Abstract. Let M_k be the k-fold branched cyclic covering of a (tame) knot of S^1 in S^3. Our main result is that the following statements are equivalent:
(1) $H_1(M_k)$ is periodic with period n, i.e. $H_1(M_k)\cong H_1(M_{k+n})$ for all k,
(2) $H_1(M_k)\cong H_1(M_{(k,n)})$ for all k,
(3) the first Alexander invariant of the knot, $\lambda_1(t)=\Delta_1(t)/\Delta_2(t)$, divides t^n-1.

1. **Introduction.** The complement C of the trefoil knot is a fibre bundle over S^1 with fibre F a punctured torus, and group $\boldsymbol{Z}_6$ ([27], [20], [31]). So $C\cong F\times I/h$ (i.e. $F\times I$ with $F\times 0$ and $F\times 1$ identified via the homeomorphism $(x, 0)\mapsto(h(x), 1)$), where $h\colon F\to F$ is a homeomorphism of period 6. If $\tilde{C}$ is the infinite cyclic covering, and C_k the k-fold cyclic covering, of C, then $\tilde{C}\cong F\times R$, and $C_k\cong F\times I/h^k$. In particular, $C_k\cong C_{k+6}$. Hence if M_k is the corresponding k-fold branched cyclic covering, then, since the homology of M_k depends only on the homology of C_k, we see that $H_1(M_k)$ is periodic in k, with period 6. (See also [4], [5], [21].)

For any knot, however, $H_1(M_k)$ is completely determined by the Alexander matrix, and so it is clear that a knot need not be a fibred knot in order that $H_1(M_k)$ be periodic. For example, there exist many nontrivial knots with trivial Alexander matrix [30], and since a fibred knot with Alexander polynomial $\Delta(t)$ and genus g must satisfy $\deg\Delta(t)=2g$ [19], none of them are fibred. But they certainly exhibit periodicity, since $H_1(M_k)=0$ for all k. We are therefore led to the problem of finding necessary and sufficient conditions on the Alexander matrix for $H_1(M_k)$ to be periodic. The main aim of the present paper is to solve this problem.

MAIN THEOREM. *Let K be a knot with first Alexander invariant $\lambda_1(t)$ (i.e. $\lambda_1(t)=\Delta_1(t)/\Delta_2(t)$, where $\Delta_i(t)$ is the ith Alexander polynomial of K). Then the following statements are equivalent*:
(1) $H_1(M_k)\cong H_1(M_{k+n})$ *for all* k,
(2) $H_1(M_k)\cong H_1(M_{(k,n)})$ *for all* k,
(3) $\lambda_1(t)\mid(t^n-1)$.

It is perhaps at first sight somewhat surprising that the existence of periodicity should depend only on the first Alexander invariant, since even the complete set

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of Alexander invariants (or equivalently, the complete set of Alexander polynomials) does not in general determine $H_1(M_k)$. (The stevedore's knot (6_1), and the knot 9_{46}, for example, have the same polynomial invariants, but in the first case, $H_1(M_2)\cong \boldsymbol{Z}_9$, whereas in the second, $H_1(M_2)\cong \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$.) The following interpretation, however, renders the result more plausible. We first show that a necessary condition for periodicity is that K have an Alexander matrix of the form $M-tI$, where M is unimodular. This means that the infinite cyclic covering $\tilde{C}$ behaves homologically as if the knot were fibred. Condition (3) is then equivalent to the statement $M^n=I$, which is just the homological analogue of the condition that the fibre homeomorphism be periodic with period n.

We also prove that the periods which can occur are precisely those integers which are not prime powers, and we show how to find all possible periods for knots of a given genus.

Finally, one of our lemmas enables us to prove in passing that a theorem of Fox relating the torsion numbers of $H_1(M_{rk})$ to those of $H_1(M_k)$ for knots of genus 1 [5] actually holds for all knots.

Theorems, lemmas, etc. will be referred to by the number of the section in which they appear.

2. **Definitions and background material.** This section consists largely of a summary of the relevant standard material. General references for §§2.1–2.7 are [3], [6], [16], and [25].

2.1. Let $K\subset S^3$ be an oriented polyhedral knot, N a regular neighborhood of K, and write $C=S^3-\operatorname{int} N$. The *infinite cyclic covering* of C, i.e. the covering associated with the kernel of the abelianization homomorphism $\pi_1(C)\to H_1(C)\cong \boldsymbol{Z}$, will be denoted by $\tilde{C}$. Identifying $J\boldsymbol{Z}$, the integral group ring of $\boldsymbol{Z}$, with Λ, the ring of Laurent polynomials in a single variable t with integral coefficients, $H_1(\tilde{C})$ becomes a finitely-generated Λ-module. Similarly, $H_1(\tilde{C};\, Q)$ is a finitely-generated Γ-module, where Γ is the ring of Laurent polynomials in t with rational coefficients.

2.2. Let R be a ring and A an R-module. Then there is an exact sequence of R-modules

$$F_2 \xrightarrow{\delta} F_1 \xrightarrow{\eta} A \longrightarrow 0,$$

where F_1, F_2 are free R-modules with bases $\{x_j\}$, $\{r_i\}$, say. A matrix $M=(m_{ij})$, $m_{ij}\in R$, is a *presentation matrix* for A as an R-module if, for some such exact sequence, M represents δ with respect to the bases $\{x_j\}$, $\{r_i\}$, i.e.

$$\delta(r_i) = \sum_j m_{ij}x_j \quad \text{for each } i.$$

In the special case $R=J$, the ring of integers, we say M presents A *as an abelian group*.

