

ONE-PARAMETER INVERSE SEMIGROUPS

BY

CARL EBERHART AND JOHN SELDEN

Abstract. This is the second in a projected series of three papers, the aim of which is the complete description of the closure of any one-parameter inverse semigroup in a locally compact topological inverse semigroup. In it we characterize all one-parameter inverse semigroups. In order to accomplish this, we construct the free one-parameter inverse semigroups and then describe their congruences.

0. Let G be a subgroup of the multiplicative group of positive real numbers and let P denote the subsemigroup of G consisting of all $x \in G$ with $x \geq 1$. Denote by $\mathcal{C}_P$ the class of all inverse semigroups H for which there is a homomorphism $f: P \rightarrow H$ such that $f(P)$ generates H (no proper inverse subsemigroup of H contains $f(P)$). We shall call such semigroups H *one-parameter inverse semigroups* and denote by $\mathcal{C} = \bigcup_P \mathcal{C}_P$ the class of all one-parameter inverse semigroups.

The class $\mathcal{C}$ contains well-known semigroups. For example, each homomorphic image of a subgroup of R , the positive real numbers, is a member of $\mathcal{C}$. Also the bicyclic semigroup B is a member of $\mathcal{C}$, as is seen by noting that B is generated by a copy of the nonnegative integers. Indeed, if H is any elementary inverse semigroup, then H^1 is generated by a homomorphic image of the nonnegative integers, and so is a one-parameter inverse semigroup.

The main purpose of this paper is to describe all one-parameter inverse semigroups. In the process of doing this, we shall construct what we term the *free one-parameter inverse semigroups* F_P , one for each subgroup G of R and its associated semigroup P . The semigroup F_P is the only inverse semigroup (up to isomorphism) generated by a subsemigroup isomorphic with P which has the property that each homomorphism $f: P \rightarrow S$, an inverse semigroup, extends uniquely to a homomorphism $\tilde{f}: F_P \rightarrow S$. In particular, every $H \in \mathcal{C}_P$ is a homomorphic image of F_P . We thus adopt the point of view that by describing F_P and the lattice of congruences of F_P for arbitrary P , we will have described all one-parameter inverse semigroups.

We shall assume a certain familiarity with the algebraic theory of semigroups, particularly inverse semigroups. (See Clifford and Preston [1].)

The existence and uniqueness of F_P is a consequence of a theorem due to McAlister [3, Theorem 33]. We were greatly aided in the actual description of F_P

Presented to the Society, August 27, 1969; received by the editors June 10, 1969.

AMS 1970 subject classifications. Primary 20M10; Secondary 20M05, 22A15.

Key words and phrases. Inverse semigroup, one-parameter inverse semigroup, bicyclic semigroup, free semigroup, lattice of congruences, freely generated, bisimple inverse semigroup, Green's relations, normal congruence, group congruence, lattice of subgroups, kernel.

Copyright © 1972, American Mathematical Society

by two results of Gluskin on elementary inverse semigroups [2, p. 24]. For the description of the congruences on F_P , the results of Reilly and Scheiblich in [4] proved useful.

Although this paper is primarily algebraic in nature, there is a natural topology on F_P with respect to which F_P is a topological inverse semigroup. This fact, together with several other comments of a topological nature, are included in remarks throughout the paper.

1. The free inverse semigroup on a set X . In this section we shall review some theory which has already been obtained by McAlister in [3].

If S is an inverse semigroup generated by a subset X , then we say that S is *freely generated* by X provided each function from X into an inverse semigroup extends to a homomorphism on S . One shows easily, using the fact that homomorphisms on inverse semigroups take inverses to inverses, that if S is freely generated by X , then each function from X into an inverse semigroup T extends to a unique homomorphism from S into T .

1.1. THEOREM. *For any nonvoid X there is one and only one inverse semigroup (up to isomorphism) I_X freely generated by X .*

Although it is not our intention to investigate them here, we remark that many interesting questions arise concerning the structure of I_X and its lattice of congruences. For example, it is not difficult to show that the smallest group congruence on I_X has the free group on X as its quotient semigroup.

Now let P be a fixed semigroup. Consider the class of pairs (f, S) where S is an inverse semigroup and f is a homomorphism from P into S so that $f(P)$ generates S . Define two pairs (f, S) and (g, T) to be equivalent provided there is an isomorphism $\phi: S \xrightarrow{\text{onto}} T$ so that $\phi f = g$. This is easily seen to be an equivalence relation on pairs. We call a pair (f, S) a *free pair* provided given any pair (g, T) there is a homomorphism $\phi: S \rightarrow T$ such that $\phi f = g$. It follows from the fact that two homomorphisms on an inverse semigroup which agree on a generating set are identical, that the homomorphism ϕ above is unique.

The next theorem establishes the existence and uniqueness of a free pair (f, S) .

1.2. THEOREM. *There is an inverse semigroup S and a homomorphism $f: P \rightarrow S$ such that (f, S) is a free pair. Furthermore any two free pairs are equivalent. The homomorphism f is 1-1 if and only if P is embeddable in an inverse semigroup.*

In case f is 1-1 we identify P with $f(P)$ and call S the inverse semigroup freely generated by the subsemigroup P and denote S by F_P . Note that F_P is characterized by the property that any homomorphism from P into an inverse semigroup extends to a unique homomorphism on F_P . In particular, any inverse semigroup generated by a homomorphic image of P is isomorphic with a quotient semigroup of F_P .

2. The free one-parameter inverse semigroups F_P . Let G be a fixed subgroup of R and let $P = \{x \in G \mid x \geq 1\}$, $P_0 = P \setminus \{1\}$. In this section we shall describe fully the structure of the semigroups F_P and F_{P_0} freely generated by the subsemigroups P and P_0 respectively.

First we construct a homomorphic image B_P of F_P which is a generalization of the bicyclic semigroup B . This construction is similar to the one found on p. 107 of Vol. 2 of [1]. Let $B_P = P \times P$ with the following operation:

$$(x, y)(z, w) = (xz/y \wedge z, yw/y \wedge z)$$

where $y \wedge z = \min \{y, z\}$. It is easily checked that the product of two elements of B_P is an element of B_P . In fact we have the following consequence of Theorems 8.43 and 8.44 of Vol. 2 of [1]:

2.1. THEOREM. B_P is a bisimple inverse semigroup which is generated by $P_0 \times 1$.

2.2. THEOREM. The real number 1 is the identity for F_P . Furthermore F_{P_0} does not have an identity and in fact is isomorphic with $F_P \setminus \{1\}$. Thus F_P is obtained from F_{P_0} by adjoining an identity.

