

THE VARIATION OF SINGULAR CYCLES IN AN ALGEBRAIC FAMILY OF MORPHISMS

BY

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Abstract. (1) Let $g: V^r \rightarrow W^m$ ($m \geq r$) be a morphism of nonsingular varieties over an algebraically closed field. Under certain conditions, one can define a cycle S_i on V with $\text{Supp}(S_i) = \{x \mid \dim_{k(x)}(\Omega_{X/Y}^1)(x) \geq i\}$.

The multiplicity of a component of S_i can be computed directly from local equations for g . If $V^r \subset \mathbf{P}^n$, and if $g: V \rightarrow \mathbf{P}^m$ is induced by projection from a suitable linear subspace of $\mathbf{P}^n$, then S_1 is $c_{m-r+1}(N \otimes \mathcal{O}(-1))$, up to rational equivalence, where N is the normal bundle of V in $\mathbf{P}^n$.

(2) Let $f: X \rightarrow S$ be a smooth projective morphism of noetherian schemes, where S is connected, and the fibres of f are absolutely irreducible r -dimensional varieties. For a geometric point $\eta: \text{Spec}(k) \rightarrow S$, and a locally free sheaf E on X , let X_η be the corresponding geometric fibre, and E_η the sheaf induced on X_η . If $E_1, \dots, E_m$ are locally free sheaves on X , and if $i_1 + \dots + i_m = r$, then the degree of the zero-cycle $c_{i_1}(E_{1\eta}) \cdots c_{i_m}(E_{m\eta})$ is independent of the choice of η .

(3) The results of (1) and (2) are used to study the behavior under specialization of a closed subvariety $V' \subset \mathbf{P}^{2r-1}$ which is the image under generic projection of a nonsingular $V^r \subset \mathbf{P}^n$.

1. Introduction. Let V^r be a nonsingular r -dimensional projective variety over an algebraically closed field k . If V is projected generically onto $V' \subset \mathbf{P}^{2r-1}$, then V' has a singular curve with finitely many points of a type known as *pinch points* (cf. §5). Suppose that $\text{char}(k) = 0$ and that V can be specialized (along with its projective embedding) to a nonsingular variety V_1 , defined over k_1 , which can be projected generically onto $V'_1 \subset \mathbf{P}^{2r-1}$. We can ask whether V'_1 has the same number of pinch points as V' . The answer is "yes" if $\text{char}(k_1) \neq 2$; if $\text{char}(k_1) = 2$, then V'_1 has half as many pinch points as V' .

In this paper we develop some techniques which enable one to answer this and other enumerative questions of a similar nature. In §2, we prove a result which says roughly that a Chern polynomial of weight r is constant in a connected family of nonsingular projective varieties of dimension r (cf. Theorem 1). In §3 we recall the definition of the dependency cycle of a set of sections of a locally free sheaf on V . Mattuck [6] has shown how to express the rational equivalence class of this cycle as a Chern class of E . We express the multiplicities of its components as the

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lengths of certain Fitting ideals (cf. Proposition 4). In the case that the set of sections is part of a Serre sequence, these results follow from Theorem 2.7 of [3]. In §4, we define, under suitable assumptions, the singular cycles S_i of a morphism $f: V^r \rightarrow W^m$ of nonsingular varieties ($m \geq r$). Intuitively, $\text{Supp}(S_i)$ is the set of points where the kernel of the tangent map has dimension $\geq i$. Our definition is stated in a form which gives immediately the multiplicities of the components of S_i . Let $V \subset \mathbf{P}^n$, and let $\pi: V \rightarrow \mathbf{P}^m$ be induced by generic projection. (See Lemma 3 for the meaning of "generic" in this context.) Theorem 2 says that the rational equivalence class of $S_1(\pi)$ is $c_{m-r+1}(N \otimes \mathcal{O}(-1))$, where N is the normal bundle of V in $\mathbf{P}^n$. Finally, §5 gives the application of the results of §§2, 3, and 4 to the problem stated in the first paragraph. We also give a concrete example to illustrate our result.

We will deal with Chern classes constructed in the rational equivalence ring $\mathcal{A}(V)$. Our references for this topic are Grothendieck's appendix to the Borel-Serre paper [4], and Séminaire Chevalley 1958, "Anneaux de Chow et applications." As usual, Grothendieck's *Eléments de géométrie algébrique* is denoted EGA.

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2. Chern classes and algebraic families. Let $f: X \rightarrow S$ be a smooth projective morphism of noetherian schemes. Assume that S is connected and that all fibres of f are absolutely connected. In particular, f is flat, and the fibre $X_s = f^{-1}(s)$ is an absolutely nonsingular irreducible projective variety over $k(s)$ for all $s \in S$, where $k(s)$ is the residue field of $\mathcal{O}_{S,s}$.

Let $E_1, \dots, E_m$ be locally free sheaves on X . For each geometric point $\eta: \text{Spec}(k) \rightarrow S$ (k is an algebraically closed field), let X_η be the corresponding geometric fibre, and let $E_{1\eta}, \dots, E_{m\eta}$ be the sheaves induced on X_η by $E_1, \dots, E_m$. We will consider the Chern classes $c_i(E_{j\eta})$, which are elements of $\mathcal{A}(X_\eta)$, the Chow ring of X_η . For a rational equivalence class, z , of zero-cycles on X_η , we will denote by $\deg_\eta(z)$ the degree of z . Finally, we note that the dimension of X_η is independent of η .

