

EQUIVARIANT BORDISM AND SMITH THEORY. III

BY
R. E. STONG

Abstract. The bordism classes of a manifold with involution is determined by means of Wu type classes in the Smith cohomology.

1. Introduction. Being given a closed differentiable manifold M^n with a differentiable involution $t: M^n \rightarrow M^n$, it was shown in [3] that the bordism class of the involution (M, t) is determined by characteristic numbers from the Smith theory of (M, t) . The object of this note is to show that the bordism class may be determined from the Smith theory of (M, t) directly.

Roughly this is analogous to a theorem of Wu [5]. For a closed manifold M^n , the Wu classes $v_i \in H^i(M; \mathbb{Z}_2)$ are defined by $\langle v_i \cup x, [M] \rangle = \langle Sq^i x, [M] \rangle$ for all x in $H^{n-i}(M; \mathbb{Z}_2)$, and then the Stiefel-Whitney classes of M are given by $w_i = \sum_j Sq^{i-j} v_j$. By Thom's theorem [4], the characteristic numbers $\langle w_{i_1} \cup \cdots \cup w_{i_r}, [M] \rangle$ with $i_1 + \cdots + i_r = n$ determine the bordism class of M . In the case of an involution, analogous Wu classes arise from operations in the Smith theory.

The author is indebted to Professor P. E. Conner for several conversations related to this work and to the National Science Foundation for financial support.

2. Smith theory characteristic classes. Being given a space X and a continuous involution $t: X \rightarrow X$, let $F \subset X$ be the fixed point set of t and X/t the orbit space. The Smith theory cohomology of the involution (X, t) is then $H_{\mathbb{Z}_2}^i(X, t; \mathbb{Z}_2) = H^i(X/t, F; \mathbb{Z}_2) \oplus H^i(F; \mathbb{Z}_2)$.

Provided the involution (X, t) is sufficiently nice, i.e. if F is an equivariant strong deformation retract of a neighborhood in X , the projection $\pi_x: X \times EZ_2 \rightarrow X$ with EZ_2 a universal space for Z_2 induces an isomorphism

$$\pi_x^*: H^*(X/t, F; \mathbb{Z}_2) \xrightarrow{\cong} H^*(X \times EZ_2/Z_2, F \times BZ_2; \mathbb{Z}_2),$$

where BZ_2 is a classifying space for Z_2 and $F \times BZ_2 = F \times EZ_2/Z_2$. This will be assumed throughout, being true for a differentiable involution.

Now let $c' \in H^1(X \times EZ_2/Z_2; \mathbb{Z}_2)$ be the characteristic class of the double cover $X \times EZ_2 \rightarrow X \times EZ_2/Z_2$, and $c \in H^1(F \times BZ_2; \mathbb{Z}_2)$ the induced class. The cup product with c' defines the Gysin homomorphism

$$\mu: H^i(X/t, F; \mathbb{Z}_2) \rightarrow H^{i+1}(X/t, F; \mathbb{Z}_2).$$

Received by the editors August 25, 1971.

AMS 1970 subject classifications. Primary 57D85; Secondary 55C35.

Copyright © 1972, American Mathematical Society

Letting $\delta: H^i(F; Z_2) \rightarrow H^{i+1}(X/t, F; Z_2)$ be the coboundary one then defines homomorphisms

$$\phi_{k+1}: H^i(F; Z_2) \rightarrow H^{i+k+1}(X/t, F; Z_2)$$

by $\phi_{k+1} = \mu^k \delta$.

Since the coboundary homomorphism

$$\delta': H^*(F \times BZ_2; Z_2) \rightarrow H^*(X \times EZ_2/Z_2, F \times BZ_2; Z_2)$$

is a $H^*(X \times EZ_2/Z_2; Z_2)$ module homomorphism, $\phi_{k+1}(\alpha) = \delta'(\pi^*(\alpha) \cup c^k)$.

Letting $I: H^i(F; Z_2) \rightarrow H_{Z_2}^i(X, t; Z_2)$ and $J: H^i(X/t, F; Z_2) \rightarrow H_{Z_2}^i(X, t; Z_2)$ be the inclusions of the summands, one defines homomorphisms

$$\psi_k: H^i(F; Z_2) \rightarrow H_{Z_2}^{i+k}(X, t; Z_2)$$

by $\psi_0 = I$, $\psi_k = J\phi_k$ if $k \geq 1$.