Proof. Since 1 is the identity of P and P generates F_P , 1 is the identity of F_P . Let S denote the inverse subsemigroup of F_P generated by P_0 , and let f be a homomorphism from P_0 into an inverse semigroup T . We assume T has an identity e , for otherwise we could adjoin it. Then f extends to a homomorphism $g: P \rightarrow T$ by defining $g(1) = e$. Now g extends to a homomorphism $\bar{g}: F_P \rightarrow T$, and $\bar{g}|S$ is clearly the sought extension of f to S . Thus S is freely generated by P_0 ; that is, $S = F_{P_0}$. Now suppose S has an identity i . Then there exist $x_1, x_2, \dots, x_n$ in P_0 such that $i = x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ where $j_k \in \{1, -1\}$ for $k = 1, 2, \dots, n$. Thus $x_1^{j_1} x_1^{-j_1} = x_1^{j_1} x_1^{-j_1} \cdot i = i$ and hence, for some $x \in P_0$, $i = xx^{-1}$ or $i = x^{-1}x$. Suppose that $i = xx^{-1}$. Let $f: P_0 \rightarrow P_0 \times 1 \subseteq B_P$ be given by $f(t) = (t, 1)$. Then f extends to a homomorphism $\bar{f}: S \rightarrow B_P$. Further $f(S) = B_P$ since $P_0 \times 1$ generates B_P . Hence $\bar{f}(i)$ is an identity for B_P and so $\bar{f}(i) = (1, 1)$. But $\bar{f}(i) = \bar{f}(xx^{-1}) = \bar{f}(x)\bar{f}(x)^{-1} = (x, 1)(1, x) = (x, x)$ and $x \neq 1$. From this contradiction we conclude that $S = F_{P_0}$ does not have an identity. In particular $1 \notin S$. Suppose $x \in F_P \setminus \{1\}$. Then there exist elements $x_1, x_2, \dots, x_n$ of P so that $x = x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ where $j_k \in \{1, -1\}$ for $k = 1, 2, \dots, n$. In fact we may assume that $x_k \in P_0$ for $k = 1, 2, \dots, n$ (this is true for at least one value of k since $x \neq 1$). Thus $x \in S$ and we have shown that $S = F_P \setminus \{1\}$. This completes the proof of this theorem.

An elementary inverse semigroup is defined to be an inverse semigroup generated by a single element. An elementary inverse semigroup may or may not be a one-parameter inverse semigroup depending on whether it has an identity; however we do have the following corollary.

2.3. COROLLARY. Suppose the given subgroup G of R is cyclic. Then F_{P_0} is an elementary inverse semigroup with the property that every elementary inverse semigroup is a homomorphic image of F_{P_0} .

Proof. This follows from 2.2 together with the fact that a homomorphism on the positive integers is determined by its value at 1.

2.4. LEMMA. *If $x \leq y$ then*

- (i) $xy^{-1} = (y/x)^{-1}yy^{-1}$,
- (ii) $y^{-1}x = y^{-1}y(y/x)^{-1}$,
- (iii) $yx^{-1} = (y/x)xx^{-1}$,
- (iv) $x^{-1}y = x^{-1}x(y/x)$.

Proof. To see (i), note that

$$\begin{aligned} xy^{-1} &= x((y/x)x)^{-1} = xx^{-1}(y/x)^{-1} = xx^{-1}(y/x)^{-1}(y/x)(y/x)^{-1} \\ &= (y/x)^{-1}(y/x)xx^{-1}(y/x)^{-1} = (y/x)^{-1}yy^{-1}. \end{aligned}$$

Part (ii) is proved similarly and (iii) and (iv) are trivial.

The next result is, in a sense, an analogue of a theorem of Gluskin [2, Lemma 1.2] and follows immediately from the above lemma.

2.5. LEMMA. *Let $x, y, z \in P$. Then the elements $xy^{-1}z$ and $x^{-1}yz^{-1}$ of F_P can also be written as follows:*

- (i)
$$\begin{aligned} xy^{-1}z &= xz/y && \text{if } y \leq x, z, \\ &= (y/x)^{-1}z && \text{if } x \leq y \leq z, \\ &= x(y/z)^{-1} && \text{if } z \leq y \leq x, \\ &= (y/x)^{-1}y(y/z)^{-1} && \text{if } x, z \leq y. \end{aligned}$$
- (ii)
$$\begin{aligned} x^{-1}yz^{-1} &= (xz/y)^{-1} && \text{if } y \leq x, z, \\ &= x^{-1}(y/z) && \text{if } z \leq y \leq x, \\ &= (y/x)z^{-1} && \text{if } x \leq y \leq z, \\ &= (y/x)y^{-1}(y/z) && \text{if } x, z \leq y. \end{aligned}$$

(iii) *There exist a, b, c in P such that $b \geq a, c$ and $x^{-1}yz^{-1} = ab^{-1}c$.*

Proof. Parts (i) and (ii) follow immediately from Lemma 2.4. Using (ii) we can write $x^{-1}yz^{-1}$ as $ab^{-1}c$ if we choose a, b and c as follows: if $y \leq x, z$ let $a=1$, $b=xz/y$, $c=1$; if $z \leq y \leq x$, let $a=1$, $b=x$, $c=y/z$; if $x \leq y \leq z$, let $a=y/x$, $b=z$, $c=1$; and if $x, z \leq y$, let $a=y/x$, $b=y$, $c=y/z$. In each case $b \geq a, c$, and a, b and c are in P .

2.6. THEOREM. $F_P = PP^{-1}P = P^{-1}PP^{-1}$ and $F_{P_0} = PP_0^{-1}P = P^{-1}P_0P^{-1}$.

Proof. It is an immediate consequence of 2.5(i) that $PP^{-1}P \subset P^{-1}PP^{-1}$. Hence $P^{-1}PP^{-1} = (PP^{-1}P)^{-1} \subset (P^{-1}PP^{-1})^{-1} = PP^{-1}P$ and so $PP^{-1}P = P^{-1}PP^{-1}$. Note also that $(PP^{-1}P)^2 = (PP^{-1}P)(PP^{-1}P) \subset P(P^{-1}PP^{-1})P = P(PP^{-1}P)P \subset PP^{-1}P$. Hence $PP^{-1}P$ is an inverse subsemigroup of F_P . Since $P \subset PP^{-1}P$, we obtain $F_P = PP^{-1}P$. Now suppose $u \in F_{P_0} = F_P \setminus \{1\}$. Then there exist $x, y, z \in P$ such that $u = x^{-1}yz^{-1}$. Now it follows from 2.5(iii) that there exist $a, b, c \in P$ with $b \geq a, c$ so

that $u = x^{-1}yz^{-1} = ab^{-1}c$. However, at least one of a, b, c is not 1, and so $b \neq 1$. This says that $u \in PP_0^{-1}P$. On the other hand, choose $xy^{-1}z$ in $PP_0^{-1}P$. Suppose $1 = xy^{-1}z$. Note that $y \neq 1$. If $x = z = 1$, then $y^{-1} = 1$. So $y = 1$ which is a contradiction. Thus, either $x \neq 1$ or $z \neq 1$. Without loss of generality, suppose $x \neq 1$. Now if $z = 1$, then $1 = xy^{-1} \in F_{P_0}$, which is a contradiction. So $z \neq 1$. Thus none of x, y , or z is 1. Therefore $1 = xy^{-1}z \in F_{P_0}$, another contradiction. Thus $xy^{-1}z \neq 1$; i.e., $xy^{-1}z \in F_{P_0}$. Hence $PP_0^{-1}P = F_{P_0}$.