THEOREM 1. *Let $f: X \rightarrow S$ and $E_1, \dots, E_m$ be as above. If $i_1, \dots, i_m$ is a sequence of positive integers such that $i_1 + \dots + i_m = \dim(X_\eta)$, and $i_j \leq \text{rk}(E_j)$, for $j = 1, \dots, m$, then the value of*

$$\deg_\eta(c_{i_1}(E_{1\eta}) \cdots c_{i_m}(E_{m\eta}))$$

is independent of the choice of the geometric point, η , of S .

REMARK. The E_i need not be distinct.

Proof. We first consider the case where the E_j are all invertible, so that $c_i(E_{j\eta}) = 0$ for $i > 1$. Thus let $r = \dim(X_\eta)$, and let $L_1, \dots, L_r$ be invertible sheaves on X (not necessarily distinct). Then

$$\deg(c_1(L_{1\eta}) \cdots c_1(L_{r\eta})) = (L_{1\eta} \cdots L_{r\eta}),$$

which is the intersection number, computed on X_η . Using the techniques of [7, Lecture 12], one shows

$$(L_{1\eta} \cdots L_{r\eta}) = \chi(\mathcal{O}_{X_\eta}) - \sum_{i=1}^r \chi(L_{i\eta}^{-1}) + \sum_{i < j} \chi(L_{i\eta}^{-1} \otimes L_{j\eta}^{-1}) - \cdots + (-1)^r \chi(L_{1\eta}^{-1} \otimes \cdots \otimes L_{r\eta}^{-1}).$$

(By induction on the dimension, one shows that our expression is correct when $L_1 \cong \mathcal{O}_V(D)$, with D very ample, and is linear in each variable.)

The fact that this intersection number is independent of η is a consequence of the following lemma.

LEMMA 1. *Let $f: X \rightarrow S$ be a projective morphism, where S is noetherian, and let E be a coherent sheaf on X which is flat over S . For a field, k , and a k -valued point $\eta: \text{Spec}(k) \rightarrow S$, let*

$$\chi(E, \eta) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X_\eta, E_\eta),$$

where E_η is the sheaf induced on the fibre X_η . If S is connected, then $\chi(E, \eta)$ is independent of the choice of k and η .

This lemma is a consequence of EGA III.7.9.11; cf. also 7.7.4, 7.7.12(i), and 7.9.3 of the same chapter.

We will now reduce the general case to the case just considered. We claim that there is a smooth projective morphism $g: Y \rightarrow X$ with connected fibres such that

- (1) For each j , g^*E_j has a filtration of locally free subsheaves

$$E_{j1} \subset E_{j2} \subset \cdots \subset E_{j,p_j} = g^*E_j,$$

where $p_j = \text{rk}(E_j)$, such that the quotients $L_{jv} = E_{jv}/E_{j,v-1}$ are invertible.

- (2) There are invertible sheaves, $\Lambda_1, \dots, \Lambda_s$, on Y and positive integers, $d_1, \dots, d_s$, such that

$$(\alpha) \quad d_1 + \cdots + d_s = d = \text{dimension of any fibre of } g,$$

$$(\beta) \quad g_{\eta*}(c_1(\Lambda_{1\eta})^{d_1} \cdots c_1(\Lambda_{s\eta})^{d_s}) = 1_{X_\eta},$$

for all geometric points $\eta: \text{Spec}(k) \rightarrow S$, where g_η is obtained by base extension.

We will first use (1) and (2) to achieve the desired reduction. From (1) we have

$$(*) \quad c((g^*E_j)_\eta) = \prod_v (1 + c_1(L_{jv\eta})).$$

Next, let $z \in \mathcal{A}(X_\eta)$, and set $y_\eta = (c_1(\Lambda_{1\eta})^{d_1} \cdots c_1(\Lambda_{s\eta})^{d_s})$. Using the projection formula (cf. [1, p. 3-17]), and (2), we obtain

$$z = z \cdot 1_{X_\eta} = z \cdot g_{\eta*}(y_\eta) = g_{\eta*}(g_\eta^*(z) \cdot y_\eta).$$

In particular, this implies that

$$(**) \quad \deg_\eta(z) = \deg_\eta(g_\eta^*(z) \cdot y_\eta)$$

if z is the class of a zero-cycle on X_η . Note that the expression on the right is the degree of a zero-cycle on Y_η .

We apply (**) in the case $z = c_{i_1}(E_{1\eta}) \cdots c_{i_m}(E_{m\eta})$ and use (*) to express $g_\eta^*(z) \cdot y_\eta$ as a polynomial of degree $r + d$ in the $c_1(\Lambda_{i\eta})$ and the $c_1(L_{j\eta})$. In this way, we reduce the question to the case of the theorem already proved.