If A is a finitely-generated R-module, and R is Noetherian, then A has a finite presentation matrix.

Two matrices M, M' with entries in R are *equivalent* (over R), written $M \sim^R M'$, if and only if they present isomorphic R-modules. This equivalence can be characterised in terms of the *elementary matrix operations.*

2.3. Let M be a (finite) presentation matrix for A. By adjoining rows of zeros if necessary, we may suppose that M is $m \times n$ with $m \geqq n$. Then the ith *elementary ideal* of A, $E_i(A)$, $i \geqq 1$, is the ideal in R generated by the $(n-i+1)$th order minors of M, with the convention that $E_i(A) = R$ if $i > n$.

2.4. If R is a principal ideal domain, then $E_i(A)$ is a principal ideal, $(\Delta_i(A))$ say, where $\Delta_i(A)$ is uniquely determined up to association (multiplication by a unit of R).

Moreover, by the structure theorem for finitely-generated modules over a principal ideal domain, A is isomorphic to a direct sum

$$R/(\lambda_1) \oplus R/(\lambda_2) \oplus \cdots \oplus R/(\lambda_n),$$

where $\lambda_{i+1} | \lambda_i$, $1 \leqq i \leqq n$. It is then clear that

$$\begin{aligned} E_i(A) = (\Delta_i(A)) &= (\lambda_i \cdots \lambda_n), \qquad 1 \leqq i \leqq n, \\ &= (1), \qquad\qquad i > n. \end{aligned}$$

Again $\lambda_i = \lambda_i(A)$ is uniquely determined up to association; any member of the associate class will be called the ith *invariant factor* of A. (We define $\lambda_i(A) = 1$, $i > n$.)

2.5. Returning to the knot situation, a presentation matrix for $H_1(\tilde{C})$ as a Λ-module will be called an *Alexander matrix* for the knot K. Now Λ is not a principal ideal domain, and so the elementary ideals E_i of $H_1(\tilde{C})$ are not necessarily principal. However, Λ is a unique factorization domain, and hence, for each i, there is a unique minimal principal ideal containing E_i, namely the ideal generated by the h.c.f. of the $(n-i+1)$th order minors of any Alexander matrix. This h.c.f. is determined only up to association (i.e. multiplication by $\pm t^r$, $r \in \boldsymbol{Z}$), but there is a unique representative of the associate class with no negative powers of t, and with positive constant term. This representative, $\Delta_i(t)$, is called the ith *Alexander polynomial* of the knot K. We write $\Delta_1(t) = \Delta(t)$, and call it simply *the Alexander polynomial* of K.

2.6. Now $H_1(\tilde{C}; Q)$ is a finitely-generated Γ-module, and Γ is a principal ideal domain, so the discussion in §2.4 applies. We can therefore define $\lambda_i(t)$, the ith *Alexander invariant* of K, to be the unique ith invariant factor of $H_1(\tilde{C}; Q)$ with no negative powers of t, positive constant term, and whose coefficients are integers with h.c.f. 1.

2.7. We thus regard $\Delta_i(t)$ and $\lambda_i(t)$ as elements of the polynomial ring $J[t]$. Note also that each $\Delta_i(t)$ and each $\lambda_i(t)$ is *primitive*, i.e. the h.c.f. of its coefficients is 1. For $\lambda_i(t)$, this follows by definition, and for $\Delta_i(t)$, it is a consequence of the well-known fact that $\Delta_i(1) = \pm 1$.

If $M(t)$ presents $H_1(\tilde{C})$ as a Λ-module, then $M(t)$ (regarding the entries now as elements of Γ) also presents $H_1(\tilde{C}; Q)$ as a Γ-module (see [25]). It follows that if E_i^Γ is the ith elementary ideal of $H_1(\tilde{C}; Q)$, then $E_i^\Gamma = (\Delta_i(t))$. But

$$\begin{aligned} E_i^\Gamma &= (\lambda_i(t)\cdots\lambda_n(t)), & 1 \leqq i \leqq n, \\ &= (1), & i > n, \end{aligned}$$

and so $\Delta_i(t)$ and $\lambda_i(t)\cdots\lambda_n(t)$ are associate in Γ. It is clear that they must therefore be associate in $Q[t]$, and hence, since each is primitive, associate in $J[t]$. Since each has positive constant term, we finally have

$$\begin{aligned} \Delta_i(t) &= \lambda_i(t)\cdots\lambda_n(t), & 1 \leqq i \leqq n, \\ &= 1, & i > n. \end{aligned}$$

Since $\Delta_i(1) = \pm 1$, we also see that $\lambda_i(1) = \pm 1$.

2.8. If $\Delta(0) = 1$, then ([23], [1]) $H_1(\tilde{C}) \cong \bigoplus_{i=1}^n \boldsymbol{Z}$, where $n = \deg \Delta(t)$. The automorphism $t\colon H_1(\tilde{C}) \to H_1(\tilde{C})$ is then described by an $n \times n$ unimodular integral matrix M, and $M - tI$ is an Alexander matrix for K. (If $n = 0$, i.e. if $\Delta(t) = 1$, then K has trivial Alexander matrix (1).)

2.9. Recall the notation of §2.1, and let C_k, $k \geqq 1$, be the covering of C associated with the kernel of the composition

$$\pi_1(C) \to H_1(C) \cong \boldsymbol{Z} \to \boldsymbol{Z}_k.$$

Then C_k is an oriented manifold with boundary $\partial C_k \cong S^1 \times S^1$, and the *$k$-fold branched cyclic covering of K*, M_k, is defined to be the closed, oriented manifold obtained by sewing back the regular neighborhood N via the homeomorphism $\partial N \to \partial C_k$ which identifies each meridian loop in $\partial N = \partial C$ with the lift in ∂C_k of its kth power. (See [4], [24], [31].)