2.7. THEOREM. *Each element of F_P can be written in one and only one way in the form $xy^{-1}z$ where $x, y, z \in P$ with $x, z \leq y$. Refer to this as the canonical representation of elements of F_P . Then if $u, v \in F_P$ with canonical representations $u = xy^{-1}z$ and $v = rs^{-1}t$, then uv has as its canonical representation*

$$uv = (xZR/y \wedge ZR)(YZRS/(Y \wedge ZR)(ZR \wedge S))^{-1}(ZRT/ZR \wedge S).$$

Proof. Let $u \in F_P$. Then by 2.6 there are elements, $a, b, c \in P$ such that $u = a^{-1}bc^{-1}$. Now using 2.5(iii) we can write $u = xy^{-1}z$ where $x, z \leq y$. To show that the representation is unique, we make use of the semigroup B_P defined earlier. Let $f, g: P \rightarrow B_P$ be the homomorphisms given by $f(x) = (x, 1)$ and $g(x) = (1, x)$. Let $\bar{f}$ and $\bar{g}$ be the extensions of f and g respectively to F_P . Now suppose that $u \in F_P$ has two representations $xy^{-1}z$ and $rs^{-1}t$ where $x, z \leq y$ and $r, t \leq s$. Then $\bar{f}(xy^{-1}z) = f(x)f(y)^{-1}f(z) = (x, 1)(1, y)(z, 1) = (x, y/z)$ and similarly $\bar{f}(rs^{-1}t) = (r, s/t)$, $\bar{g}(xy^{-1}z) = (y/x, z) = \bar{g}(rs^{-1}t) = (s/r, t)$. Hence $r = x$, $s = y$ and $z = t$ and thus the representation is unique.

To establish the rule for multiplication, let $u, v \in F_P$ with representations (not necessarily canonical) $u = xy^{-1}z$ and $v = rs^{-1}t$. It then follows from 3.4(ii) that

$$\begin{aligned} uv &= x(ys/zr)^{-1}t && \text{if } zr \leq s, y, \\ &= xy^{-1}(zrt/s) && \text{if } s \leq zr \leq y, \\ &= (xZR/y)s^{-1}t && \text{if } y \leq zr \leq s, \\ &= (xZR/y)(ZR)^{-1}(ZRT/s) && \text{if } s, y \leq zr. \end{aligned}$$

Now since $y \wedge zr \leq xZR$ and $zr \wedge s \leq zrt$ it follows that $xZR/(y \wedge zr)$, $zrt/(zr \wedge s)$, and $yzrt/((y \wedge zr)(zr \wedge s))$ are all in P . It is a simple matter to check using the four cases above that in fact,

$$uv = (xZR/y \wedge ZR)(YZRS/(Y \wedge ZR)(ZR \wedge S))^{-1}(ZRT/ZR \wedge S).$$

Further, if $xy^{-1}z$ and $rs^{-1}t$ are canonical; i.e. if $x, z \leq y$ and $r, t \leq s$ then it is easily checked that $xZR/y \wedge zr$, $zrt/zr \wedge s \leq yzrs/(y \wedge zr)(zr \wedge s)$ and so the representation for the product above is canonical. This completes the proof.

2.8. COROLLARY. *The elements of $F_{P_0} = F_P \setminus \{1\}$ consist precisely of those elements of F_P whose canonical representation $xy^{-1}z$ is such that $y \neq 1$.*

Proof. Let $u \in F_{P_0}$ and let $xy^{-1}z$ be its canonical representation. If $y = 1$ then $x = z = 1$ and so $u = 1$. Hence $y \neq 1$. Conversely, if $xy^{-1}z \in F_P$ with $x, z \leq y \neq 1$, then $xy^{-1}z \in PP_0^{-1}P = F_{P_0}$, by 2.6. Q.E.D.

Using 2.7 and 2.8 we immediately obtain the following parametrization theorem for F_P and F_{P_0} .

2.9. COROLLARY. Let $T_P = \{(x, y, z) \mid x, y, z \in P \text{ with } x, z \leq y\}$. Define an operation on T_P by

$$(x, y, z)(r, s, t) = (xzt/y \wedge zr, yzrs/(y \wedge zr)(zr \wedge s), zrt/zr \wedge s).$$

Then the map $\phi: F_P \rightarrow T_P$ defined by $\phi(u) = (x, y, z)$ for $u \in F_P$ with canonical representation $u = xy^{-1}z$ is an isomorphism from F_P onto T_P . Further if $T_{P_0} = T_P \setminus \{(1, 1, 1)\}$, then $\phi|_{F_{P_0}}$ is an isomorphism from F_{P_0} onto T_{P_0} .

2.10. REMARK. If T_P is given the subspace topology from the product space $P \times P \times P$, where P is given the subspace topology from R with the usual topology, then it is easily seen that the multiplication and inversion on T_P are continuous; that is, T_P is a topological inverse semigroup. This follows from the fact that multiplication and inversion on R and the $\wedge$ operation on P are all continuous operations. Hence there is a natural topology on F_P making F_P into a topological inverse semigroup. Indeed, F_P is freely generated by P even in the topological sense; that is, any continuous homomorphism from P into a topological inverse semigroup S extends to a unique continuous homomorphism from F_P into S .

The idempotent structure of F_P is determined next.

2.11. LEMMA. Let $u \in F_P$ with canonical representation $u = xy^{-1}z$. Then the canonical representation of u^{-1} is $(y/z)y^{-1}(y/x)$.

Proof. Note $y/z, y/x \in P$. Also note $u^{-1} = z^{-1}yx^{-1}$. Hence by 2.5(ii) $u^{-1} = (y/z)y^{-1}(y/x)$.

For $x \in P$, let $e_x = xx^{-1}$ and $f_x = x^{-1}x$, and let $E = \{e_x \mid x \in P\}$, $F = \{f_x \mid x \in P\}$. Note $E, F \subseteq E_P$, the set of idempotents of F_P .

