

SMOOTH EXTENSIONS IN INFINITE DIMENSIONAL BANACH SPACES

BY
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Abstract. If B is $l_p(\omega)$ or $c_0(\omega)$ we show B has the following extension property. Any homeomorphism from a compact subset M of B into B may be extended to a homeomorphism of B onto B which is a C^∞ diffeomorphism on $B \setminus M$ to its image in B . This is done by writing B as a direct sum of closed subspaces B_1 and B_2 both isomorphically isometric to B so that the natural projection of K into B_1 along B_2 is one-to-one (see H. H. Corson, contribution in *Symposium on infinite dimensional topology*, Ann. of Math. Studies (to appear)). With K , B , B_1 and B_2 as above a homeomorphism of B onto itself is constructed which leaves the B_1 -coordinates of points in B unchanged, carries K into B_1 and is a C^∞ diffeomorphic map on $B \setminus K$. From these results the extension theorem may be proved by standard methods.

1. Introduction, outline and preliminaries. Our main result is the following extension theorem.

THEOREM 4.1. *Let B be $C_0(\omega)$ or $l_p(\omega)$ and let M and N be compact subsets of B . Let f be a homeomorphism mapping M onto N . There is an extension f^* of f to a homeomorphism of B onto itself such that f^* restricted to $B \setminus M$ is a C^∞ diffeomorphism of $B \setminus M$ onto $B \setminus N$.*

We start with the following Splitting Theorem which is an immediate consequence of the results and techniques of Corson [6].

SPLITTING THEOREM (CORSON). *Let B be $C_0(\omega)$ or $l_p(\omega)$. Let K be a σ -compact subset of B . Then B may be written as an internal direct sum $B = B_1 + B_2$ where B_1 and B_2 are both isomorphic to B and where the projection π_1 of B into B_1 along B_2 yields a one-to-one map when restricted to K .*

This theorem will be applied to a compact set K and $\pi_1(K)$ will be denoted K_1 . In this case π_1 yields a homeomorphism of K onto K_1 . Further, there is a continuous mapping g of K_1 into B_1 such that $K = \{k_1 + g(k_1) \mid k_1 \in K_1\}$. By a slight abuse of the language we call $\{k_1 + g(k_1) \mid k_1 \in K_1\}$ the graph of g .

Thus we see that the compact set K is the graph of a continuous function g from some set K_1 which lies in a subspace B_1 of infinite deficiency in B into B_2 a complement of B_1 in B .

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The above observation about compact sets in certain infinite dimensional Banach spaces leads to a proof of the extension theorem by fairly standard methods once we have established the following somewhat technical result on smoothly flattening graphs into subspaces.

THEOREM 3.1. *Let B be the direct internal sum of Banach spaces B_1 and B_2 . Let each of B_1 and B_2 satisfy at least one of the conditions of being separable or admitting C^∞ partitions of unity subordinate to any open cover. Let K_1 be a compact subset of B_1 and let $K = \{k_1 + g(k_1) \mid k_1 \in K_1\}$ where g is a continuous map of K_1 into B_2 . Then there is a homeomorphism d of B onto B such that*

(i) $d(k) = \pi_1(k)$ for all $k \in K$, where π_1 is the natural projection of B into B_1 along B_2 ,

(ii) $\pi_1 \circ d = \pi_1$,

(iii) d restricted to $B \setminus K$ is a C^p diffeomorphism of $B \setminus K$ onto $B \setminus K_1$.

If both spaces allow C^∞ norms or are separable, then p in (iii) may be taken to be ∞ .

If one merely wants d to be a homeomorphism, d may be defined by $d(x_1 + x_2) = x_1 + x_2 - g^*(x_1)$, where g^* is any continuous extension of g to a map from X_1 to X_2 . Such an extension of g is easily constructed. This shows that the topological version of Theorem 3.1 follows immediately from the Splitting Theorem. The difficulties which must be overcome here arise from the requirement of smoothness.

The reader will notice that the constructions presented here may be used in spaces other than $c_0(\omega)$ and $p(\omega)$. For separable spaces one need only check that the spaces admit an appropriate splitting (see Corson [6]). For nonseparable spaces one also needs smooth partitions of unity to carry through the construction of §3. For the constructions of the other sections it is sufficient that the space admit some suitable smooth function (for instance, a continuous norm which is C^p except at the origin). The theory of the existence of smooth partitions of unity (Bonic and Frampton [5]) applies to separable spaces. However, we have stated our results so as to take advantage of any extension of this theory to nonseparable spaces.

The splitting and flattening theorems above were devised to take compact subsets of infinite dimensional Banach spaces into subspaces of infinite deficiency. Once a compact set has been carried into a subspace of infinite deficiency it may be extracted by a refinement of the process used by Bessaga [3]. In this way with some effort and delicacy one may prove that the removal of closed locally compact subsets from a smooth infinite dimensional B -manifold M leaves the differentiable structure of M unchanged for many Banach spaces B (for details, see the author's dissertation [20]).