2.10. LEMMA. *For any knot K, and for any integers k, r ($\geqq 1$), there exists an epimorphism $\phi\colon H_1(M_{rk}) \to H_1(M_k)$.*

The referee has pointed out that this is an immediate consequence of the fact that the projection $M_{rk} \to M_k$ induces an epimorphism $\pi_1(M_{rk}) \to \pi_1(M_k)$, which in turn can be proved by an argument similar to that given in [9, p. 331]. At any rate, Lemma 2.10 seems to be well known, and is usually attributed to R. H. Fox.

2.11. Although the Alexander invariants do not completely determine $H_1(M_k)$, they do give some information. (See [10], [28], [7], [18], but beware of errors in the last two references.) We denote the first Betti number of M_k by $\beta_1(M_k)$, and adopt the convention that the order of an infinite group is 0.

PROPOSITION (GOERITZ, FOX). *Let K be a knot with Alexander polynomial $\Delta(t)$ and ith Alexander invariant $\lambda_i(t)$. Then*

(i) order $H_1(M_k) = |\prod_{j=1}^k \Delta(\rho^j)|$, *where ρ is a primitive kth root of unity,*

(ii) $\beta_1(M_k) = \sum_i \mu_i(k)$, *where $\mu_i(k)$ is the number of distinct kth roots of unity which are roots of $\lambda_i(t)$.*

2.12. Suppose that K has an Alexander matrix $M(t)=M-tI$. Then if T_k is the $k\times k$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & 1 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

$M(T_k)$ presents $H_1(M_k)$ as an abelian group ([10], [18], [4], [7], [29]) and, by a sequence of elementary matrix operations entirely analogous to those in [24] (see also [29]), it can be shown that $M(T_k) \sim^J M^k-I$, i.e. M^k-I presents $H_1(M_k)$ as an abelian group.

This can also be seen as follows. From the exact sequence

$$H_1(\tilde{C}) \xrightarrow{t^k-1} H_1(\tilde{C}) \longrightarrow H_1(C_k) \longrightarrow \boldsymbol{Z} \longrightarrow 0$$

(see [25]), and the fact that $H_1(C_k)\cong H_1(M_k)\oplus \boldsymbol{Z}$ ([2], [7]), we get the exact sequence

$$H_1(\tilde{C}) \xrightarrow{t^k-1} H_1(\tilde{C}) \longrightarrow H_1(M_k) \longrightarrow 0.$$

Now if $M=(m_{ij})$ is $n\times n$, then $H_1(\tilde{C})$ is generated (as a Λ-module) by $\{x_1, \ldots, x_n\}$, say, with relations

$$tx_i = \sum_j m_{ij}x_j, \qquad 1 \leqq i \leqq n.$$

It is then clear that, as an abelian group, $H_1(\tilde{C})$ is freely generated by $\{x_1, \ldots, x_n\}$. Since $t^k-1\colon H_1(\tilde{C}) \to H_1(\tilde{C})$ is represented with respect to this basis by M^k-I, it follows that M^k-I presents $H_1(M_k)$ as an abelian group.

2.13. Again suppose that K has an Alexander matrix of the form $M-tI$, and let $\lambda_1(t)$ be its first Alexander invariant. Then it is well known that $\lambda_1(t)$ is also the minimum polynomial of M (see, for example, [14, p. 397] or [17, p. 20]). Essential use will be made of this fact in the proof of the Main Theorem.

2.14. Let $\phi_m(t)$ be the mth *cyclotomic polynomial*, i.e. the monic polynomial whose roots are the primitive mth roots of 1. Then, in fact, $\phi_m(t) \in J[t]$, and, since it is monic, it is primitive. It is also irreducible, and hence if $f(t) \in J[t]$, and some root of $f(t)$ is a primitive mth root of 1, it follows that $\phi_m(t)|f(t)$ in $J[t]$. (See [14, p. 206].) The degree of $\phi_m(t)$ is $\phi(m)$, the number of integers k such that $1\leqq k\leqq m$ and $(k, m)=1$. It is also easy to show (using, for example, the identities in [14, pp. 206–207]) that $\phi_1(1)=0$, $\phi_{p^r}(1)=p$ if p is prime and $r>0$, and $\phi_n(1)=1$ if n has at least two distinct prime factors.

Now if $\lambda(t)$ is an Alexander invariant of some knot, then $\lambda(1)=\pm 1$ (see §2.7), and so the above remarks show that $\lambda(t)$ can be written uniquely as

$$\lambda(t) = \psi(t)\prod_{i=1}^{s}(\phi_{m_i}(t))^{a_i}, \tag{2.14.1}$$

where $\psi(t)\in J[t]$ and no root of $\psi(t)$ is a root of unity, each m_i has at least two distinct prime factors, the m_i's are all distinct, and $a_i>0$.

We shall require this later, but for the moment let us pause to note the following elementary consequences.

THEOREM. *For any knot K, the first Betti number of M_k satisfies*

(i) $\beta_1(M_k)\equiv 0 \pmod 2$ *for all k,*

(ii) $\beta_1(M_k)\equiv 0 \pmod 4$ *if k is odd,*

(iii) $\beta_1(M_k)=0$ *if $k=p^r$, p prime.*

Proof. Let $\lambda(t)$ be some Alexander invariant of K, and let $\mu(k)$ be the number of distinct kth roots of unity which are roots of $\lambda(t)$. Then, after Proposition 2.11(ii), it will be sufficient to prove the statements corresponding to (i), (ii) and (iii) for $\mu(k)$.