2.12. THEOREM. Let $u \in F_P$ with canonical representation $xy^{-1}z$. Then $u \in E_P$ if and only if $y = xz$. Furthermore, each element of E can be written in one and only one way in the form $e_x f_x$ for some $x, z \in P$. Thus E_P is the direct sum of the two subsemilattices E and F . Also $e_x f_y \leq e_u f_v$ if and only if $u \leq x$ and $v \leq y$.

Proof. Suppose $u \in E_P$ and $xy^{-1}z$ is the canonical representation of u . Then by 2.9, $u = u^{-1} = (y/z)y^{-1}(y/x)$. Hence $(y/z) = x$, that is, $y = xz$. On the other hand, if $y = xz$ then $xy^{-1}z = (xx^{-1})(z^{-1}z) = e_x f_z \in E_P$. Hence to establish the last statement we need only show the uniqueness of the representation. So suppose $x, z, r, t \in P$ with $xx^{-1}z^{-1}z = e_x f_z = e_r f_t = rr^{-1}t^{-1}t$. Then, using the homomorphisms $\bar{f}$ and $\bar{g}$ of 2.7 we see that $f(xx^{-1}z^{-1}z) = f(x)f(x)^{-1}f(z)^{-1}f(z) = (x, 1)(1, x)(1, z)(z, 1) = (x, x) = \bar{f}(rr^{-1}t^{-1}t) = (r, r)$ and similarly $\bar{g}(xx^{-1}z^{-1}z) = (z, z) = \bar{g}(rr^{-1}t^{-1}t) = (t, t)$. Hence $x = r$ and $z = t$. The last assertion follows easily upon noting that $e_x e_u = e_{x \vee u}$. 2.13 follows immediately from 2.12 and the fact that $F_{P_0} = F_P \setminus \{1\}$.

2.13. COROLLARY. The idempotents of F_{P_0} are precisely those elements of F_P which can be written (uniquely) in the form $e_x f_z$ where $\{x, z\} \cap P_0 \neq \emptyset$.

Next we determine Green's relations (confer with [1]) on F_P .

2.14. THEOREM. Let $u, v \in F_P$ with canonical representations $u = xy^{-1}z$ and $v = rs^{-1}t$. Then

- (i) $u \mathcal{R} v$ if and only if $x=r$ and $y=s$,
- (ii) $u \mathcal{L} v$ if and only if $y=s$ and $z=t$,
- (iii) $u \mathcal{H} v$ if and only if $x=r$, $y=s$ and $z=t$,
- (iv) $u \mathcal{D} v$ if and only if $y=s$.

Proof. (i) We know $u \mathcal{R} v$ if and only if $uu^{-1} = vv^{-1}$. But

$$uu^{-1} = (xy^{-1}z)((y/z)y^{-1}(y/x)) = xy^{-1}(y/x)$$

and similarly $vv^{-1} = rs^{-1}(s/t)$. Hence by 2.7 $uu^{-1} = vv^{-1}$ if and only if $x=r$ and $y=s$.

(ii) Analogous to (i).

(iii) Follows immediately from (i) and (ii).

(iv) Suppose $u \mathcal{D} v$. Then there is an element w of F with $u \mathcal{R} w$ and $w \mathcal{L} v$. Let $ab^{-1}c$ be the canonical representation of w . Then by (i) $y=b$ and by (ii) $b=s$. Hence $y=s$. On the other hand, if $y=s$ let $w = xy^{-1}t$. Then $u \mathcal{R} w$ by (i) and $w \mathcal{L} v$ by (ii). Hence $u \mathcal{D} v$. This completes the proof of 2.12.

From 2.14 we get that there is a $\mathcal{D}$ -class D_y for each element y of D : $D_y = \{xy^{-1}z \mid x, z \in P \text{ with } x, z \leq y\}$. Note also that $E_P \cap D_y = \{e_x f_z \mid xz = y\}$. Hence the $\mathcal{D}$ -class D_y can be pictured as in Figure 1.

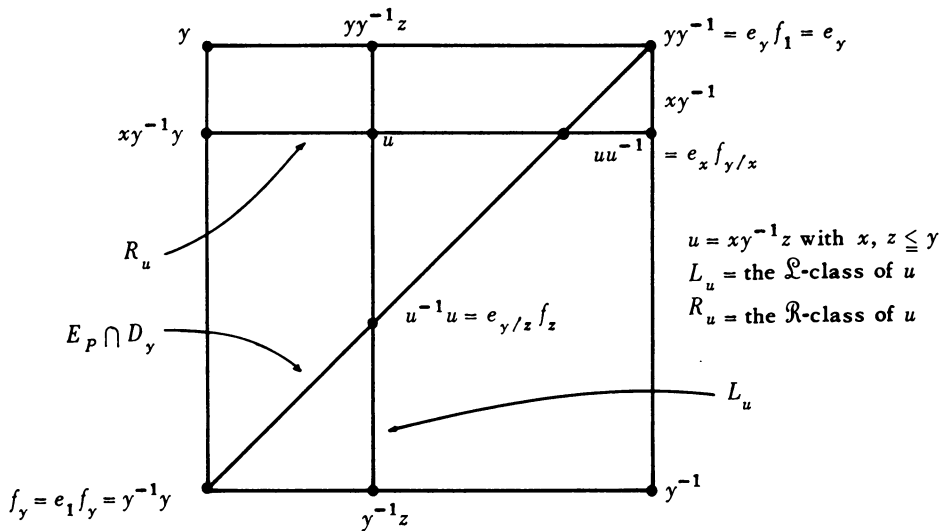


FIGURE 1

It may be helpful to visualize F_P as in Figure 2.

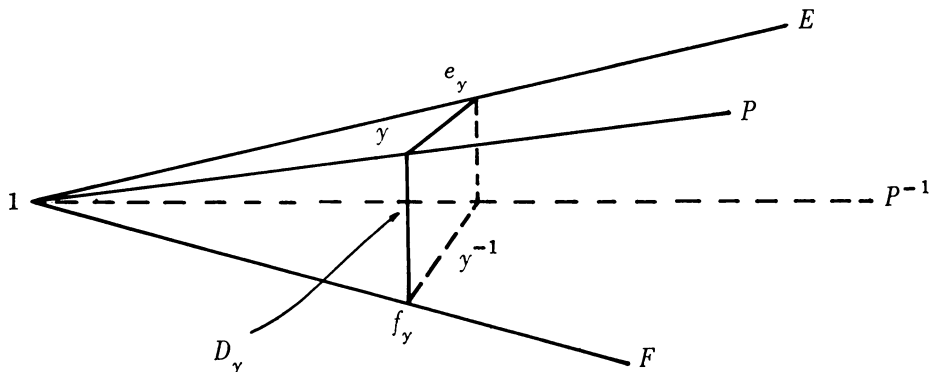


FIGURE 2

Note that the idempotents of F_P lie in a plane which cuts F_P into two pieces.