Several authors have considered or used results on the negligibility of various spacial sorts of subsets of various infinite dimensional manifolds. For the topological case, see Bessaga's notes [4] which survey the field. This author does not know of such a survey in the differentiable case. However, we have the work of Bessaga [3], Eells and Kuiper [11], Martens [16], and Cutler [7] to mention a few.

Results of this sort must be compared with those obtained by a very different and much deeper approach. A series of results starting with Kuiper's paper showing that the general linear group of $l_2(\omega)$ is contractible [13] has led to the characterization up to diffeomorphism of most common infinite dimensional Banach manifolds M by the homotopy type of M . (See papers of Kuiper and Burghlelea [14], Moulis [17], Eells and Elworthy [10] for results on manifolds. See papers of Neubauer [18], Edelstein, Mitjagin and Semenov [9] for papers on contractibility of general linear groups of Banach spaces.) This deep theory together with results known in the topological case leads to the fact that removal of closed σ -compact subsets does not change the C^∞ structure of such manifolds.

The approach to negligibility presented here has the advantage of being more elementary. It may be extended to nonseparable spaces in so far as these spaces admit suitable splittings (one only needs B_1 to be of infinite deficiency in B) and smooth partitions of unity. The needed splitting theorem is not hard to prove for nonseparable spaces. However, the question of partitions of unity in such spaces is still open even when the spaces have smooth norms. The techniques used here do not lead to the stronger classification of manifolds mentioned above.

The following outlines the steps in establishing the extension theorem. In §1 we introduce the notion of smooth approximation and prove a series of lemmas leading to our main lemma on flattening graphs into subspaces (Lemma 2.4). In §3 we show that continuous functions on compact sets in the context of Theorem 3.1 always satisfy the smooth approximability condition of Lemma 2.4. This leads immediately to a proof of Theorem 3.1. In §4 we use the Splitting Theorem of Corson and the remarks about compact sets and graphs of this introductory section to show that Theorem 3.1 may be applied in a symmetrical way to obtain a special result on flattening compact sets in $c_0(\omega)$ or $l_p(\omega)$. From this result the proof of Theorem 4.1 is direct.

The author expresses his debt to Harry Corson who suggested the problem which led to these results. Professor Corson made many helpful strategic and tactical suggestions during this work. In particular, he pointed out that it was possible to obtain extension results from flattening results by the methods used here. The outline of this extension process in the topological case comes from Klee [12]. The author's thesis and its results given here depend upon the general methods and to some extent upon specific results of Klee's work on infinite dimensional topology. It is my pleasure to acknowledge this debt to Professor Klee and his work.

Almost all of the notation used is standard. The symbol $\mathbf{R}$ stands for the real numbers and when N appears for an index set it denotes the positive integers. If a function is p times continuously differentiable we say it is C^p . If a function is C^p for all p in N we say it is C^∞ . We will use Df , f' , or $(d/dx)f$ to denote the derivative of a function f , the notation chosen will be appropriate to the context. The abbreviations cl and int will be used for the topological operations of closure and interior.

2. Smooth approximation and the flattening of graphs. Let X and Y be normed linear spaces and let U be an open subset of X . We say a sequence of functions $\{f_n \mid n \in \mathbf{N}\}$ from X into Y smoothly approximates the function f from X to Y on U provided

- (i) $\|f_n(x) - f(x)\|$ tends to zero uniformly in x as n tends to infinity, and
- (ii) f_n is eventually constant as a function of n locally on U (i.e. for each $u \in U$ there is a neighborhood $U(u)$ of u and an integer $n(u)$ such that $f_k(x) = f_m(x)$ for all $x \in U(u)$ and all integers $k, m \geq n(u)$).

LEMMA 2.1. *Let X and Y be normed linear spaces and let the map f of X into Y be smoothly approximable on U by a sequence of C^p functions. There is a function F from $\mathbf{R} \times X$ into Y such that*

- (i) $F(r, \cdot)$ is eventually constant as a function of r as r tends to 1 locally on U .
- (ii) F is continuous and $F(r, x)$ is C^∞ in r and C^p in x if $r < 1$ or $x \in U$.
- (iii) $F(r, \cdot) = f(\cdot)$ if $r \geq 1$.
- (iv) $\|D_1 F(r, x)\| \leq \frac{1}{2}$ for all r in $\mathbf{R}$ and x in X .