Now suppose (X, t) behaves like a differentiable involution on an n -dimensional manifold. Specifically, suppose there is a fundamental class $[X, t] \in H_n^{Z_2}(X, t; Z_2)$ and the fixed set F is the disjoint union of subspaces F^k ($0 \leq k \leq n$), so that there is a fundamental class $[F^k] \in H_k(F^k; Z_2)$ with respect to which F^k satisfies Poincaré duality.

The formulas $\langle v_i \cup x, [F^k] \rangle = \langle Sq^i x, [F^k] \rangle$ for all $x \in H^{k-i}(F^k; Z_2)$ and $w_i = \sum_j Sq^{i-j} v_j$ then define Stiefel-Whitney classes $w_i \in H^i(F; Z_2)$, following Wu. Further, there are classes $\alpha_i \in H^i(F; Z_2)$ defined on F^k by $\langle \alpha_i \cup x, [F^k] \rangle = \langle \psi_{n-k+i} X, [X, t] \rangle$ for all $x \in H^{k-i}(F^k; Z_2)$.

For any pair of sequences $(i_1, \dots, i_r), (j_1, \dots, j_s)$, one may then compute the number $\langle w_{i_1} \cup \dots \cup w_{i_r} \cup \alpha_{j_1} \cup \dots \cup \alpha_{j_s}, [F] \rangle \in Z_2$, which is the value on $[F^k]$ where $k = i_1 + \dots + i_r + j_1 + \dots + j_s$. These will be called the *Wu-Stiefel-Whitney numbers* of the involution (X, t) .

The main result of this paper is

PROPOSITION. *If (M^n, t) is a differentiable involution on a closed n -manifold, then (M^n, t) bounds if and only if all of the Wu-Stiefel-Whitney numbers of (M^n, t) are zero.*

By the results of Conner and Floyd [2, Chapter IV], the involution bounds if and only if all of the characteristic numbers $\langle w_{i_1} \cup \dots \cup w_{i_r} \cup w'_{j_1} \cup \dots \cup w'_{j_s}, [F] \rangle$ vanish, where w'_i is the i th Stiefel-Whitney class of the normal bundle ν of F in M . Hence it suffices to show

LEMMA. *The class $\alpha_i \in H^i(F; Z_2)$ is the i th Stiefel-Whitney class of the complement of ν and is given by $w'_i + P_i(w'_1, \dots, w'_{i-1})$ for a universal polynomial P_i .*

Thus, the Stiefel-Whitney classes of the normal bundle of the fixed set in M are determined by the Wu classes α_i or the operations ψ_k .

3. Proof of the Lemma. Now suppose $X = M^n$ is a closed n -dimensional manifold and t is a differentiable involution. The fixed set F is then a disjoint union of closed manifolds, and F^k is the union of the k -dimensional components of F . Let ν^{n-k} be the normal bundle of F^k in M and identify a neighborhood of F in M with the disjoint union of the disc bundles $D(\nu^{n-k})$, with t agreeing with multiplication by -1 in the discs.

Let \bar{M} be obtained from M by deleting the interiors of the sets $D(\nu^{n-k})$, so that \bar{M} is a manifold with boundary, whose boundary $\partial\bar{M}$ is the union of the sphere bundles $S(\nu^{n-k})$. (Note. $D(\nu^0) = F^n$ and $S(\nu^0)$ is empty.) The involution t acts freely on \bar{M} , so that \bar{M}/t is a manifold with boundary, whose boundary is $(\partial\bar{M})/t$ which is the union of the real projective space bundles $RP(\nu^{n-k})$. Let $p: RP(\nu^{n-k}) \rightarrow F^k$ denote the projection.

Since F^k is an equivariant strong deformation retract of $D(\nu^{n-k})$, one has $H^*(M/t, F; Z_2) \cong H^*(M/t, \bigcup_k D(\nu^{n-k})/t; Z_2)$ and by excision this is

$$H^*(\bar{M}/t, \partial\bar{M}/t; Z_2).$$

Henceforth, these groups will be identified. In particular, the coboundary homomorphism

$$\delta': H^*(F \times BZ_2; Z_2) \rightarrow H^*(M/t, F; Z_2)$$

is identical with the composite of

$$(p \times q)^*: H^*(F \times BZ_2; Z_2) \rightarrow H^*(\partial\bar{M}/t; Z_2),$$

where p is projection on F and q classifies the double cover $\partial\bar{M} \rightarrow \partial\bar{M}/t$ or $S(\nu^{n-k}) \rightarrow RP(\nu^{n-k})$, and the coboundary homomorphism

$$\delta'': H^*(\partial\bar{M}/t; Z_2) \rightarrow H^*(\bar{M}/t, \partial\bar{M}/t; Z_2).$$

Also, in $H_n^{Z_2}(M, t; Z_2) \cong H_n(\bar{M}/t, \partial\bar{M}/t; Z_2) \oplus H_n(F^n; Z_2)$, the fundamental class of $[M, t]$ is the sum of the fundamental classes $[\bar{M}/t, \partial\bar{M}/t]$ and $[F^n]$.