Next we determine the ideal structure of F_P . For $y \in R$, let

$$I_y = \bigcup \{D_t \mid t \geq y \text{ and } t \in P\}$$

and let

$$I_y^\circ = \bigcup \{D_t \mid t > y \text{ and } t \in P\}.$$

2.15. THEOREM. *For each $y \in P$, I_y and I_y° are ideals of F_P . Conversely, if I is an ideal of F_P , then there is an element $y \geq 1$ of R such that $I = I_y$ or $I = I_y^\circ$. Consequently the ideals of F_P are totally ordered with respect to set inclusion.*

Proof. The fact that I_y and I_y° are ideals of F_P follows readily from the rule for multiplication expressed in 2.7. On the other hand, if I is an ideal of F_P , then let y denote the greatest lower bound of the set of all $t \in P$ such that $D_t \cap I \neq \emptyset$. It is not difficult to show that if $D_t \cap I \neq \emptyset$, then $D_{t_1} \subset I$ for all $t_1 \in P$, and hence $I = I_y$ if $D_y \cap I \neq \emptyset$ or $I = I_y^\circ$ if $D_y \cap I = \emptyset$. Q.E.D.

2.16. REMARK. If we give F_P the natural topology described in 2.10 then the closed ideals are the ones which can be written in the form I_y .

3. The lattice of congruences on F_P . In this section as in the last, G is an arbitrary subgroup of R , the multiplicative group of positive reals, and $P = \{x \in G \mid x \geq 1\}$. We shall describe here the structure of the lattice of congruences on the free one-parameter inverse semigroup F_P , and hence obtain a description of every one-parameter inverse semigroup.

The set $\Lambda(S)$ of congruences on a semigroup S is well known to be a complete lattice with respect to the operations

$$\sigma \wedge \rho = \sigma \cap \rho \quad \text{and} \quad \sigma \vee \rho = \bigcap \{\delta \in \Lambda(S) \mid \cup \rho \sigma \subset \delta\}.$$

The largest (resp. smallest) congruence on S , which is $S^2 = S \times S$ (resp. $\Delta S^2 = \{(x, x) \mid x \in S\}$), is denoted by 1 (resp. 0). The θ relation on $\Lambda(S)$, first defined and studied on regular semigroups S by Reilly and Scheiblich [4] provides a useful aid in visualizing $\Lambda(S)$. The relation is defined by $\sigma \theta \rho$ if and only if $\sigma \cap E^2 =$

$\rho \cap E^2$, where E is the set of idempotents on S . It is shown in [4] that if S is an inverse semigroup, then θ is a lattice congruence on $\Lambda(S)$. The θ -class of 1 is the set of group congruences on S ; the θ -class of 0 is the set of idempotent-separating congruences; in general, each θ -class is a complete lattice of commuting congruences on S .

A congruence ω on E , the idempotents of an inverse semigroup S , is *normal* provided whenever $e \omega f$, then $xex^{-1} \omega xfx^{-1}$ for all $x \in S$. The normal congruences on E are precisely those congruences ω on E such that $\omega = \sigma \cap E^2$ for some $\sigma \in \Lambda(S)$. In fact one sees that $\Lambda(S)/\theta$ is isomorphic with the lattice of normal congruences on E , under the map induced by the map from $\Lambda(S)$ to the normal congruences on E given by $\sigma \rightarrow \sigma \cap E^2$.

As a first step in describing $\Lambda(F_P)$, we shall determine the normal congruences on E_P , the set of idempotents of F_P . Recall 2.12, which says that E_P is the direct sum of $E = \{xx^{-1} \mid x \in P\}$ and $F = \{x^{-1}x \mid x \in P\}$.

3.1. LEMMA. *Let $x, y, t \in P$. Then*

- (i)
$$te_x f_y t^{-1} = \begin{cases} e_{txf_y/t} & \text{if } t \leq y \\ e_{tx} & \text{if } y \leq t \end{cases} = e_{xtf_y/y \wedge t},$$
- (ii)
$$t^{-1}e_x f_y t = \begin{cases} e_{x/lt} f_{ty} & \text{if } t \leq x \\ f_{ty} & \text{if } x \leq t \end{cases} = e_{x/x \wedge t} f_{ty}.$$

Proof. This follows from the rule for multiplication expressed in 2.7.

Let A and B denote the relations on E_P defined by $e_x f_y A e_r f_s$ if and only if $x=r$ and $e_x f_y B e_r f_s$ if and only if $y=s$. These are clearly congruence relations on E_P . Furthermore, it is also clear that $A \vee B = E_P^2$ and $A \wedge B = \Delta E_P^2$. Let I be an ideal of F_P , and let $IA = (A \cap I^2) \cup \Delta E_P^2$, $IB = (B \cap I^2) \cup \Delta E_P^2$, and $IE_P^2 = (E_P^2 \cap I^2) \cup \Delta E_P^2$. We see immediately that IA , IB , and IE_P^2 are all congruences on E_P also.

3.2. THEOREM. *Each of the above congruences on E_P is normal. As a set of normal congruences, they form a lattice with the structure as indicated in the diagram below:*

(I, J ideals of F_P with $J \subseteq I$)

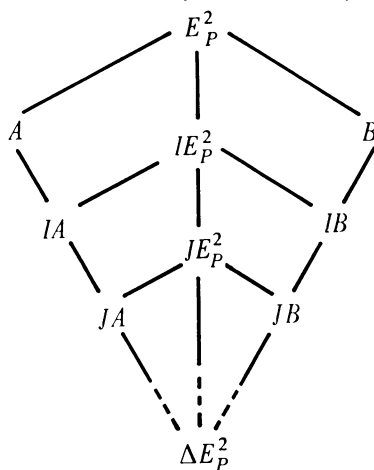


FIGURE 3

Proof. If I is an ideal of F_P and ω is a normal congruence on E_P , then $I\omega = (\omega \cap I^2) \cup \Delta E_P^2$ is clearly a normal congruence on E_P , since it is the intersection of the two normal congruences ω and $(I^2 \cap E_P^2) \cup \Delta E_P^2$. Hence the only assertion requiring proof is that A and B are normal. To see this, let $u = ab^{-1}c \in F_P$ and note that by 3.1

$$ue_x f_y u^{-1} = e_{acx/b \wedge cx} f_{(by/y \wedge c)/(a \wedge (by/(y \wedge c)))}.$$

From this we see that A and B are normal. Q.E.D.