We now prove the existence of Y . In the case $m = 1$, write $E = E_1$, and take $Y = \text{Flag}(E)$, the flag bundle of E over X (cf. [1, pp. 4-18 and 4-19.]). Thus, if $p = \text{rk}(E)$, we have a sequence of morphisms

$$Y = P_{p-1} \rightarrow P_{p-2} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 = X,$$

such that $P_1 = P(E)$, and $P_{i+1} = P(F_i)$, for $i \geq 1$, F_i being defined by the exactness of $0 \rightarrow F_i \rightarrow q_i^*(F_{i-1}) \rightarrow \mathcal{O}_{P_i}(1) \rightarrow 0$, where $q_i: P_i \rightarrow P_{i-1}$, and $F_0 = E$. Now, it is known that

$$(***) \quad \begin{aligned} q_{i\eta*}((c_1(\mathcal{O}_{P_{i\eta}}(1)))_\eta)^e &= 0, & e < p-i, \\ &= 1, & e = p-i \end{aligned}$$

(cf. [1, p. 4-13]). Letting $g_i: Y \rightarrow P_i$ be the composition $P_{p-1} \rightarrow \cdots \rightarrow P_i$, we set $\Lambda_i = g_i^*(\mathcal{O}_{P_i}(1))$ and $d_i = p - i$. Writing $g = g_0$ and using (**) and the projection formula repeatedly, we find

$$g_{\eta*}(c_1(\Lambda_{1\eta})^{p-1} \cdots c_1(\Lambda_{p-1,\eta})) = 1_{X_\eta}.$$

In the case $m > 1$, we proceed by induction on m . Suppose that $q: Z \rightarrow X$ and the invertible sheaves $\Lambda_1, \dots, \Lambda_s$ on Z have properties (1) and (2) relative to $E_1, \dots, E_{m-1}$. Let $Y = \text{Flag}(q^*E_m)$, and let $r: Y \rightarrow Z$. We pull back $\Lambda_1, \dots, \Lambda_s$ to $r^*\Lambda_1, \dots, r^*\Lambda_s$ on Y and form $p-1$ other invertible sheaves on Y by the process used in the case $m = 1$, where $p = \text{rk}(E_m)$. Setting $g = q \circ r$, one uses the projection formula and the fact that $g_{\eta*} = q_{\eta*} \circ r_{\eta*}$ to check that the sheaves constructed on Y have property (2).

3. Dependency cycles. In this section, V will be a nonsingular quasi-projective variety defined over an algebraically closed field k , and E will be a locally free sheaf on V , of rank $p \leq \dim(V)$. We will recall the definition of the dependency cycle of a set of sections of E (cf. Mattuck [6, §4]), and we will give an expression for the multiplicities of the components of this cycle.

Let $\hat{E} = P(\mathcal{O}_V \oplus E^*)$ (cf. EGA II, §4), where $E^* = \mathcal{H}om(E, \mathcal{O}_V)$, and let $\pi: \hat{E} \rightarrow V$ be the natural projection. Let $\sigma_0: V \rightarrow \hat{E}$ correspond to the surjection $\mathcal{O}_V \oplus E^* \rightarrow \mathcal{O}_V$ which restricts to the identity on $\mathcal{O}_V$ and to the zero map on E^* . Thus $V \cong \sigma_0(V)$. Let $s \in \Gamma(V, E)$, and let $\sigma: V \rightarrow \hat{E}$ correspond to the surjection $\mathcal{O}_V \oplus E^* \rightarrow \mathcal{O}_V$ which restricts to the identity on $\mathcal{O}_V$ and to the map dual to $s: \mathcal{O}_V \rightarrow E$ on E^* . Under the assumption that $\sigma^{-1}(\sigma_0(V)) = \{x \in V \mid s(x) = 0\}$ is of pure codimension p in V , we say that *the cycle of zeros of s is defined*; this cycle is defined to be $\sigma^*(\sigma_0(V))$ and is denoted $s^*(0)$.

Let us write $s^*(0) = \sum \mu_Z Z$; the sum ranges over the irreducible components of $\sigma^{-1}(\sigma_0(X))$. For a fixed Z , let x be the generic point of Z , and let U be a neighborhood of x on which E is free. On U , we can write $s = \sum_{i=1}^p f_i e_i$, where $f_i \in \Gamma(U, \mathcal{O}_V)$, and $\{e_1, \dots, e_p\}$ is a basis of $\Gamma(U, E)$.

PROPOSITION 1. *With the above notation, $\mu_Z = \ell_A(A/(f_1, \dots, f_p))$, where $A = \mathcal{O}_{V,x}$, and $\ell_A(M)$ denotes the length of the A -module M .*

Proof. Since $\sigma_0(V)$ and $\sigma(V)$ are locally complete intersections on $\hat{E}$, one can check that $\ell_A(A/(f_1, \dots, f_p))$ is the multiplicity of the corresponding component of $\sigma_0(V) \cdot \sigma(V)$. Q.E.D.

PROPOSITION 2. *If $s \in \Gamma(V, E)$ is such that $s^*(0)$ is defined, then $s^*(0) = c_p(E)$ in $\mathcal{A}(V)$.*

The proof can be modeled directly after one given by Grothendieck [4, Theorem 2].