Now recall that $H^*(RP(\nu^{n-k}); Z_2)$ is the free $H^*(F^k; Z_2)$ module via p^* on $1, c, \dots, c^{n-k-1}$, where $c = (p \times q)^*(c)$, and the algebra structure is given by the relation

$$c^{n-k} + c^{n-k-1}p^*w'_1 + \dots + c^{n-k-r}p^*w'_r + \dots + p^*w'_{n-k} = 0.$$

In particular, for $a \in H^*(F^k; Z_2)$

$$\begin{aligned} \langle p^*a \cup c^j, [RP(\nu^{n-k})] \rangle &= 0, & j < n-k-1, \\ &= \langle a, [F^k] \rangle, & j = n-k-1. \end{aligned}$$

One may now use the relation $c^{n-k} = c^{n-k-1}p^*w'_1 + \dots + p^*w'_{n-k}$ inductively to write

$$\begin{aligned} c^{n-k+i-1} &= c^{i-1}(c^{n-k}) = c^{i-1}(c^{n-k-1}p^*w'_1 + \dots + c^{n-k-i}p^*w'_i + \dots) \\ &= c^{n-k+i-2}p^*w'_1 + \dots + c^{n-k-1}p^*w'_i + \text{terms } c^j p^*(x) \\ &\hspace{15em} \text{with } j < n-k-1, \end{aligned}$$

so that if $c^{n-j} = c^{n-k-1}p^*(x_j) + \text{terms } c^q p^*(y), q < n-k-1$, then $x = w'_1 x_{i-1} + \cdots + w'_{i-1} x_1 + w'_i$, which is the inductive formula for the i th Stiefel-Whitney class of the complement of ν , and expresses x_i inductively as $x_i = w'_i + P_i(w'_1, \dots, w'_{i-1})$.

Now let $k < n$ and $x \in H^{k-i}(F^k; \mathbb{Z}_2)$. Then

$$\begin{aligned} \langle \alpha_i \cup x, [F^k] \rangle &= \langle \psi_{n-k+i} x, [M, t] \rangle = \langle J\phi_{n-k+i} x, [M, t] \rangle \\ &= \langle \phi_{n-k+i} x, [\bar{M}/t, \partial \bar{M}/t] \rangle \\ &= \langle \delta'(\pi_F^*(x) \cup c^{n-k+i-1}), [\bar{M}/t, \partial \bar{M}/t] \rangle \\ &= \langle \delta''(p^*(x) \cup c^{n-k+i-1}), [\bar{M}/t, \partial \bar{M}/t] \rangle \\ &= \langle p^*(x) \cup c^{n-k+i-1}, [\partial \bar{M}/t] \rangle \\ &= \langle p^*(x) \cup c^{n-k+i-1}, [RP(\nu^{n-k})] \rangle \\ &= \left\langle c^{n-k-1} p^*(x \cup (w'_i + P_i)) + \sum_{j < n-k-1} c^j p^*(xy), [RP(\nu^{n-k})] \right\rangle \\ &= \langle (w'_i + P_i) \cup x, [F^k] \rangle \end{aligned}$$

so $\alpha_i = w'_i + P_i(w'_1, \dots, w'_{i-1})$ in F^k for $k < n$.

For $k = n$, the homomorphism $\delta: H^*(F^n; \mathbb{Z}_2) \rightarrow H^*(M/t, F; \mathbb{Z}_2)$ is zero, so ψ_i is zero on $H^*(F^n; \mathbb{Z}_2)$ for $i > 0$. Thus $\alpha_i = 0 = w'_i + P_i(w'_1, \dots, w'_{i-1})$ for $i > 0$. For $i = 0$ and $x \in H^n(F^n; \mathbb{Z}_2)$,

$$\langle \alpha_0 \cup x, [F^n] \rangle = \langle \psi_0 x, [M, t] \rangle = \langle Ix, [M, t] \rangle = \langle x, [F^n] \rangle$$

so $\alpha_0 = 1 = w'_0$.