3.3. LEMMA. Suppose ω is a normal congruence on E_P , and suppose $x_0, y_0, t_0 \in P$ with $t_0 \neq 1$. Let I denote the ideal $I_{x_0 y_0} = \bigcup \{D_t \mid t \geq x_0 y_0\}$ of F_P . Then

- (i) if $e_{x_0} f_{y_0} \omega e_{x_0} f_{y_0 t_0}$, then $IA \subseteq \omega$,
- (ii) if $e_{x_0} f_{y_0} \omega e_{x_0 t_0} f_{y_0}$, then $IB \subseteq \omega$.

Proof. (i) Suppose $x, y, t \in P$ with $xy \geq x_0 y_0$. We wish to show that $e_x f_y \omega e_x f_{yt}$. Note that $e_x f_y = x f_{xy} x^{-1}$ and $e_x f_{yt} = x f_{xyt} x^{-1}$; hence the result follows if $f_{xy} \omega f_{xyt}$. To see this, first note that $f_{x_0 y_0} = x_0^{-1} e_{x_0} f_{y_0 x_0} \omega x_0^{-1} e_{x_0} f_{y_0 t_0} x_0 = f_{x_0 y_0 t_0}$. Hence $f_{x_0 y_0 t_0} = t_0^{-1} f_{x_0 y_0} t_0 \omega t_0^{-1} f_{x_0 y_0 t_0} t_0 = f_{x_0 y_0 t_0^2}$, and so $f_{x_0 y_0} \omega f_{x_0 y_0 t_0^n}$. Inductively, we have that $f_{x_0 y_0} \omega f_{x_0 y_0 t_0^n}$ for each positive integer n . Now choose n so large that $x_0 y_0 t_0^n \geq xy t \geq xy$. Then since ω is a congruence on E_P ,

$$f_{xy} = f_{xy} \cdot f_{x_0 y_0} \omega f_{xy} \cdot f_{x_0 y_0 t_0^n} = f_{x_0 y_0 t_0^n}$$

and

$$f_{xyt} = f_{xyt} \cdot f_{x_0 y_0} \omega f_{xyt} \cdot f_{x_0 y_0 t_0^n} = f_{x_0 y_0 t_0^n}.$$

Hence $f_{xy} \omega f_{xyt}$ and the proof of (i) is complete. The proof of (ii) is analogous.

3.4. THEOREM. Let ω be a nonzero normal congruence on E_P . Then there is an ideal I of F_P such that ω is one of the congruences IA , IB , or IE_P^2 . Consequently the lattice shown in 3.2 is the lattice of all normal congruences on E_P .

Proof. Since $\omega \neq \Delta E_P^2$, there exist $x, y, r, s \in P$ with $x \neq r$ or $y \neq s$ such that $e_x f_y \omega e_r f_s$. Suppose $x \neq r$; say $x < r$. Then since $e_x f_{y \vee s} = e_x f_y (f_{y \vee s}) \omega e_r f_s (f_{y \vee s}) = e_r f_{y \vee s}$, we have by 3.3 that $I_{x(s \vee y)} B \subseteq \omega$. Similarly, if $y < s$, then $I_{(x \vee r)y} A \subseteq \omega$. In any event, at least one of the sets $L = \{t \in P : I_t A \subseteq \omega\}$ and $R = \{T \in P : I_T B \subseteq \omega\}$ is nonvoid.

Suppose $R = \emptyset$ and $L \neq \emptyset$. Let $I_L = \bigcup \{I_t : t \in L\}$ and note that $I_L A = \bigcup \{I_t A : t \in L\} \subseteq \omega$. So let $e_x f_y \omega e_r f_s$; $x = r$, otherwise $R \neq \emptyset$. Assume $y < s$. Then $(e_x f_y, e_r f_s) \in I_{xy} A$. But by 3.3, $I_{xy} A \subseteq \omega$ so $xy \in L$; hence $I_{xy} A \subseteq I_L A$. Therefore $\omega = I_L A$. By an analogous argument we conclude that if $L = \emptyset$, then $R \neq \emptyset$, so $I_R B = \omega$ where $I_R = \bigcup \{I_t : t \in R\}$.

If neither L nor R is void, then we claim $L = R$ and $\omega = I_L E_P^2$. To see that $L = R$, let $t \in L$. Choose any $t_0 \in R$. Then $(e_t f_1, e_{t_0} f_{t_0}) \in I_t A \subseteq \omega$ as $t \in L$; also $(e_t f_{t_0}, e_{t_0} f_{t_0}) \in I_{t_0} B \subseteq \omega$ and $(e_{t_0} f_{t_0}, e_{t_0} f_1) \in I_{t_0} A \subseteq \omega$. So $(e_t f_1, e_{t_0} f_1) \in \omega$. By 3.3 we conclude that $I_t B \subseteq \omega$; i.e. $t \in R$. Thus $L \subseteq R$. Similarly $R \subseteq L$. So $L = R$.

Note that $I_L E_P^2 \subseteq \omega$ since $I_L A \subseteq \omega$ and $I_R B \subseteq \omega$, and $I_L A \vee I_L B = I_L E_P^2$. Now suppose $e_x f_y \omega e_r f_s$. If $x=r$ and $y=s$, then $(e_x f_y, e_r f_s) \in \Delta E_P^2 \subseteq I_L E_P^2$. Without loss of generality assume $x \neq r$, say $x > r$. If $y=s$, then $(e_x f_y, e_r f_y) \in \omega$, so $I_{ry} B \subseteq \omega$. Thus $I_{ry} B \subseteq I_L E_P^2$, so $(e_x f_y, e_r f_s) = (e_x f_y, e_r f_y) \in I_L E_P^2$. Similarly for the case $x < r$. A similar argument shows if $x=r$ and $y \neq s$, then $(e_x f_y, e_r f_s) \in I_L E_P^2$. Now if $x \neq r$ and $y \neq s$, w.l.o.g. assume $x > r$. Then $e_x f_y \omega e_x f_s$, and hence $e_x f_s \omega e_r f_s$. By 3.3, this implies $I_{x(y \wedge s)} A \subseteq \omega$ and $I_{rs} B \subseteq \omega$, so $I_x(y \wedge s) A$ and $I_{rs} B \subseteq I_L E_P^2$. Therefore, $e_x f_y (I_L E_P^2) e_x f_s (I_L E_P^2) e_r f_s$, so $(e_x f_y, e_r f_s) \in I_L E_P^2$, and $\omega \subseteq I_L E_P^2$. This completes the proof.

Now that we have determined the lattice of normal congruences on E_P (and hence the lattice $\Lambda(F_P)/\theta$), we concentrate on determining each θ -class of $\Lambda(F_P)$. If ω is a normal congruence on E_P then the θ -class belonging to ω is the set of all congruences $\sigma \in \Lambda(F_P)$ such that $\sigma \cap E_P^2 = \omega$.