Proof. Let h_n be a sequence of nondecreasing infinitely differentiable functions from $\mathbf{R}$ to $\mathbf{R}$ satisfying

$$\begin{aligned} h_n(x) &= 0 & \text{if } x \leq 1 - 2^{1-n}, \\ h_n(x) &= 1 & \text{if } x \geq 1 - 2^{-n} \end{aligned} \quad \text{for all } n \text{ in } \mathbf{N}.$$

Let a_n be a sequence of real numbers such that $h'_n(x) \leq a_n$ for all x in $\mathbf{R}$ and all n in $\mathbf{N}$. By our hypothesis we may find a sequence f_n of C^p functions smoothly approximating f on U satisfying $\|f_n - f\| \leq \frac{1}{2} a_k^{-2}$ for all $k \leq n+1$ for all n in $\mathbf{N}$. We define F by

$$F = f_1 + \sum h_{n+1} \cdot (f_{n+1} - f_n) \quad (n \in \mathbf{N}).$$

Since the series defining F has only finitely many nonzero terms in some neighborhood $\mathbf{R} \times U(x)$ of a point (r, x) with x in U and since each $h_k(r)$ eventually becomes identically 1 as r tends to 1 property (i) is true. Similarly if $r < 1$, the series defining F consists of only finitely many terms on the neighborhood $(-\infty, \frac{1}{2}(r+1)) \times X$ of (r, x) . Since each term of the series is C^∞ in r and C^p in x , it follows that the function F which is a finite sum in a neighborhood of any point (r, x) such that $r < 1$ or $x \in U$ is C^∞ in r and C^p in x on $\{(r, x) \in (\mathbf{R} \times X) \mid r < 1 \text{ or } x \in U\}$ as required in (ii).

The mean value theorem ensures that $a_k \geq 2^k$. The following estimate shows that the series defining F is absolutely and uniformly convergent on $\mathbf{R} \times X$.

$$\begin{aligned} \sum \|h_{k+1}(r)(f_{k+1}(x) - f_k(x))\| & \\ &\leq \sum |h_{k+1}(r)| \|f_{k+1}(x) - f_k(x)\| & (k \in \mathbf{N}) \\ &\leq \sum 1(\|f_{k+1}(x) - f(x)\| + \|f(x) - f_k(x)\|) & (k \in \mathbf{N}) \\ &\leq \sum (\frac{1}{2} a_{k+1}^{-2} + \frac{1}{2} a_k^{-2}) & (k \in \mathbf{N}) \\ &< \infty & (k \in \mathbf{N}). \end{aligned}$$

Hence F is the sum of an absolutely and uniformly convergent series of continuous functions and therefore F is a continuous function on $\mathbf{R} \times X$. Thus (ii) is established. If $r \geq 1$, then $h_{k+1}(r) = 1$ for all k in $\mathbf{N}$ and the series defining $F(r, \cdot)$ is a telescoping series with limit $f(\cdot)$. Thus (iii) is established.

The following estimate shows that the derived series is absolutely and uniformly convergent; hence, by a standard theorem (Dieudonné [8, 8.6.5]) term by term differentiation is justified and $\|D_1 F(r, x_1)\| \leq \frac{1}{2}$. We have

$$\begin{aligned} \sum \|h'_{n+1}(r)(f_{n+1}(x_1) - f_n(x_1))\| & \\ & \leq \sum |h'_{n+1}(r)|(\|f_{n+1} - f\| + \|f - f_n\|) \quad (n \in \mathbf{N}) \\ & \leq \sum a_{n+1}(\frac{1}{2}a_{n+1}^{-2} + \frac{1}{2}a_n^{-2}) \quad (n \in \mathbf{N}) \\ & \leq \sum a_n^{-1} \quad (n \in \mathbf{N}) \\ & \leq \frac{1}{2} \quad (n \in \mathbf{N}), \end{aligned}$$

since $a_n \geq 2^n$ for n in $\mathbf{N}$. This establishes (iv).

The context and notation of the next lemma are as follows. The normed linear space X is the direct internal sum of subspaces X_1 and X_2 . We assume the natural projections of X into X_1 and X_2 have operator norm less than or equal to 1. There is a continuous function f mapping X_1 to X_2 and $Gf = \{x_1 + f(x_1) \mid x_1 \in X_1\}$ is called the graph of f . The set Gf is a closed subset of $X = X_1 + X_2$ since f is continuous.