Let $q \leq p$, and let $s_1, \dots, s_q \in \Gamma(V, E)$. Now, $s_1, \dots, s_q$ define a map of $\mathcal{O}_V$ -modules, $\phi: \mathcal{O}_V^q \rightarrow E$; by duality we get $\phi^*: E^* \rightarrow \mathcal{O}_V^q$. Let $P = V \times \mathbf{P}^{q-1}$, and let $\pi: P \rightarrow V$ and $\rho: P \rightarrow \mathbf{P}^{q-1}$ be the two projections. Then ϕ^* gives rise to a map of $\mathcal{O}_P$ -modules: $\pi^* E^* \rightarrow \rho^* \mathcal{O}(1)$. Tensoring with $\mathcal{O}(-1)$ and dualizing, we obtain $s \in \Gamma(P, \pi^* E \otimes \mathcal{O}(1))$. We will make two assumptions:

- (1) $s^*(0)$ is defined;
- (2) π sends every irreducible component of $s^*(0)$ to a subvariety of codimension $p - q + 1$ in V .

When these assumptions are satisfied, we define the *dependency cycle*, $D(\Sigma)$ of the set $\Sigma = \{s_1, \dots, s_q\}$ to be $\pi_*(s^*(0))$.

PROPOSITION 3. *Let $\Sigma \subset \Gamma(V, E)$ and $s \in \Gamma(P, \pi^* E \otimes \mathcal{O}(1))$ be as above. If $D(\Sigma)$ is defined, then $D(\Sigma) = c_{p-q+1}(E)$ in $\mathcal{A}(V)$.*

For a proof, see Mattuck [6, Theorem 2].

We will now give a local description of the section $s \in \Gamma(P, \pi^* E \otimes \mathcal{O}(1))$. Let $U = \text{Spec}(A) \subset V$ be an affine open set on which E is free, and let $\{e_1, \dots, e_p\}$ be a basis of $\Gamma(U, E)$. Thus, $s_i = \sum_{j=1}^p f_{ij} e_j$, where $f_{ij} \in \Gamma(U, \mathcal{O}_V)$, for $i = 1, \dots, q$. Let $T_1, \dots, T_q$ be a basis of $\Gamma(\pi^{-1}U, \mathcal{O}(1))$, and let U_m be the open subset where $T_m \neq 0$, for $1 \leq m \leq q$. It is easy to verify:

$$(*) \quad s|_{U_m} = \left(\sum_{i=1}^q \sum_{j=1}^p (t_i/t_m) f_{ij} e_j \right) \otimes T_m,$$

where $T_i|_{U_m} = (t_i/t_m) \cdot (T_m|_{U_m})$. This implies

$$(**) \quad s|_{(\pi^{-1}U)} = \sum_{i,j} f_{ij} (e_j \otimes T_i).$$

Relation $(*)$ also implies that $x \in \text{Supp}(D(\Sigma))$ iff $s_1, \dots, s_q$ become dependent in $E \otimes k(x)$.

Assume that $D(\Sigma)$ is defined, and let $D(\Sigma) = \sum \mu_Z \cdot Z$. For a fixed Z , let x be a generic point of Z , and choose $U \subset V$, as above, so that $x \in U$.

PROPOSITION 4. *With the above notation and assumptions, $\mu_Z = \ell_A(A/I)$, where $A = \mathcal{O}_{V,x}$, and I is the ideal in A generated by the $q \times q$ minors of the $q \times p$ matrix (f_{ij}) . Thus, I is the 0th Fitting ideal of $\text{Coker}(\phi_x)$, where $\phi: \mathcal{O}_V^q \rightarrow E$ is defined by $s_1, \dots, s_q$.*

Proof. Every $q \times q$ minor of the matrix (f_{ij}) vanishes along $W = Z \cap U$. Assumption (2) implies that some $(q-1) \times (q-1)$ minor of (f_{ij}) is nonzero at x . We may assume that this minor is nonzero at all points of U . Thus, some subset of Σ , say $\{s_1, \dots, s_{q-1}\}$, is a subset of a basis of $\Gamma(U, E)$.

Let $J \subset \Gamma(U, \mathcal{O}_V)$ be generated by the $q \times q$ minors of the matrix (f_{ij}) . Since $E|_U$ is free, J is the $(p-q)$ th Fitting ideal of $\text{Coker}(\Gamma(U, \mathcal{O}_V)^q \rightarrow \Gamma(U, E))$. Hence J is independent of the choice of basis of $\Gamma(U, E)$ (cf. Fitting [2, Hauptsatz]), and we may assume that $e_i = s_i$, for $i = 1, \dots, q-1$. Therefore J is generated by the $p-q+1$ elements $f_{qq}, \dots, f_{qp}$. Moreover, the relation $(**)$ becomes

$$s|(\pi^{-1}U) = \sum_{j=1}^{q-1} (e_j \otimes T_j + f_{qj}(e_j \otimes T_q)) + \sum_{j=q}^p f_{qj}(e_j \otimes T_q).$$