Now all the roots of a cyclotomic polynomial are distinct, and no two distinct cyclotomic polynomials have a common root. Hence (see (2.14.1)) $\mu(k)=\sum\phi(m_i)$, the sum taken over those i such that $m_i|k$. If $k=p^r$, p prime, the only divisors of k are p^s, $0\leqq s\leqq r$, and so (iii) follows immediately.

Also, since ϕ is multiplicative (in the number theoretic sense that if $(n_1, n_2)=1$ then $\phi(n_1n_2)=\phi(n_1)\phi(n_2)$), it is easy to show that if the prime factors of a typical m_i are $p_1, \ldots, p_t$, then $\prod_{j=1}^{t}(p_j-1)|\phi(m_i)$. Since no m_i is a power of 2, it follows that $\phi(m_i)$ is always even, and so we get (i). Finally, if k is odd, each m_i which divides k must be odd. Such an m_i then has at least two distinct odd prime factors, and so $\phi(m_i)\equiv 0 \pmod 4$. This proves (ii).

REMARK. Comparison of (i) and (ii) is interesting in connection with a theorem of Plans [21] which states that if k is odd, $H_1(M_k)$ is always a direct double. (See also [11].)

Finally, let us note that since $\{\alpha^{-1}: \alpha \text{ is a primitive } m\text{th root of unity}\}=\{\alpha: \alpha \text{ is a primitive } m\text{th root of unity}\}$, $\phi_m(t)=\pm t^{\phi(m)}\phi_m(t^{-1})$. But if $m>1$, $\phi_m(1)\neq 0$, and so we must have $\phi_m(t)=t^{\phi(m)}\phi_m(t^{-1})$.

3. An algebraic lemma and a theorem of Fox.

3.1. LEMMA. *Let R be a principal ideal domain, and suppose A, B are finitely-generated R-modules such that there exists an epimorphism $\phi: A\to B$.*

Then $\lambda_r(B)|\lambda_r(A)$ for all r.

Proof. Suppose $\lambda_r(A)\neq 1$, $1\leqq r\leqq n$, and $\lambda_r(A)=1$, $r>n$. Then it is well known that $n=m(A)$, the minimal number of generators of A. Let us also write, for $x\in R$, $xA=\{xa: a\in A\}$. Then clearly $\lambda_r(xA)=\lambda_r(A)/(\lambda_r(A), x)$, where $(\lambda_r(A), x)$ is the h.c.f. of $\lambda_r(A)$ and x. Therefore, $m(xA)=n-$the number of $\lambda_r(A)$, $1\leqq r\leqq n$, which divide x.

Now the existence of an epimorphism $\phi: A \to B$ implies that $m(B) \leqq m(A)$, and hence, $m(xB) = n -$ the number of $\lambda_r(B)$, $1 \leqq r \leqq n$, which divide x.

But the existence of ϕ also implies more generally that $m(xB) \leqq m(xA)$, for all $x \in R$. It follows that, for all $x \in R$,

(3.1.1) The number of $\lambda_r(B)$, $1 \leqq r \leqq n$, which divide $x \geqq$ the number of $\lambda_r(A)$, $1 \leqq r \leqq n$, which divide x.

Let $\lambda_{s+1}(A)$ be the first $\lambda_r(A) \neq 0$. Then we certainly have $\lambda_r(B) | \lambda_r(A)$ for $1 \leqq r \leqq s$. Now take $x = \lambda_{s+1}(A)$ in (3.1.1). Since $\lambda_{r+1}(A) | \lambda_r(A)$ for all r, this gives the number of $\lambda_r(B)$, $1 \leqq r \leqq n$, which divide $\lambda_{s+1}(A) \geqq n - s$.

Since $\lambda_{r+1}(B) | \lambda_r(B)$ for all r, this implies that $\lambda_{s+1}(B) | \lambda_{s+1}(A)$. Now repeat the argument, successively taking $x = \lambda_{s+2}(A), \ldots, \lambda_n(A)$ in (3.1.1), and finally note that $\lambda_r(A) = \lambda_r(B) = 1$ if $r > n$.

Lemma 3.1 seems to be fairly well known, and other proofs (using presentation matrices, for example) can be given. The above proof was shown to me by J. H. Conway, whom I thank.

In §4.1 we shall require the following fact about finitely-generated abelian groups. We state it here as an immediate consequence of the preceding lemma.

COROLLARY. *Let A, B be finitely-generated abelian groups such that there exist epimorphisms $\phi: A \to B$ and $\psi: B \to A$. Then $A \cong B$.*

REMARK. Lemma 3.1 also shows that Corollary 3.1 is true for finitely-generated R-modules, where R is any principal ideal domain. In fact, the result holds for finitely-generated modules over any commutative ring with an identity [26].

3.2. Lemma 3.1 is really a digression to enable us to generalize to all knots a result proved by Fox [5, Theorem 4] for knots of genus 1. If M_k is the k-fold branched cyclic covering of some knot, then $H_1(M_k) \cong \boldsymbol{Z}_{\tau_1} \oplus \boldsymbol{Z}_{\tau_2} \oplus \cdots$, where $\tau_{i+1} | \tau_i$ for all i, and $\tau_i = 1$, $i \geqq n$, for some n. (We do not, of course, exclude the possibility $\tau_i = 0$.) Let us write $\tau_i = \tau_i(M_k)$.

THEOREM. *Let K be any knot, and k, r any integers ($\geqq 1$). Then $\tau_i(M_k) | \tau_i(M_{rk})$ for all i.*

Proof. Lemmas 2.10 and 3.1.