Let I be an arbitrary ideal of F_P . In the next three theorems we shall determine the θ -class belonging to IE_P^2 . Let f denote the inclusion map of P into G and let $\tilde{f}$ denote the extension of f to F_P . Note that $\tilde{f}(xy^{-1}z) = xz/y$, and that $\tilde{f}|I$ is onto G .

3.5. THEOREM. *A congruence σ on I is a group congruence if and only if there is a subgroup N of G such that for each $u, v \in I$ (with canonical representations $u = xy^{-1}z$, $v = rs^{-1}t$), $u \sigma v$ if and only if $xzs/rt y \in N$.*

Proof. Let σ be a group congruence on I , and consider the following diagram:

$$\begin{array}{ccc}
 & G & \\
 \tilde{f}|I \uparrow & \text{---} h_\sigma \text{---} & \\
 I & \xrightarrow{\bar{\sigma}} & I/\sigma
 \end{array}
 \qquad h_\sigma \tilde{f} = \bar{\sigma}$$

In order to check that the homomorphism h_σ exists, we note that if $\tilde{f}|I(xy^{-1}z) = \tilde{f}|I(rs^{-1}t)$, then $xzs = rty$. Hence $\bar{\sigma}(xy^{-1}z) = \bar{\sigma}(rs^{-1}t)$. Since $\tilde{f}|I$ is onto, there is a unique homomorphism induced which we call h_σ . Now let $N = \ker h_\sigma$ and note that $xy^{-1}z \sigma rs^{-1}t$ if and only if $\bar{\sigma}(xy^{-1}z) = \bar{\sigma}(rs^{-1}t)$ if and only if $h_\sigma \tilde{f}(xy^{-1}z) = h_\sigma \tilde{f}(rs^{-1}t)$ if and only if $h_\sigma(xz/y) = h_\sigma(rt/s)$ if and only if $xz/y \div rt/s = xzs/rt y \in \ker h_\sigma = N$.

Conversely suppose N is a subgroup of G . Let σ_N be the relation on I defined by $xy^{-1}z \sigma_N rs^{-1}t$ if and only if $xzs/rt y \in N$, where $y \geq x, z$ and $s \geq r, t$ and $xy^{-1}z, rs^{-1}t \in I$. It is readily checked that σ_N is a congruence on I using the fact that N is a group.

To see that σ_N is a group congruence we need only show I/σ_N has only one idempotent. So let e, f be idempotents in I . Then by 2.10, $e = x(xz)^{-1}z$ and $f = r(rt)^{-1}t$ for some x, z, r , and t in P . Since $xz(rt)/rt(xz) = 1 \in N$ we have that $e \sigma_N f$. Thus I/σ_N is a group.

3.6. THEOREM. *The correspondences $\sigma \rightarrow \ker h_\sigma$ and $N \rightarrow \sigma_N$ described in 3.1 between the lattice of group congruences on I and the lattice of subgroups of G are mutually inversive lattice isomorphisms.*

Proof. Let σ be a group congruence on I , and let $\delta = \sigma_{\ker h_\sigma}$. Now as in 3.5 $xy^{-1}z \sigma rs^{-1}t$ if and only if $xzs/rty \in \ker h_\sigma$. But from the definition of δ , $xy^{-1}z \delta rs^{-1}t$ if and only if $xzs/rty \in \ker h_\sigma$. Hence $\sigma_{\ker h_\sigma} = \sigma$. On the other hand, let N be a subgroup of G . Let $u, v \in I$ with canonical representations $u = xy^{-1}z$ and $v = rs^{-1}t$. Now $u \sigma_N v$ if and only if $xzs/rty \in N$. Also using the induced homomorphism h_{σ_N} , $u \sigma_N v$ if and only if $xzs/rty \in \ker h_{\sigma_N}$. Hence $N = \ker h_{\sigma_N}$. Hence the correspondences are mutually inversive functions. To complete the proof we need only show that the correspondence $N \rightarrow \sigma_N$ is a lattice homomorphism.

Let N and M be subgroups of G . It will suffice to show that $N \subseteq M$ if and only if $\sigma_N \subseteq \sigma_M$. Now it is clear that $N \subseteq M$ implies $\sigma_N \subseteq \sigma_M$. Conversely if $\sigma_N \subseteq \sigma_M$ let x be in N with $x = y/z$ such that $y, z \in P$. Then $(1, y, 1) \sigma_N (1, z, 1)$ implies $(1, y, 1) \sigma_M (1, z, 1)$. Thus $x \in M$ and $N \subseteq M$. This completes the proof of 3.6.

3.7. THEOREM. *The θ -class belonging to the normal congruence IE_P^2 is isomorphic with the lattice of subgroups of G under the correspondence $N \rightarrow \sigma_N \cup \Delta F_P^2$.*

Proof. Let Γ denote the θ -class belonging to IE_P^2 , Ω the lattice of subgroups of G , and Δ the lattice of group congruences on I . By 3.6 the function from Ω onto Δ taking N to σ_N is a lattice isomorphism. Hence we only need show that the function from Δ to Γ taking δ to $\delta \cup \Delta F_P^2$ is a 1-1 onto lattice isomorphism.

To see that this function is 1-1 and onto, let $\delta \cup \Delta F_P^2 = \delta'$ for $\delta \in \Delta$ and $\rho \cap I^2 = \rho^*$ for $\rho \in \Gamma$. Clearly $\delta' \in \Gamma$ and $\rho^* \in \Delta$. Also one sees without difficulty that $(\delta')^* = \delta$, for $\delta \in \Delta$. On the other hand if $\rho \in \Gamma$, then to show that $(\rho^*)' = \rho$ we need only show that whenever $u, v \in F_P$ with $u \neq v$ and $u \rho v$ then $u, v \in I$. We consider two cases: (1) If $u \notin I, v \in I$, then $uu^{-1} \notin I$ and $vv^{-1} \in I$. Also $uu^{-1} \rho vv^{-1}$. However this is impossible since $\rho \cap E_P^2 = IE_P^2$. (2) If $u \notin I, v \notin I$, then $uu^{-1}, vv^{-1}, u^{-1}u, v^{-1}v \notin I$; but $uu^{-1} \rho vv^{-1}$, so $uu^{-1} = vv^{-1}$ since $\rho \cap E_P^2 = IE_P^2$. Similarly $u^{-1}u = v^{-1}v$. However this implies that u and v are $\mathcal{H}$ related and so by 2.14 we conclude that $u = v$, a contradiction. This shows that $(\rho^*)' = \rho$. Hence the functions $\delta \rightarrow \delta'$ and $\rho \rightarrow \rho^*$ are mutually inversive functions; and thus $\sigma_N \rightarrow \sigma_N \cup \Delta F_P^2$ is a 1-1 onto function.