Thus, $\text{Supp}(s^*(0) \cap U) \subset U_q$. Now, $(*)$ becomes

$$s|U_q = \sum_{j=1}^{q-1} (t_j + f_{qj})(e_j \otimes T_q) + \sum_{j=q}^p f_{qj}(e_j \otimes T_q),$$

where we have set $t_q = 1$. Let Z' be the unique component of $s^*(0)$ lying above Z , let y be the generic point of Z' , and let $B = \mathcal{O}_{\tilde{E},y}$. If $\mu_{Z'}$ is the multiplicity of Z' in $s^*(0)$, Proposition 1 implies that $\mu_Z = \ell_B(B/I^*)$, where

$$I^* = (t_1 + f_{q1}, \dots, t_{q-1} + f_{q,q-1}, f_{qq}, \dots, f_{qp})B.$$

Further, we have isomorphisms

$$B/I^* \cong (A/I)[T_1, \dots, T_{q-1}]/(T_1 + \bar{f}_{q1}, \dots, T_{q-1} + \bar{f}_{q,q-1}) \cong A/I.$$

This implies that (i) A and B have the same residue field, and (ii) $\ell_A(A/I) = \ell_B(B/I^*)$. Now, (i) implies that $\pi_*(Z') = Z$. Using (ii), we obtain $\mu_Z = \ell_A(A/I)$. Q.E.D.

4. The cycles S_i . Let $f: V^r \rightarrow W^m$ be a morphism of nonsingular varieties over the algebraically closed field k , where $m \geq r$. Let $\Omega_{V/W}^1$ be the sheaf of relative differentials, and let $S_i \subset V$ be the closed subset $\{x \mid \dim_{k(x)}(\Omega_{V/W}^1 \otimes k(x)) \geq i\}$, for each $i \geq 1$. We will say that S_i has the proper codimension iff every irreducible component has codimension $i(m-r+i)$. If S_i has the proper codimension, we define the cycle S_i by

$$S_i = \sum \nu_Z \cdot Z;$$

the sum ranges over all components, and $\nu_Z = \ell_{\mathcal{O}_x}(\mathcal{O}_x/J)$, where x is the generic point of Z , and J is the $(i-1)$ st Fitting ideal of $(\Omega_{V/W}^1)_x$. We must check that ν_Z is finite. We have an exact sequence of $\mathcal{O}_x$ -modules

$$\Omega_{\mathcal{O}_y/k}^1 \otimes \mathcal{O}_x \xrightarrow{\phi} \Omega_{\mathcal{O}_x/k}^1 \longrightarrow \Omega_{\mathcal{O}_x/\mathcal{O}_y}^1 \longrightarrow 0,$$

where $y=f(x)$. The first two terms in this sequence are free of rank m and r respectively. If we choose suitable bases for these modules, then ϕ is described by an $m \times r$ matrix, and J is generated by its minors of rank $r-i+1$. Since Z is the only component of S_i containing x , the maximal ideal $\mathfrak{m}_x$ is the only associated prime of J . Therefore ν_Z is finite. We will call the cycles S_i the *singular cycles* of f .

LEMMA 2. *Suppose that S_1 is of the proper codimension. Let Z be a component of S_1 , with $Z \not\subset S_2$, and with generic point x . Then $\nu_Z = \ell_{\mathcal{O}_x}((\Omega_{V/W}^1)_x)$.*

Proof. Since $(\Omega_{V/W}^1)_x$ is generated by one element, $(\Omega_{V/W}^1)_x \cong \mathcal{O}_x/J$, where J is the 0th Fitting ideal. Q.E.D.

If x is a closed point of V , then $x \in S_i$ iff $\dim_k(\mathfrak{m}_x/(\mathfrak{m}_y\mathcal{O}_x + \mathfrak{m}_x^2)) \geq i$, where $y=f(x)$, and $\mathfrak{m}_x$ and $\mathfrak{m}_y$ are the maximal ideals of $\mathcal{O}_x$ and $\mathcal{O}_y$. In particular, suppose that $V \subset \mathbf{P}^n$ and that $f=\pi: V \rightarrow \mathbf{P}^m$ is induced by projection from an $(n-m-1)$ -subspace $L \subset \mathbf{P}^n$ such that $L \cap V = \emptyset$. For a closed point $x \in V$, it follows that $x \in S_i$ iff $\dim(L \cap t_{V,x}) \geq i-1$, where $t_{V,x}$ is the r -subspace of $\mathbf{P}^n$ tangent to V at x . (This can be checked using the techniques of the proof of Proposition 3 of [8].)

LEMMA 3. *Let $r \leq m < n$, and let $V \subset \mathbf{P}^n$ be nonsingular. Then there is a dense open subset of the Grassmann variety $G=G(n, n-m-1)$ consisting of linear subspaces $L \subset \mathbf{P}^n$ such that $L \cap V = \emptyset$, and $S_i(\pi)$ is purely of codimension $i(m-r+i)$ for all i , where $\pi: V \rightarrow \mathbf{P}^m$ is induced by projection from L .*

Proof. For each i , consider the correspondence $Z_i \subset V \times G$ consisting of all (x, L) such that $\dim(L \cap t_{V,x}) \geq i-1$. By a counting of constants which uses standard facts about Schubert cycles on G (cf. [5, Chapter XIV, §2]), one finds that $\dim(Z_i) = \dim(G) + r - i(m-r+i)$. Q.E.D.