4. Periodicity.

4.1. THEOREM. (i) *$H_1(M_{k+n}) \cong H_1(M_k)$ for all k if and only if $H_1(M_{(k,n)}) \cong H_1(M_k)$ for all k.*

(ii) *$H_1(M_{k+n}; Q) \cong H_1(M_k; Q)$ for all k if and only if $H_1(M_{(k,n)}; Q) \cong H_1(M_k; Q)$ for all k.*

Proof. Since the "if" statements are obvious, we confine our attention to the converses.

(i) Given k, there exist integers r, s, both > 0, such that $rk = (k, n) + sn$. Then

$H_1(M_{rk}) = H_1(M_{(k,n)+sn}) \cong H_1(M_{(k,n)})$. But by Lemma 2.10 there exist epimorphisms $H_1(M_{rk}) \to H_1(M_k)$, $H_1(M_k) \to H_1(M_{(k,n)})$. So, by Corollary 3.1, $H_1(M_k) \cong H_1(M_{(k,n)})$.

(ii) As in (i), the hypothesis implies that $\beta_1(M_{rk}) = \beta_1(M_{(k,n)})$. Again, because of the epimorphisms mentioned above (or, alternatively, by Proposition 2.11(ii)), we have $\beta_1(M_{rk}) \geqq \beta_1(M_k) \geqq \beta_1(M_{(k,n)})$. Hence $\beta_1(M_k) = \beta_1(M_{(k,n)})$, and the result follows.

4.2. In this section we show that $H_1(M_k; Q)$ is always periodic, and that the period depends only on $\Delta(t)$.

Up to sign, $\Delta(t)$ can be written uniquely as $\Phi(t)\Psi(t)$, where $\Phi(t), \Psi(t) \in J[t]$, all the roots of $\Phi(t)$ are roots of unity, and no root of $\Psi(t)$ is a root of unity (see §2.14). The complete description of periodicity of $H_1(M_k; Q)$ is then contained in Theorem 4.1(ii) and the following:

THEOREM. *$H_1(M_k; Q) \cong H_1(M_{(k,n)}; Q)$ for all k if and only if all the roots of $\Phi(t)$ are nth roots of unity.*

Proof. Suppose all the roots of $\Phi(t)$ are nth roots of unity. Then if $\lambda(t)$ is one of the Alexander invariants of K, $\lambda(t)|\Delta(t)$, and so if $\lambda(t)$ is factorized as in (2.14.1), we see that each $m_i|n$. Hence, for any k, $m_i|k$ if and only if $m_i|(k, n)$. It follows that if $\mu(k)$ is the number of distinct kth roots of unity which are roots of $\lambda(t)$, we have $\mu(k) = \mu((k, n))$ (see the proof of Theorem 2.14). Then, by Proposition 2.11(ii), $\beta_1(M_k) = \beta_1(M_{(k,n)})$.

Conversely, suppose that $\beta_1(M_k) = \beta_1(M_{(k,n)})$ for all k, and let $\lambda(t)$ be a typical Alexander invariant as before. Then $\mu(k) = \sum \phi(m_i)$ over those i such that $m_i|k$, and $\mu((k, n)) = \sum \phi(m_i)$ over those i such that $m_i|(k, n)$. But every i which occurs in the second sum must also occur in the first, and so since $\beta_1(M_k) = \beta_1(M_{(k,n)})$ for all k we must have that $m_i|k$ if and only if $m_i|(k, n)$, for all k. In particular, taking $k = m_i$ shows that $m_i|(m_i, n)$, and therefore $m_i|n$. This holds for each m_i, and for each Alexander invariant $\lambda(t)$, and hence all the roots of $\Phi(t)$ are nth roots of unity.

4.3. After Theorem 4.1(i), it suffices, to prove the Main Theorem, to establish the equivalence of (1) and (3). One half is fairly easy:

THEOREM. *Let K be a knot with first Alexander invariant $\lambda_1(t)$. Then if $\lambda_1(t)|(t^n - 1)$, $H_1(M_k) \cong H_1(M_{k+n})$ for all k.*

Proof. Since $\lambda_i(t)|\lambda_1(t)$ for all i, the hypothesis implies that $\Delta(t)|(t^n - 1)^m$, for some m, and so $\Delta(0) = 1$. If $\Delta(t) = 1$, then $H_1(M_k) = 0$ for all k. So suppose $\Delta(t) \neq 1$, and let $M - tI$ be an Alexander matrix for K (see §2.8). By §2.13, $\lambda_1(M) = 0$, and so $M^n = I$. Hence $M^k - I = M^{k+n} - I$, which implies, by §2.12, that $H_1(M_k) \cong H_1(M_{k+n})$.

REMARK. The above proof is similar to the argument in [18, Corollary 5.3.3]. Note, however, that in the latter, for the proof to be valid, the hypothesis "... all the roots of the Alexander polynomial are νth roots of unity" should be replaced by "... the Alexander polynomial divides $t^\nu - 1$."

4.4. To complete the proof of the Main Theorem, we have to show that (1) implies (3). As a first step towards this, we now prove that if some root of $\Delta(t)$ is not a root of unity, then $H_1(M_k)$ is not periodic. In fact we prove slightly more:

THEOREM. *If K is a knot with Alexander polynomial $\Delta(t)$, and some root of $\Delta(t)$ is not a root of unity, then for any integer N there exists k such that $H_1(M_k)$ is finite and* order $H_1(M_k) > N$.