This completes the proof of the lemma.

One may now note that the classes defined actually give all of the Smith theory characteristic classes of the tangent bundle. Specifically, let (BO_n, S) be the classifying space for n -plane bundles with involution, $i: BO_n \rightarrow BO_{n+1}$ the classifying map for $\gamma_n \oplus 1$, the Whitney sum with a trivial bundle with trivial action, and $\tau: M^n \rightarrow BO_n$ the equivariant classifying map for the tangent bundle of M .

Now BO_n is the limit of actions on Grassmannians, so may be assumed nice near the fixed set, with the fixed set $FBO_n = \bigcup_{j+k=n} BO_j \times BO_k$, where the $+1$ eigenbundle over $BO_j \times BO_k$ has dimension j , and the -1 eigenbundle has dimension k . Then under i , $BO_j \times BO_k$ is sent into $BO_{j+1} \times BO_k$. The classifying map τ sends F^q into $BO_q \times BO_{n-q}$ with the classes $w_i(F^q)$ and α_j completely determining the cohomology homomorphism.

Now examining the summand $H^*(BO_n/\mathbb{Z}_2, FBO_n; \mathbb{Z}_2)$ in the Smith theory of (BO_n, S) , one may first form the product with $E\mathbb{Z}_2$. Then $BO_n \times E\mathbb{Z}_2/\mathbb{Z}_2$ is homotopy equivalent to $BO_n \times B\mathbb{Z}_2$, since both classify n -plane bundles with involution over a double cover, and the inclusion of the fixed component $BO_n \times BO_0$ induces a homotopy equivalence $BO_n \times BO_0 \times B\mathbb{Z}_2 \rightarrow BO_n \times E\mathbb{Z}_2/\mathbb{Z}_2$. Thus, the homomorphism

$$\delta': \bigoplus_{j+k=n; k \neq 0} H^*(BO_j \times BO_k \times B\mathbb{Z}_2; \mathbb{Z}_2) \rightarrow H^*(BO_n/\mathbb{Z}_2, FBO_n; \mathbb{Z}_2)$$

is an isomorphism. This is compatible with i^* .

Thus, under the tangent map $\tau: M \rightarrow BO_n$, or its stabilization to BO , the induced homomorphism on Smith theory is determined by the classes $w_i(F^q)$ and α_j , for every Smith theory class is obtained from products of these by applying the homomorphisms ψ_k and summing.

4. Remarks. The results obtained obviously give

COROLLARY. *If $f: (M^n, t) \rightarrow (N^n, s)$ is an equivariant map of differentiable involutions on closed n -manifolds inducing isomorphisms on Smith theory, then (M^n, t) and (N^n, s) are bordant as involutions.*

This generalizes a result obtained in a discussion with Professor Conner:

PROPOSITION. *If two differentiable involutions (M^n, t) and (N^n, s) have the same equivariant homotopy type then they are bordant as involutions.*

Proof (Direct argument). Let $f: (M^n, t) \rightarrow (N^n, s)$ be an equivariant homotopy equivalence. Then $f: M \rightarrow N$ is a homotopy equivalence and $f: F_M \rightarrow F_N$ is a homotopy equivalence. Then $f^*(w') = f^*(w(N)/w(F_N)) = w(M)/w(F_M) = w'$ and $f^*(w(F_N)) = w(F_M)$ so the involutions have the same Conner-Floyd characteristic numbers, and so are bordant.

Finally, the interested reader is encouraged to look at Professor Conner's paper [1]. While there is no direct overlap, there are definite interrelations with this paper. It may clarify the relation between Smith theory and equivariant bordism.

REFERENCES

1. P. E. Conner, *Diffeomorphisms of period two*, Michigan Math. J. **10** (1963), 341–352. MR **28** #2562.
2. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., Band 33, Academic Press, New York; Springer-Verlag, Berlin, 1964. MR **31** #750.
3. R. E. Stong, *Equivariant bordism and Smith theory. I*, Trans. Amer. Math. Soc. **159** (1971), 417–426.
4. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86. MR **15**, 890.
5. Wu Wen-tsün, *Classes caractéristiques et i -carrés d'une variété*, C.R. Acad. Sci. Paris **230** (1950), 508–511. MR **12**, 42.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22904