Henceforth we will fix an $(n-m-1)$ -subspace $L \subset \mathbf{P} = \mathbf{P}^n$, such that $L \cap V = \emptyset$, and $S_i = S_i(\pi)$ is of the proper codimension for all i , where $\pi: V \rightarrow \mathbf{P}^m$ is induced by projection from L . We will also fix a basis, $\{T_0, \dots, T_n\}$ of $\Gamma(P, \mathcal{O}(1))$, such that L is given by $T_0 = \dots = T_m = 0$.

THEOREM 2. *Let V, L , and π be as above. Then*

$$S_1 = c_{m-r+1}(N \otimes \mathcal{O}_V(-1)) \quad \text{in } \mathcal{A}(V),$$

where N is the normal bundle of V in $\mathbf{P}^n$, viz., $N = (I/I^2)^* = \mathcal{H}om_{\mathcal{O}_V}(I/I^2, \mathcal{O}_V)$, where $I = \text{Ker}(\mathcal{O}_P \rightarrow \mathcal{O}_V)$.

Proof. For $j=0, \dots, n$, let T_j be as above, and let $U_j \subset P$ be the open set $\{x \mid T_j(x) \neq 0\}$. Choose $t_0=1, t_1, \dots, t_n \in k(P)$ such that $T_i = (t_i/t_j)T_j$ on U_j . Thus,

$\Gamma(U_j, \mathcal{O}_P)$ is the polynomial ring $k[t_0/t_j, \dots, t_n/t_j]$. For $0 \leq i \leq n$ and $i \neq j$, let D_{ij} be the derivation of $\Gamma(U_j, \mathcal{O}_P)$ given by $D_{ij}(t_v/t_j) = \delta_{iv}$ (Kronecker delta), and let $D_{jj} = -\sum_{i \neq j} (t_i/t_j) D_{ij}$. We can extend these derivations to derivations of the function field $k(P) = k(t_1, \dots, t_n)$ and check that $D_{ij} = (t_j/t_h) D_{ih}$ for all i, j, h .

For each (i, j) , D_{ij} induces an element $\bar{D}_{ij} \in \Gamma(U_j, (I/I^2)^*)$. We define

$$s_{ij} = \bar{D}_{ij} \otimes T_j^{-1} \in \Gamma(U_j, N \otimes \mathcal{O}(-1)).$$

Since $\bar{D}_{ij}|_{U_j \cap U_h} = (t_j/t_h) \bar{D}_{ih}|_{U_j \cap U_h}$, the various s_{ij} (for each i) fit together to give sections $s_0, \dots, s_n \in \Gamma(V, N \otimes \mathcal{O}(-1))$.

Let x be any closed point of V . Since $L \cap V = \emptyset$, we have $T_q(x) \neq 0$ for some $q \leq m$. We may assume $q = 0$; thus $x \in U_0$. Let $g_1, \dots, g_{n-r}$ generate I in a neighborhood of x , and let $\sigma_i: I/I^2 \rightarrow \mathcal{O}_V$ be given locally by $\sigma_i([g_j]) = \delta_{ij}$. With these notations, we find that

$$s_i = \sum_{j=1}^{n-r} (\partial g_j / \partial t_i) (\sigma_j \otimes T_0^{-1})$$

in some neighborhood of x . (The g_j are polynomials in $t_1, \dots, t_n$.) In particular, it follows that $s_{m+1}, \dots, s_n$ become dependent in $(N \otimes \mathcal{O}(-1)) \otimes k(x)$ iff $x \in S_1$. To see this, we note that $L \cap t_{V,x} \neq \emptyset$ iff there are elements $b_{m+1}, \dots, b_n \in k$, not all zero, such that $\sum_{j=m+1}^n b_j \cdot (\partial g_j / \partial t_i)(x) = 0$, for $i = 1, \dots, n-r$. Since S_1 is purely of codimension $m-r+1 = (n-r) - (n-m) + 1$, it follows that $D(\Sigma)$ is defined ($\Sigma = \{s_{m+1}, \dots, s_n\}$), and $\text{Supp}(D(\Sigma)) = \text{Supp}(S_1)$. Proposition 4 implies that $D(\Sigma) = \sum \mu_Z Z$, where $\mu_Z = \ell_{\mathcal{O}_x}(\mathcal{O}_x/J')$, with x the generic point of Z , and J' the ideal in $\mathcal{O}_x$ generated by the minors of order $(n-m-1)$ of the $(n-r) \times (n-m-1)$ matrix $(\partial g_i / \partial t_j)_{1 \leq i \leq n-r, m+1 \leq j \leq n}$. Finally, Proposition 3 implies that $D(\Sigma) = c_{m-r+1}(N \otimes \mathcal{O}_V(-1))$ in $\mathcal{A}(V)$.