Proof. Write $\Delta(t) = c\prod_{r=1}^{n}(\alpha_r - t)$, where c is an integer. Then, if ρ is a primitive kth root of unity,

$$\prod_{j=1}^{k}\Delta(\rho^j) = c^k\prod_{j=1}^{k}\left(\prod_{r=1}^{n}(\alpha_r-\rho^j)\right)$$
$$= c^k\prod_{r=1}^{n}\left(\prod_{j=1}^{k}(\alpha_r-\rho^j)\right) = c^k\prod_{r=1}^{n}(\alpha_r^k-1).$$

Now it is a well-known theorem of Kronecker (see, for example, [22, p. 118]), that if all the roots of a monic polynomial $\in J[t]$ have unit modulus, then they are roots of unity. Hence either $|c|>1$, or $|c|=1$ and some α_r has $|\alpha_r|\neq 1$. But if $|c|=1$, then $\prod_{r=1}^{n}\alpha_r = \pm\Delta(0)$ is a (nonzero) integer, and therefore, if there exists an α_r with $|\alpha_r|\neq 1$, there exists an α_r with $|\alpha_r|>1$. (In fact, of course, $\Delta(t)$ is always symmetric, so the roots occur in inverse pairs but it is not necessary, for the present argument, to assume this.) Now if $|\alpha|<1$, $|\alpha^k-1|\to 1$ as $k\to\infty$, and if $|\alpha|>1$, $|\alpha^k-1|\to\infty$ as $k\to\infty$. Also, if $|c|>1$, $|c^k|\to\infty$ as $k\to\infty$. We can therefore always separate out the roots of unit modulus and write

$$\left|\prod_{j=1}^{k}\Delta(\rho^j)\right| = f(k)\prod_{r=1}^{m}|\alpha_r^k-1|,$$

where $0\leqq m\leqq n$, $|\alpha_r|=1$, $r=1,\ldots,m$, and $f(k)\to\infty$ as $k\to\infty$. Note that if $m=0$ (i.e. if no root of $\Delta(t)$ has unit modulus) then in fact we have proved that

$$\text{order } H_1(M_k)\to\infty \quad \text{as } k\to\infty.$$

If $m\geqq 1$, let us write $\alpha_r = \exp(2\pi i x_r)$, $r=1,\ldots,m$. So $\alpha_r^k = \exp(2\pi i k x_r)$. Then it is a theorem of Dirichlet [13, p. 170] that, given $\varepsilon>0$, there exist infinitely many integers k' such that each of $k'x_1,\ldots,k'x_m$ is distance $<\varepsilon$ from an integer. Also, since $\Delta(1)=\pm 1\neq 0$, no x_r is an integer, so if we let ε_r be the least distance of x_r from an integer, and define $\varepsilon = \min\{\varepsilon_1/2,\ldots,\varepsilon_m/2\}$, we have $\varepsilon>0$. Dirichlet's theorem then gives infinitely many k' such that each $k'x_r$ is within ε of an integer, and hence the distance of $(k'+1)x_r$ from an integer is $>\varepsilon$. Write $k=k'+1$. Then, for $r=1,\ldots,m$, $|\alpha_r^k-1| = |\exp(2\pi i k x_r)-1| > |\exp(2\pi i\varepsilon)-1| = a$, say, where $a>0$. Now $f(k)\to\infty$ as $k\to\infty$, and so, given any integer N, there exists k_0 such that $f(k)>N/a^m$ if $k>k_0$. Also, by the above, there exists $k>k_0$ such that $\prod_{r=1}^{m}|\alpha_r^k-1| > a^m$. Then, for such a k, $|\prod_{j=1}^{k}\Delta(\rho^j)|>N$, as required.

REMARK. The precise behaviour of order $H_1(M_k)$ as $k \to \infty$ is likely to be rather difficult to determine in general, since if $\Delta(t)$ has a root α of unit modulus, this contributes a factor $|\alpha^k - 1| = |\exp(2\pi i k x) - 1|$, say, which gets arbitrarily small infinitely often (by taking $m = 1$ in Dirichlet's theorem). We therefore ask

Question. If no root of $\Delta(t)$ is a root of unity, and $\Delta(t) \neq 1$, does $|\prod_{j=1}^{k} \Delta(\rho^j)| \to \infty$ as $k \to \infty$?

A related question is whether $|\prod_{j=1}^{k} \Delta(\rho^j)|$ can ever be 1 for $k > 1$ (if some root of $\Delta(t)$ is not a root of unity), i.e. whether M_k can ever be a homology sphere for $k > 1$. For knots of genus 1, this is answered negatively in [5, Theorem 3], but in general, it is easy to see that homology spheres can occur. For example, if $\Delta(t)$ is any Alexander polynomial whose roots are not all roots of unity, then $\Delta(t^k)$ is also an Alexander polynomial ([24], [15]) whose roots are not all roots of unity. But order $H_1(M_k) = |\prod_{j=1}^{k} \Delta(\rho^{kj})| = |\prod_{j=1}^{k} \Delta(1)| = 1$.

It is also easy to construct other examples, such as $1 - t^2 + t^3 - t^4 + t^6$, and $1 - t^2 - t^3 - t^4 + t^6$, which have order $H_1(M_2) = |\Delta(-1)| = 1$.

4.5. Theorem 4.4 shows that if $H_1(M_k)$ is periodic, then all the roots of $\Delta(t)$ are roots of unity. The converse is not true, as we shall see later, but we do have the following:

THEOREM. *If all the roots of* $\Delta(t)$ *are nth roots of unity, then*

(i) $H_1(M_k; Q) \cong H_1(M_{(k,n)}; Q)$ *for all* k,

(ii) order $H_1(M_k)$ = order $H_1(M_{(k,n)})$ *for all* k,

(iii) *if* $H_1(M_k)$ *is finite, then* $H_1(M_k) \cong H_1(M_{(k,n)})$.