LEMMA 2.2. *Let X , X_1 and X_2 be as above. Let $f: X_1 \rightarrow X_2$ be the uniform limit of C^p functions. Assume the norm of X_2 is C^p on $X_2 \setminus \{0\}$. (Alternatively, assume X_2 is separable.) There is a continuous function ϕ from X to $[0, 1]$ which is C^p (alternatively, C^∞) on $Y = X \setminus Gf$ such that $\|D_2 \phi\| \leq \frac{1}{2}$ and $\phi(x) = 1$ exactly when $x \in Gf$.*

Proof. Let a, b, c and d be positive numbers with $a < b$. Let h be a nonincreasing C^∞ function from $\mathbf{R}$ to $\mathbf{R}$ satisfying

$$\begin{aligned} c &\geq h(r) = h(0) > 0 && \text{whenever } r \leq a, \\ h(r) &= 0 && \text{whenever } r \geq b, \\ |h'(r)| &\leq d && \text{for all } r \text{ in } \mathbf{R}. \end{aligned}$$

Let ε be a positive number and let g be a C^p function from X_1 to X_2 such that $\|g(x_1) - f(x_1)\| < \varepsilon$. Then $\psi(x_1 + x_2) = h(\|x_2 - g(x_1)\|)$ is a nonnegative C^p function on X satisfying

$$\begin{aligned} (*) \quad c &\geq \psi(x_1 + x_2) = h(0) > 0 && \text{if } \|x_2 - f(x_1)\| \leq a - \varepsilon, \\ \psi(x_1 + x_2) &= 0 && \text{if } \|x_2 - f(x_1)\| \geq b + \varepsilon, \\ \|D_2 \psi(x_1 + x_2)\| &\leq d && \text{for all } (x_1 + x_2) \in \dot{X}. \end{aligned}$$

The nonnegativity and first two properties of ψ are evident. Off of

$$Gg = \{x_1 + g(x_1) \mid x_1 \in X_1\}$$

the function ψ is a composition of C^p functions, while at each point of Gg the function ψ is locally constant. Thus ψ is at least C^p on X . The bound on the norm of $D_2\psi$ is established by taking D_2 by the usual methods (Lang [15]) and using the fact that the operator norm of the derivative of the norm of any Banach space is less than or equal to 1.

We fix sequences a_n, b_n, c_n and ε_n of positive numbers such that

$a_n < b_n$ and $b_{n+1} + \varepsilon_{n+1} < a_n - \varepsilon_n$ for all n in N ,

b_n tends to zero as n tends to infinity,

$1 \geq \sum c_n$ ($n \in N$), and

$\frac{1}{2} \geq \sum d_n$ ($n \in N$),

(e.g., $a_n = 2^{-2n}$, $b_n = 2a_n$, $c_n = 2^{-4n}$, $d_n = \frac{1}{2}c_n$, and $\varepsilon_n = 2^{-4(n+1)}$).

We take a sequence ψ_n of nonnegative C^p functions on X satisfying conditions (*) with $a = a_n$, $b = b_n$, $c = c_n$, $d = d_n$ and $\varepsilon = \varepsilon_n$. We set

$$\psi^*(x) = \sum \psi_n(x) \quad (n \in N).$$

The conditions ensure that ψ^* is an absolutely and uniformly convergent sum of continuous functions and hence continuous.

Since f is the uniform limit of C^p functions, f is continuous. If $(x_1 + x_2) \in Y$, then $x_2 \neq f(x_1)$ and for some n $\|x_2 - f(x_1)\| > b_n + \varepsilon_n$. Since this inequality holds at $x_1 + x_2$ and since the functions involved are continuous, it holds in a neighborhood of $x_1 + x_2$. But this inequality implies ψ_k vanishes for $k \geq n$. Hence ψ^* is locally a finite sum of C^p functions and thus C^p on Y .

Each point $x_1 + x_2$ of Gf satisfies $0 = \|x_2 - f(x_1)\| < a_n - \varepsilon_n$ for all n in N . Thus ψ^* equals the constant

$$d^* = \sum \psi_n(x_1 + f(x_1)) \quad (n \in N)$$

on Gf . Clearly, $d^* > \psi^*(x_1 + x_2)$ if $(x_1 + x_2) \in Y$, since $\psi_n(x_1 + f(x_1))$ is positive and $\psi_n(x_1 + x_2) \leq \psi_n(x_1 + f(x_1))$ for all n in N while only finitely many of the terms $\psi_n(x_1 + x_2)$ are nonzero. Thus ψ^* attains its supremum d^* precisely on the set Gf .

The derived series for $D_2\psi^*$ is absolutely and uniformly convergent in view of the bounds on $\|D_2\psi_n\|$. Thus, term by term differentiation is justified and $\|D_2\psi^*\| \leq \frac{1}{2}$. The function ψ^* has most of the properties we require. To obtain a function having all the desired properties, it is sufficient to set $\phi = \psi^* + (1 - d^*)$. This insures that the supremum of ϕ which is obtained only on Gf is equal to 1. It is clear that ϕ is as differentiable as ψ^* and that $\|D_2\phi\| = \|D_2\psi^*\| \leq \frac{1}{2}$. Thus ϕ has all the properties required in the lemma.