Since S_1 and $D(\Sigma)$ have the same irreducible components, it will follow that $S_1 = D(\Sigma)$ if we can show that the ideal J' defined above is the 0th Fitting ideal of $\Omega_{\mathcal{O}_x/\mathcal{O}_y}^1$ ($y = \pi(x)$). This will complete the proof. Thus, let $U = P^n - L = U_0 \cup \dots \cup U_m$. The projection $\pi: P^n - L \rightarrow P^m$ has the property that $\pi|_{U_i}$ looks like the projection $A^m \times A^{n-m} \rightarrow A^m$. Hence in the exact sequence of $\mathcal{O}_V$ -modules

$$I/I^2 \xrightarrow{\delta} \Omega_{U/P^m}^1 \otimes \mathcal{O}_V \longrightarrow \Omega_{V/P^m}^1 \longrightarrow 0,$$

the first two terms are free. Moreover, if $x \in U_0$, and if $g_1, \dots, g_{n-r}$ generate I_x , then δ is given by

$$\delta([g_i]) = \sum_{j=m+1}^n (\partial g_i / \partial t_j) (dt_j \otimes 1),$$

for $i = 1, \dots, n-r$. We conclude that J' is the 0th Fitting ideal of Ω_{V/P^m}^1 . Q.E.D.

COROLLARY 1. *With the assumptions of Theorem 2,*

$$S_1 = \sum_{j=0}^{m-r+1} \binom{m+1}{j} \gamma_{q-j} h^j \quad \text{in } \mathcal{A}(V),$$

where h is the divisor class of a hyperplane section, $(1 - \gamma_1 + \gamma_2 - \cdots + (-1)^r \gamma_r) = c(\Omega_V^1)^{-1}$, and $q = m - r + 1$.

Proof. With $P = \mathbf{P}^n$, let D_{ij} be the derivations of $k(P)$ defined above. Let $\Theta = (\Omega_P^1)^*$, and let $s_0, \dots, s_n \in \Gamma(P, \Theta \otimes \mathcal{O}_P(-1))$ satisfy $s_i|U_j = D_{ij} \otimes T_j^{-1}$. The s_i give a map $\psi: \mathcal{O}_P^{n+1} \rightarrow \Theta \otimes \mathcal{O}_P(-1)$ and thus an exact sequence

$$0 \rightarrow \Omega_P^1 \rightarrow \mathcal{O}_P(-1)^{n+1} \rightarrow \mathcal{O}_P \rightarrow 0.$$

This gives $c(\Omega_P^1) = (1 - h)^{n+1}$.

On V we have an exact sequence

$$0 \rightarrow (I/I^2) \rightarrow \Omega_P^1 \otimes \mathcal{O}_V \rightarrow \Omega_V^1 \rightarrow 0.$$

Thus $c(N^*) = j^*(c(\Omega_P^1)) \cdot c(\Omega_V^1)^{-1}$, where $j: V \rightarrow P$. The rest of the computation will be omitted.

COROLLARY 2. *If $m = r = 1$, and if the curve $V \subset \mathbf{P}^n$ has genus g and degree d , then*

$$2g - 2 = -2d + \sum_{x \in S_1(\pi)} \ell_{\mathcal{O}_x}((\Omega_{V/\mathbf{P}^1}^1)_x).$$

This is a special case of Hurwitz' formula for the genus change under a morphism of curves. For a proof, use the exact sequence

$$0 \rightarrow I/I^2 \rightarrow j^*(\Omega_{\mathbf{P}^n/k}^1) \rightarrow \Omega_{V/k}^1 \rightarrow 0$$

(with $j: V \rightarrow \mathbf{P}^n$) to show that $\deg(N \otimes \mathcal{O}(-1)) = 2g + 2d - 2$. Lemma 2 shows that the summation on the right side of the formula also is $\deg(N \otimes \mathcal{O}(-1))$.

GENERALIZATION. It is also possible to express the cycles S_i , $i > 1$, in terms of Chern classes. Thus let $G' = G'(n, n - r - 1)$ be the Grassmannian which parameterizes $(n - r)$ -quotients of rank $n + 1$ free sheaves, and let $\Phi: \mathcal{O}_{G'}^{n+1} \rightarrow E$ be the universal surjection. There is a morphism $u: V \rightarrow G'$ such that Φ pulls back to $\psi: \mathcal{O}_V^{n+1} \rightarrow N \otimes \mathcal{O}_V(-1)$. The Chern classes of E can be expressed in terms of Schubert cycles on G' . On the other hand, it seems clear that the Schubert cycles which pull back to the cycles S_i can be expressed in terms of the Fitting ideals of Coker (Φ) . Thus, one should obtain formulas which are similar to formula (10) on p. 357 of [5].

5. Enumeration of pinch points. Let V^r be a nonsingular projective variety over an algebraically closed field k . One can find a projective embedding $V \subset \mathbf{P}^n$ such that there is a finite morphism $\pi: V \rightarrow \mathbf{P}^{2r-1}$ induced by projection from an $(n - 2r)$ -subspace $L \subset \mathbf{P}^n$ satisfying

(I) $S_1(\pi)$ is purely 0-dimensional, and $\pi|S_1(\pi)$ is injective. Moreover, if $r \geq 2$, then $\pi^{-1}(\pi(x)) = \{x\}$ for all $x \in S_1(\pi)$.