To see that it is a lattice isomorphism, let $\delta, \sigma \in \Delta$. Then $\delta \vee \sigma = \delta \circ \sigma$, since $\delta \circ \sigma = \sigma \circ \delta$. Also $\delta' \vee \sigma' = \delta' \circ \sigma'$ according to [4]. So $(\delta \vee \sigma)' = (\delta \circ \sigma) \cup \Delta F_P^2$, and $\delta' \vee \sigma' = (\delta \cup \Delta F_P^2) \circ (\sigma \cup \Delta F_P^2)$. From this it follows that $(\delta \vee \sigma)' = \delta' \vee \sigma'$; hence $\sigma_N \rightarrow \sigma_N \cup \Delta F_P^2$ preserves $\vee$. Since the inverse of this function clearly preserves $\wedge$, we conclude that $\sigma_N \rightarrow \sigma_N \cup \Delta F_P^2$ is a lattice isomorphism.

3.8. COROLLARY. *For each subgroup N of G , let σ^N denote the relation on F_P defined by $u \sigma^N v$ if and only if $u = v$, or $u, v \in I$ and $xzs/rty \in N$, where $xy^{-1}z$ and $rs^{-1}t$ are the canonical representations of u and v respectively. Then σ^N is a member*

of the θ -class belonging to IE_P^2 . Furthermore if M is a subgroup of G then $\sigma^N \vee \sigma^M = \sigma^{NM}$ and $\sigma^N \cap \sigma^M = \sigma^{N \cap M}$.

Now we shall determine the θ -class belonging to IA and IB . It turns out that they are both degenerate. Let $g, h: P \rightarrow B_P$ be the homomorphisms given by $g(x) = (x, 1)$ and $h(x) = (1, x)$. Let $\bar{g}, \bar{h}: F_P \rightarrow B_P$ denote the extensions of g and h , and let α, β be the congruences on F_P determined by $\bar{g}, \bar{h}$ respectively. Note that $u \alpha v (u \beta v)$ if and only if $x=r$ and $yt=sz$ ($z=t$ and $yr=sx$) where $xy^{-1}z$ and $rs^{-1}t$ are the canonical representations of u and v . Let $I\alpha = (\alpha \cap I^2) \cup \Delta F_P^2$ ($I\beta = (\beta \cap I^2) \cup \Delta F_P^2$). It is readily checked that $I\alpha$ ($I\beta$) is a congruence on F_P lying in the θ -class belonging to IA (IB).

3.9. THEOREM. The θ -class belonging to IA (IB) has $I\alpha$ ($I\beta$) as its only member.

Proof. Let Γ denote the θ -class belonging to IA , and let ρ and σ denote the largest and smallest elements of Γ respectively. It follows from Theorem 4.2 of [4] that for $u, v \in F_P$ with canonical representations $xy^{-1}z$ and $rs^{-1}t$ respectively that $u \sigma v$ if and only if $uu^{-1}(IA)vv^{-1}$ and $eu=ev$ for some $e \in E_P$ such that $e IA uu^{-1}$. To prove the theorem we need only show that $u \rho v$ implies $u \sigma v$. So suppose $u \rho v$. Then $u^{-1} \rho v^{-1}$ so $uu^{-1} \rho vv^{-1}$. Thus $e_x f_{y/x} = uu^{-1}(IA)vv^{-1} = e_r f_{s/r}$ and so $x=r$. Also $e_{y/z} f_z = u^{-1}u(IA)v^{-1}v = e_{s/t} f_t$ and so $yt=sz$. Now let $e = e_x f_{s/y}$ and note that $eu=ev$ and $e IA uu^{-1}$. Hence $u \sigma v$, and we conclude that $\sigma = \rho = I\alpha$. The proof that the θ -class belonging to IB contains only $I\beta$ is analogous.

The following corollary sums up the information contained in 3.7 and 3.9. For an arbitrary ideal I of F_P and an arbitrary congruence σ on F_P , let $I\sigma$ denote the congruence $(\sigma \cap I^2) \cup \Delta F_P^2$ on F_P . The *top* of $\Lambda(F_P)$, T , is the set of group congruences on F_P together with the two congruences α and β .

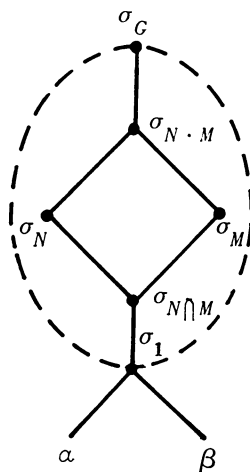


FIGURE 4

3.10. COROLLARY. *Every nonzero congruence σ on F_p can be written uniquely in the form $I\delta$ for some $\delta \in T$ and some ideal I of F_p . Furthermore for ideals I and J of F_p and γ and δ in T , $I\gamma \subset J\delta$ if and only if $I \subset J$ and $\gamma \subset \delta$.*

3.11. REMARK. If we consider F_p with the topology described in 2.10, then it is natural to ask what the closed congruences on F_p are. It is not hard to see that 1 , 0 , α and β are closed. Also the group congruence σ_N is closed if and only if N is cyclic, and if I is an ideal of F_p and $\sigma \in T$ then $I\sigma$ is closed if and only if I is closed and σ is closed.

Several additional pieces of information can be obtained from the preceding theorems. We state them below.

3.12. COROLLARY. *$\Lambda(F_p)$ is a nonmodular lattice.*

3.13. COROLLARY. *All one-parameter inverse semigroups except those of the form F_p have a kernel (i.e. minimal ideal). In particular, if I is an ideal of F_p then $F_p/I\alpha$ and $F_p/I\beta$ have a kernel isomorphic with B_p and $F_p/I\sigma_N$ has a kernel isomorphic with G/N .*

3.14. COROLLARY. *The lattice of congruences on F_{p_0} is isomorphic with the complement of the top of $\Lambda(F_p)$ under the mapping $\sigma \rightarrow \sigma \cup \{(1, 1)\}$.*

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vols. 1, 2, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R. I., 1961, 1967. MR 24 #A2627; MR 36 #1558.
2. L. M. Gluskin, *Elementary generalized groups*, Mat. Sb. **41** (83) (1957), 23–36. (Russian) MR 19, 836.
3. D. B. McAlister, *A homomorphism theorem for semigroups*, J. London Math. Soc. **43** (1968), 355–366. MR 37 #329.
4. N. R. Reilly and H. E. Scheiblich, *Congruences on regular semigroups*, Pacific J. Math. **23** (1967), 349–360. MR 36 #2725.

UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506