In this construction we may use any C^p function $N(\cdot)$ on X_2 in place of the norm $\|\cdot\|$ of X_2 provided N is a norm and $N(x) \leq \|x\|$ for all x in X_2 . To establish the alternative reading it suffices to show that any separable Banach space Y admits a possibly weaker norm $N(\cdot)$ with the above properties. To do this let Y be a separable Banach space and let $\{y_n \mid n \in N\}$ be a sequence of points of norm 1 in Y whose range is dense in the unit sphere of Y . We may construct a sequence of

linear functionals $\{f_n \mid n \in N\}$ on Y each of norm 1 such that $f_n(y_n) = 1$ for $n \in N$. It is easily seen that

$$N(y) = \left(\sum_{n=1}^{\infty} 2^{-n} (f_n(y))^2 \right)^{1/2}$$

is a C^∞ norm on Y with the desired properties. This shows that every separable space admits a smaller and possibly weaker C^∞ norm and thus establishes the alternative reading of the lemma. An alternative approach is to use the fact that any separable Banach space is isometrically isomorphic with a closed subspace of $C[0, 1]$ which admits a C^∞ norm of the desired sort, namely the $\mathcal{L}_2$ norm.

We need the following version of Banach's contraction principle. We omit the proof which is simply an exercise in the properties of contraction maps.

LEMMA 2.3. *Let X_1 be a normed linear space and X_2 be a Banach space. Let $X = X_1 + X_2$ and let h be a continuous function from X to X_2 such that*

$$\|h(x_1 + x_2) - h(x_1 + x'_2)\| \leq \frac{1}{2} \|x_2 - x'_2\|$$

for all $x_1 \in X_1$ and $x_2, x'_2 \in X_2$. Let d be a mapping defined by $d(x_1 + x_2) = x_1 + x_2 + h(x_1 + x_2)$. Then d is a homeomorphism of X onto itself, and d is a C^p diffeomorphism when restricted to any open set on which h is C^p .

These lemmas lead to a proof of our main lemma.

LEMMA 2.4. *Let X_1 be a normed linear space and X_2 a Banach space with C^p norm. (Alternatively, let X_2 be a separable Banach space.) Let $f: X_1 \rightarrow X_2$ be smoothly approximable by C^p functions on U . Let Gf be the graph of f . Let $K_1 = X_1 \setminus U$ and let $K = \{k_1 + g(k_1) \mid k_1 \in K_1\}$. Then there is a homeomorphism d of $X_1 + X_2 = X$ onto itself such that*

$$(i) \quad d(Gf) = \pi_1(Gf),$$

$$(ii) \quad d \circ \pi_1 = \pi_1,$$

$$(iii) \quad d \text{ restricted to } X \setminus K \text{ is a } C^p \text{ diffeomorphism of } X \setminus K \text{ onto } X \setminus K_1.$$

(Alternatively,

$$(iii) \quad d \text{ restricted to } X \setminus K \text{ is a } C^\infty \text{ diffeomorphism of } X \setminus K \text{ onto } X \setminus K_1.)$$

Proof. By the conditions of the theorem there exist functions F and ϕ satisfying the conditions in Lemmas 2.1 and 2.2 with respect to X_1 , X_2 , and f . We define $h(x_1 + x_2)$ for $(x_1 + x_2) \in X$ by

$$h(x_1 + x_2) = F(\phi(x_1 + x_2), x_1).$$

The function h defined above is the composition of continuous functions and thus continuous. If $(x_1 + x_2) \notin Gf$ then ϕ is C^p at $x_1 + x_2$ by the conditions of Lemma 2.2, and F is C^p at $(\phi(x_1 + x_2), x_1)$ because $\phi(x_1 + x_2) < 1$. Thus, h is C^p on $X \setminus Gf$. If $(a_1 + a_2) \in (Gf \setminus K)$, then $a_1 \in U$. Thus, as in Lemma 2.1, there is a neighborhood $U(a_1)$ of a_1 and an $r(a_1) < 1$ such that $F(r, x_1) = f(x_1)$ if $r > r(a_1)$ and $x_1 \in U(a_1)$.

Now $(a_1 + a_2) \in Gf$ so $\phi(a_1 + a_2) = 1$. Since ϕ is continuous, there is a neighborhood V of $a_1 + a_2$ such that $\phi(x_1 + x_2) > r(a_1)$ and $x_1 \in U(a_1)$ for all $(x_1 + x_2) \in V$. For $(x_1 + x_2) \in V$,

$$F(\phi(x_1 + x_2), x_1) = F(\tfrac{1}{2}(r(a_1) + 1), x_1) = f(x_1).$$

Thus h is C^p on V since $\tfrac{1}{2}(r(a_1) + 1)$ is a C^∞ function less than 1 on V and $F(r, x_1)$ is a C^p function of x_1 if $r < 1$. This shows h is a C^p function on $(X \setminus Gf) \cup (Gf \setminus K) = X \setminus K$. It also shows that if $(x_1 + x_2) \in (Gf \setminus K)$ and V is as above, then $D_2 h = 0$ on V since h does not depend on the second coordinate within V .