((i) is just a special case of Theorem 4.2, which is included here for completeness.)

The proof of (ii) will require the following lemma, in which $\phi_m(t)$ as usual denotes the mth cyclotomic polynomial.

LEMMA. *Let* ρ *be a primitive* k*th root of unity, and* σ *a primitive* (k, m)*th root of unity. Then*

$$\prod_{j=1}^{k} \phi_m(\rho^j) = \prod_{j=1}^{(k,m)} \phi_m(\sigma^j).$$

Proof. Let $\alpha_1, \ldots, \alpha_{\phi(m)}$ be the primitive mth roots of unity, so that we have $\phi_m(t) = \prod_{i=1}^{\phi(m)} (t - \alpha_i)$. Then (as in the proof of Theorem 4.4)

$$\prod_{j=1}^{k} \phi_m(\rho^j) = \prod_{i=1}^{\phi(m)} (1 - \alpha_i^k).$$

(Actually, we have introduced a factor $(-1)^{\phi(m)(k+1)}$, but, since $\phi(m)$ is even if $m \geqq 3$, this makes no difference unless $m = 1$ or 2 and k is even, and in this case, the product is zero anyway.) Similarly,

$$\prod_{j=1}^{(k,m)} \phi_m(\sigma^j) = \prod_{i=1}^{\phi(m)} (1 - \alpha_i^{(k,m)}).$$

Now since α_i is a primitive mth root of unity, α_i^k is a primitive dth root of unity, where $d = m/(k, m)$, and so $\prod_{i=1}^{\phi(m)} (t - \alpha_i^k)$ must be some power of $\phi_d(t)$; comparing

degrees, we see that this power must be $\phi(m)/\phi(d)$ [8]. The same argument with (k, m) instead of k, and the fact that $((k, m), m)=(k, m)$, then show that

$$\prod_{i=1}^{\phi(m)}(t-\alpha_i^k) = \prod_{i=1}^{\phi(m)}(t-\alpha_i^{(k,m)}),$$

which gives the desired result on putting $t=1$.

Proof of Theorem 4.5. The hypothesis implies that $\Delta(t)$ is a product of cyclotomic polynomials $\phi_m(t)$, where $m|n$. For each such m, we then have $((k, n), m) =(k, m)$, and so, by Lemma 4.5,

$$\prod_{j=1}^{k}\phi_m(\rho^j) = \prod_{j=1}^{(k,m)}\phi_m(\sigma^j) = \prod_{j=1}^{(k,n)}\phi_m(\tau^j),$$

where ρ, σ, τ are primitive kth, (k, m)th, (k, n)th roots of unity, respectively. (ii) now follows from Proposition 2.11(i).

To prove (iii), recall that by Lemma 2.10 there exists an epimorphism $\phi\colon H_1(M_k) \to H_1(M_{(k,n)})$, and then note that if $H_1(M_k)$ is finite, ϕ must be an isomorphism, because of (ii).

4.6. Before proving the converse of Theorem 4.3, let us discuss the following example, which shows that Theorem 4.5 is the best possible in the sense that all the roots of $\Delta(t)$ being roots of unity is not in general enough to guarantee periodicity.

Take any knot with Alexander matrix $M(t)=((1-t+t^2)^2)$ ([15], [12]). Then all the roots of $\lambda_1(t)=\Delta(t)$ are 6th roots of unity, but $\lambda_1(t)\nmid(t^6-1)$. Now $(1-t+t^2)^2 =1-2t+3t^2-2t^3+t^4$, and so clearly $M(t) \sim^{\Lambda} M-tI$, where

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -3 & 2 \end{pmatrix}.$$

By explicit computation of M^6 and induction on r, it can be shown that

$$M^{6r} = \begin{pmatrix} -(2r-1) & 0 & 0 & -2r \\ 2r & -(6r-1) & 6r & -4r \\ 4r & -6r & 6r+1 & -2r \\ 2r & 0 & 0 & 2r+1 \end{pmatrix}.$$

Therefore

$$M^{6r}-I = \begin{pmatrix} -2r & 0 & 0 & -2r \\ 2r & -6r & 6r & -4r \\ 4r & -6r & 6r & -2r \\ 2r & 0 & 0 & 2r \end{pmatrix} \sim^{J} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6r & 0 \\ 0 & 0 & 0 & 2r \end{pmatrix}$$

which shows that

$$H_1(M_{6r}) \cong Z \oplus Z \oplus Z_{6r} \oplus Z_{2r},$$

and so $H_1(M_k)$ is not periodic.

4.7. The converse of Theorem 4.3 is proved by showing that if $\lambda_1(t) \nmid (t^n-1)$, and Theorem 4.4 does not apply, then we have a situation not unlike that of §4.6.

THEOREM. *Let K be a knot with first Alexander invariant $\lambda_1(t)$, where all the roots of $\lambda_1(t)$ are nth roots of unity, but $\lambda_1(t) \nmid (t^n-1)$. Then the order of the torsion subgroup of $H_1(M_{rn}) \to \infty$ as $r \to \infty$.*

Proof. The hypothesis implies that for some $m \geqq 1$, $\lambda_1(t) | (t^n-1)^{m+1}$, but $\lambda_1(t) \nmid (t^n-1)^m$.

Let $M-tI$ be an Alexander matrix for K (see §2.8), and write $N=M^n-I$. Then, by §2.13, $N^{m+1}=0$, $N^m \neq 0$.