Computing $D_2 h$ we have

$$D_2 h(x_1 + x_2) = D_1 F(\phi(x_1 + x_2), x_1) \circ D_2 \phi(x_1 + x_2).$$

The bounds on the norms of $D_1 F$ and $D_2 \phi$ give us $\|D_2 h(x_1 + x_2)\| \leq \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$ when $(x_1 + x_2) \notin Gf$. Since h is continuous and $X \setminus Gf$ is dense in X , we may use the mean value theorem (Corollary 1, Proposition 17, Chapter 1, §4 of Lang [15]) to show that the inequality

$$\|h(x_1 + x_2) - h(x_1 + x'_2)\| \leq \tfrac{1}{4} \|x_2 - x'_2\|$$

holds for all $x_2, x'_2 \in X_2$ for all $x_1 \in X_1$. Thus Lemma 2.3 applies to the function $d(x_1 + x_2) = x_1 + x_2 - h(x_1 + x_2)$ defined for all $(x_1 + x_2) \in X$. It is clear from the definition of d that (ii) holds.

Applying Lemma 2.3 to the function d we see that d is a homeomorphism of X onto itself. Since d is C^p on $X \setminus K$ it is a C^p diffeomorphism of $X \setminus K$ onto $d(X \setminus K_1)$.

If $(x_1 + x_2) \in Gf$, then $h(x_1 + x_2) = F(1, x_1) = f(x_1) = x_2$ and

$$d(x_1 + x_2) = x_1 + x_2 - h(x_1 + x_2) = x_1 + x_2 - x_2 = x_1.$$

This establishes (i). In view of the above paragraph, to complete the proof it is sufficient to show that $d(X \setminus K) = X \setminus K_1$, or equivalently $d(K) = K_1$. But this follows from (i) and the definitions of K and K_1 .

Lemma 2.4 outlines conditions under which certain kinds of subsets K of $X = X_1 + X_2$ may be carried smoothly into X_1 . We will see that the conditions of this lemma are easily satisfied when the subset K is compact and X is an infinite dimensional Banach space. There are two requirements for applying Lemma 2.4. First, the set K must be the graph of a continuous function from X_1 to X_2 and second, that function must be smoothly approximable on $U = X_1 \setminus \pi_1(K)$. The Splitting Theorem shows circumstances under which compact sets may be regarded as graphs of continuous functions. We now take up the problem of smooth approximation.

3. Smooth approximation of functions. Let B be a Banach space which is the internal direct sum of subspaces B_1 and B_2 . Let K_1 be a subset of B_1 and let g be a continuous function from K_1 to B_2 . Let $K = \{k_1 + g(k_1) \mid k_1 \in K_1\}$. When may g be extended to a function f mapping B_1 into B_2 so that F, B_1, B_2 , and K_1 satisfy the hypotheses of Lemma 2.4?

We show that it is sufficient to assume K (and hence K_1) is compact and that each of B_1 and B_2 is either separable or admits C^p partitions of unity subordinate to any open cover. On B_2 either of these assumptions is sufficient to apply Lemma 2.4. We show below that either of these assumptions on B_1 will insure that g has an extension f satisfying the conditions above.

Let B_1 admit C^p partitions of unity. We will construct a series of C^p functions f_n whose partial sums approximate g on K_1 and smoothly approximate $f = \sum_{n=1}^{\infty} f_n$ on $B_1 \setminus K_1$. Thus f is an extension of g of the desired sort.

Since K_1 is compact and g is continuous, g is uniformly continuous on K_1 . Let $\mathcal{V}_1$ be a finite cover of K_1 by open sets in B_1 such that $\|f(k) - f(k')\| \leq 1$ for all $k, k' \in V \cap K$ for all $V \in \mathcal{V}_1$. We further assume that the diameter of each $V \in \mathcal{V}_1$ is less than or equal to 1. For each $V \in \mathcal{V}_1$ select a $k_1(V) \in V$. Let $\{\phi_V \mid V \in \mathcal{V}_1\} \cup \{\phi_U\}$ be a C^p partition of unity subordinate to the cover $\mathcal{V}_1 \cup \{U\}$ of B_1 . Define f_1 by

$$f_1 = \sum g(k_1(V))\phi_V \quad (V \in \mathcal{V}_1).$$

Then

$$\begin{aligned} \|g(k) - f_1(k)\| &= \left\| \sum_{V \in \mathcal{V}_1} g(k_1(V))\phi_V(k) - \sum_{V \in \mathcal{V}_1} f(k)\phi_V(k) \right\| \\ &\leq \sum_{V \in \mathcal{V}_1} \|g(k_1(V)) - f(k)\| |\phi_V(k)| \\ &\leq \sum_{V \in \mathcal{V}_1} 1 \cdot \phi_V(k) \leq 1 = 2^0 \end{aligned}$$

for every $k \in K$. This defines a C^p function f_1 which approximates g on K to within 1 and vanishes off of the neighborhood $\bigcup \mathcal{V}_1$ of K . By the same process, we may produce a C^p function f_2 which approximates $g - f_1$ on K to within $\frac{1}{2}$ on K and which vanishes off of $\bigcup \mathcal{V}_2$, where $\mathcal{V}_2$ is an open cover of K in B_1 none of whose sets have diameter greater than $\frac{1}{2}$. In the construction we will have

$$\begin{aligned} \|f_2(x)\| &= \left\| \sum_{V \in \mathcal{V}_2} (g(k_2(V)) - f_1(k_2(V)))\phi_V(x) \right\| \\ &\leq \sum_{V \in \mathcal{V}_2} \|g(k_2(V)) - f_1(k_2(V))\| \phi_V(x) \leq \sum_{V \in \mathcal{V}_2} 1 \phi_V(x) \leq 1. \end{aligned}$$

Continuing in this way the sequence f_n is constructed so that

- (i) $\|g(k) - \sum_{k=1}^n f_n(k)\| \leq 2^{1-n}$ for all $k \in K$ and $n = 1, 2, 3, \dots$,
- (ii) $\|f_n(b_1)\| \leq 2^{1-n}$ for $b_1 \in B_1$ and $n = 2, 3, 4, \dots$,
- (iii) f_n vanishes off of the set $V_n = \bigcup \mathcal{V}_n$ and $\bigcap_{n=1}^{\infty} V_n = K$.

The final condition may be assured by insisting that the diameters of the $V_n \in \mathcal{V}_n$ be less than 2^{1-n} . These conditions clearly ensure that $\sum f_n$ ($n \in \mathbb{N}$) is uniformly convergent on B_1 and that $\sum_{i=1}^n f_i$ approximates $\sum_{i=1}^{\infty} f_i$ smoothly by C^p functions on $B_1 \setminus K$. Furthermore, it is clear that $g(k) = f(k) = \sum_{i=1}^{\infty} f_i(k)$ for all $k \in K$. This completes the case where B_1 admits C^p partitions of unity.

In the case where B_1 is separable we proceed similarly using weak open sets in place of strong open sets. This technique is suggested by the methods of Anderson [1]. Since K is compact the weak and strong topologies on K agree. Since $B_1 \setminus K$ is Lindelöf in the strong topology and the identity map from the strong to weak topology is continuous, U is Lindelöf in the weak topology. Since the weak topology is regular, K is a weak G_δ . In fact, we see that there is a family of weak open sets $\{U_n \mid n \in \mathbb{N}\}$ such that $K = \bigcap U_n$ ($n \in \mathbb{N}$). To construct our sequence $\{f_n \mid n \in \mathbb{N}\}$ we take a weak cover $\mathcal{V}_1^*$ of K by a finite family of weakly open subsets of B_1 such that $\|g(k) - g(k')\| < 1$ for all $k, k' \in V^* \cap K$ where $V^* \in \mathcal{V}_1^*$. With $\{U_n \mid n \in \mathbb{N}\}$ as above, set $\mathcal{V}_1 = \{U_1 \cap V^* \mid V^* \in \mathcal{V}_1^*\}$. Only a finite number of linear functionals are required to define the weak open sets of $\mathcal{V}_1$; thus B_1 may be mapped into a finite dimensional space E_1 by a linear mapping h so that each $V \in \mathcal{V}_1$ is the inverse image of an open set, $h(V)$, in E_1 . Take a C^∞ partition of unity subordinate to the cover $\{g(V) \mid V \in \mathcal{V}_1\} \cup (E_1 \setminus h(K))$ of E_1 . This C^∞ partition of unity on E_1 induces a C^∞ partition of unity on B_1 subordinate to $\mathcal{V}_1 \cup h^{-1}(E_1 \setminus h(K))$. We proceed as in the construction of f_1 . The construction proceeds by induction as before, the only difference being that instead of insisting that the elements of $\mathcal{V}_n$ be of diameter less than 2^{1-n} , we insist that the elements of $\mathcal{V}_n$ be contained in U_n . This assures that condition (iii) is fulfilled since $K = \bigcap U_n$ ($n \in \mathbb{N}$). The above prepares the way for a proof of the following.

THEOREM 3.1. *Let B be the direct internal sum of Banach spaces B_1 and B_2 . Let each of B_1 and B_2 satisfy at least one of the conditions of being separable or admitting C^p partitions of unity subordinate to any open cover. Let K_1 be a compact subset of B_1 and let $K = \{k_1 + g(k_1) \mid k_1 \in K_1\}$ where g is a continuous map of K_1 into B_2 . Then there is a homeomorphism d of B onto B such that*

- (i) $d(k) = \pi_1(k)$ for all $k \in K$, where π_1 is the natural projection of B into B_1 along B_2 ,
- (ii) $\pi_1 \circ d = \pi_1$,
- (iii) d restricted to $B \setminus K$ is a C^p diffeomorphism of $B \setminus K$ onto $B \setminus K_1$.