A presentation matrix for $H_1(M_{rn})$ as an abelian group is

$$M^{rn}-I = (N+I)^r - I = \sum_{i=1}^{r} \binom{r}{i} N^i = \sum_{i=1}^{m} \binom{r}{i} N^i \quad \text{if } r \geqq m.$$

But certainly $\sum_{i=1}^{m} \binom{r}{i} N^i \neq 0$ (multiplication by N^{m-1} gives rN^m, which is $\neq 0$), and so the order of the torsion subgroup of $H_1(M_{rn})$ is some positive multiple of the h.c.f. of the entries of $\sum_{i=1}^{m} \binom{r}{i} N^i$. It is not hard to show, however, that for fixed m, $h(r) = \text{hcf}\{\binom{r}{1}, \binom{r}{2}, \ldots, \binom{r}{m}\} \to \infty$ as $r \to \infty$, and since $h(r)$ must divide each entry of $\sum_{i=1}^{m} \binom{r}{i} N^i$, this completes the proof.

4.8. **Proof of Main Theorem.** Theorem 4.1(i) shows that (1) and (2) are equivalent, and Theorem 4.3 shows that (3) implies (1). To prove that (1) implies (3), we first note that by Theorems 4.4 and 4.7, (1) implies that $\lambda_1(t) | (t^{n'}-1)$, for some n'. Then all the roots of $\Delta(t)$ are n'th roots of unity, and so it follows from Theorem 4.2 that in fact all the roots of $\Delta(t)$ must be nth roots of unity. It only remains to note that, since $\lambda_1(t) | (t^{n'}-1)$, $\lambda_1(t)$ has no repeated roots, and so we conclude that $\lambda_1(t) | (t^n-1)$.

4.9. If $H_1(M_k) \cong H_1(M_{k+n})$ for all k, and there does not exist n' with $0 < n' < n$ such that $H_1(M_k) \cong H_1(M_{k+n'})$ for all k, we say that the *proper period* of $H_1(M_k)$ is n.

In [21] it is shown that for knots of genus 1, the only possible proper period of $H_1(M_k)$ (other than 1) is 6. Moreover, 6 does occur as a proper period for a knot of genus 1, namely the trefoil.

THEOREM. *There exists a knot K for which $H_1(M_k)$ has proper period n if and only if $n=1$, or n has at least 2 distinct prime factors.*

Proof. If $n=p^r$, p prime, $r>0$, then $\lambda_1(t) \nmid (t^n-1)$ (see §2.14) and so for no knot can $H_1(M_k)$ have period n.

To prove the converse, first note that any knot with $\Delta(t)=1$ [30] gives period 1 ($H_1(M_k)=0$ for all k) and so we are left to consider the case where n has at least 2 distinct prime factors. By §2.14 and [15] or [12], there exists a knot with Alexander matrix $(\phi_n(t))$. Then $H_1(M_k)$ has period n, and to see that the proper period must

be n, it is enough to note that since the roots of $\phi_n(t)$ are the primitive nth roots of unity, $\phi_n(t) \nmid (t^{n'}-1)$ if $0<n'<n$.

4.10. The following theorem states precisely which proper periods can occur for knots of a given genus.

THEOREM. *There exists a knot K of genus g for which $H_1(M_k)$ has proper period n if and only if $n=1$, or $n=\operatorname{lcm}\{m_i: i=1,\ldots,r\}$, where the m_i's are all distinct, each has at least 2 distinct prime factors, and $\sum_{i=1}^{r}\phi(m_i)\leqq 2g$.*

Proof. First we dispose of the case $n=1$ by observing that for any g there exists a knot of genus g with $\Delta(t)=1$. For example, since any untwisted doubled knot [30] has genus 1 and $\Delta(t)=1$, we can simply take the connected sum of any g untwisted doubled knots.

If K is a knot for which $H_1(M_k)$ has proper period >1, then $\lambda_1(t)=\prod_{i=1}^{r}\phi_{m_i}(t)$, where the m_i's are distinct, and each has at least 2 distinct prime factors. The least n such that $\lambda_1(t)|(t^n-1)$ is then $n=\operatorname{lcm}\{m_i: i=1,\ldots,r\}$, and hence this is the proper period. Since $2g\geqq\deg\Delta(t)\geqq\deg\lambda_1(t)=\sum_{i=1}^{r}\phi(m_i)$, one half of the theorem follows.

Conversely, given such a set $\{m_i: i=1,\ldots,r\}$, there exists a knot K of genus $\frac{1}{2}\sum_{i=1}^{r}\phi(m_i)$ with Alexander matrix $(\prod_{i=1}^{r}\phi_{m_i}(t))$ [12]. Taking the connected sum of K with $g-\frac{1}{2}\sum_{i=1}^{r}\phi(m_i)$ untwisted doubled knots then gives a knot of genus g for which $H_1(M_k)$ has proper period $n=\operatorname{lcm}\{m_i: i=1,\ldots,r\}$.

4.11. For a given genus g, it is possible from Theorem 4.10 (in theory at any rate) to determine all proper periods. For example, since it is easy to check that $\phi(6)=2$, $\phi(10)=\phi(12)=4$, $\phi(14)=\phi(18)=6$, $\phi(15)=\phi(20)=\phi(24)=\phi(30)=8$, and $\phi(m)\geqq 10$ for any other m with at least 2 distinct prime factors, we get the following table for the first few values of g:

genus	proper periods
1	1, 6;
2	the above, and 10, 12;
3	the above, and 14, 18, 30;
4	the above, and 15, 20, 24, 42, 60.

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