If both spaces allow C^∞ norms or are separable, then p in (iii) may be taken to be ∞ .

Proof. Let K_1, g, B, B_1 and B_2 be as in the hypothesis of Theorem 3.1. By previous arguments we know that g may be extended to a function f mapping B_1 into B_2 such that f is smoothly approximable by at least C^p functions on $U = B_1 \setminus K_1$. Applying Lemma 2.5 we obtain the mapping d which satisfies the conclusions of Theorem 3.1.

4. Smooth extensions. We may now easily prove the following result on extensions.

THEOREM 4.1. *Let B be $C_0(\omega)$ or $l_p(\omega)$ and let M and N be compact subsets of B . Let f be a homeomorphism mapping M onto N . There is an extension f^* of f to a homeomorphism of B onto itself such that f^* restricted to $B \setminus M$ is a C^∞ diffeomorphism of $B \setminus M$ onto $B \setminus N$.*

Proof. First we consider a special case. Let B be a separable Banach space written as the direct internal sum of subspaces B_1 and B_2 . Let M be a compact subset of B_1 and N a compact subset of B_2 . Let g be a homeomorphism from M to N . We show g may be extended to a homeomorphism g^* of B onto B such that g^* restricted to $B \setminus M$ is a C^∞ diffeomorphism of $B \setminus M$ onto $B \setminus N$.

To construct g^* we apply Theorem 3.1 with B_1 and B_2 as above and with $K = \{m + g(m) \mid m \in M\}$ and $K_1 = \pi_1(K) = M$. The function g of Theorem 3.1 is as above. By Theorem 3.1 there is a homeomorphism d_1 of B onto itself satisfying

(i) $d_1(k) = d_1(\pi_1(k)) = g(\pi_1(k)) = \pi_1(k)$ for all $k \in K$ or, equivalently, $d_1(m + g(m)) = m$ for all $m \in M$,

(ii) $\pi_1 \circ d_1 = \pi_1$,

(iii) d_1 restricted to $B \setminus K = \{m + g(m) \mid m \in M\}$ is a C^∞ diffeomorphism of $B \setminus K$ onto $B \setminus K_1 = B \setminus M$.

Interchanging the roles of B_1 and B_2 and thinking of K as the graph of g^{-1} , $K = \{n + g^{-1}(n) \mid n \in N\} = \{m + g(m) \mid m \in M\}$, we apply Theorem 3.1 again to obtain a homeomorphism d_2 of B onto itself satisfying

(i') $d_2(n + g^{-1}(n)) = n$ for all $n \in N$,

(ii') $\pi_2 \circ d_2 = \pi_2$,

(iii') d_2 restricted to $B \setminus K$ is a C^∞ diffeomorphism of $B \setminus K$ onto $B \setminus N$.

A calculation shows that $d_2 \circ (d_1^{-1})(m) = d_2(m + g(m)) = g(m)$ for all $m \in M$. Thus $g^* = d_2 \circ (d_1^{-1})$ is an extension of g to a homeomorphism of B onto B . Conditions (iii) and (iii') show that g^* restricted to $B \setminus M$ is a C^∞ diffeomorphism of $B \setminus M$ onto $B \setminus N$.

To complete the proof of Theorem 4.1 we reduce the general extension problem to the special case considered above by using the hypothesis that B is $C_0(\omega)$ or $l_p(\omega)$. Let f , M , N and B be as in the hypotheses of Theorem 4.1. By the Splitting Theorem we write B as the direct sum of isomorphic subspaces B_1 and B_2 so that π_1 determines a homeomorphism of $K = M \cup N$ onto $K_1 = \pi_1(K)$. Then there is a continuous function g mapping K_1 into B_2 such that $K = \{k_1 + g(k_1) \mid k_1 \in K_1\}$. Since B is separable so are B_1 and B_2 . Theorem 3.1 in its alternative reading applies. Let d be a homeomorphism of B onto itself satisfying the conclusions of the alternative reading of Theorem 3.1.

Since B_1 and B_2 are isomorphic we can and will let h denote a linear isometry of B_1 onto B_2 . The result established above now applies to the homeomorphism $g = h \circ d \circ f \circ (d^{-1})$ of $d(M)$ onto $h(d(N))$. Thus we may extend g to g^* as above. Calculation shows that $f^* = (d^{-1}) \circ (h^{-1}) \circ g^* \circ d$ is an extension of f which has the properties required in the conclusion of Theorem 4.1. Thus Theorem 4.1 is proved.

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