(II) If $x \in S_1(\pi)$ and $y = \pi(x)$, then $\hat{\mathcal{O}}_{V,x}$ and $\hat{\mathcal{O}}_{\mathbf{P}^{2r-1},y}$ can be identified with formal power series rings $B = k[[t_1, \dots, t_r]]$ and $A = k[[t_1, \dots, t_{2r-1}]]$ so that π induces the

homomorphism $f: A \rightarrow B$ given by

$$\begin{aligned} f(t_i) &= t_i && \text{for } i = 1, \dots, r-1, \\ f(t_i) &= t_{i-r+1}t_r && \text{for } i = r, \dots, 2r-2, \\ f(t_{2r-1}) &= t_r^2 + t_r^3. \end{aligned}$$

(If $\text{char}(k) \neq 2$ we can replace $t_r^2 + t_r^3$ by t_r^2 .)

In fact, if $r \geq 2$, a suitable embedding may be found by replacing any given embedding by the embedding determined by hypersurface sections of degree $d \geq 2$, and Theorem 3 of [8] states that if L is chosen generically, then π has the following properties which imply (I) and (II). *If $V' = \pi(V)$, then V' is birational to V , $\text{Sing}(V')$ is purely of dimension 1, and V' has singular branches at only finitely many closed points $y \in V'$. If V' has a singular branch at y , then $\hat{\mathcal{O}}_{V',y}$ is isomorphic to $f(A)$, where $f: A \rightarrow B$ is as above. (Recall that if $\{x\} = \pi^{-1}(y)$, then $x \in S_1(\pi)$ iff y has a singular branch at y .)*

If $r = 1$, and V is of genus g , one embeds V by using a complete linear system of degree $\geq 2g + 3$ and uses techniques like those used in the proof of Theorem 3 of [8] to obtain (I) and (II).

PROPOSITION 5. *Let $\pi: V \rightarrow \mathbf{P}^{2r-1}$ be as above, and let $S_1 = \sum_x \nu_x \cdot x$, where the summation extends over all points of $\text{Supp}(\Omega_{V/\mathbf{P}^{2r-1}}^1)$.*

If $\text{char}(k) \neq 2$, then $\nu_x = 1$ for all x .

If $\text{char}(k) = 2$, then $\nu_x = 2$ for all x .

Proof. Let $x \in S_1(\pi)$ and $y = \pi(x)$. Since π is finite, $\mathcal{O}_x \cong R_{\mathfrak{m}}$, where R is a semi-local ring which is a finite $\mathcal{O}_y$ -module, and $\mathfrak{m}$ is maximal. Hence, $\hat{R} \cong R \otimes_{\mathcal{O}_y} \hat{\mathcal{O}}_y$, and $\hat{\mathcal{O}}_x \cong R/\mathfrak{a}$, where $\mathfrak{a}$ is generated by idempotent elements. Therefore, $\Omega_{\hat{\mathcal{O}}_x/\hat{\mathcal{O}}_y}^1 \cong \Omega_{\hat{\mathcal{O}}_x/\mathcal{O}_y}^1 \otimes \hat{\mathcal{O}}_x$, so that the 0th Fitting ideal of $\Omega_{\hat{\mathcal{O}}_x/\hat{\mathcal{O}}_y}^1$ is $I \cdot \hat{\mathcal{O}}_x$, where I is the 0th Fitting ideal of $\Omega_{\mathcal{O}_x/\mathcal{O}_y}^1$. Let $A = k[[t_1, \dots, t_{2r-1}]]$, $B = k[[t_1, \dots, t_r]]$, and let $f: A \rightarrow B$ be given as above. Then $\ell_{\mathcal{O}_x}(\mathcal{O}_x/I) = \ell_B(B/J)$, where J is the 0th Fitting ideal of $\Omega_{B/A}^1$. We have an exact sequence of B -modules

$$\hat{\Omega}_{A/k}^1 \hat{\otimes}_A B \xrightarrow{u} \hat{\Omega}_{B/k}^1 \longrightarrow \Omega_{B/A}^1 \longrightarrow 0.$$

(Cf. EGA 0_{IV}, 20.7.17.3 and 0_I, 7.3.5.) The first two terms are free, and u is given by

$$\begin{aligned} u(dt_i \otimes 1) &= dt_i && \text{for } i = 1, \dots, r-1, \\ u(dt_i \otimes 1) &= t_r dt_{i-r+1} + t_{i-r+1} dt_r && \text{for } i = r, \dots, 2r-2, \\ u(dt_{2r-1} \otimes 1) &= (2t_r + 3t_r^2) dt_r. \end{aligned}$$

The 0th Fitting ideal is thus $J = (t_1, \dots, t_{r-1}, t_r)$ if $\text{char}(k) \neq 2$, and

$$J = (t_1, \dots, t_{r-1}, t_r^2)$$

if $\text{char}(k) = 2$. The length of B/J is 1 (resp. = 2) if $\text{char}(k) \neq 2$ (resp. = 2). Q.E.D.

REMARK. As a consequence of Proposition 5, the number of points in $S_1(\pi)$ is independent of the choice of projection center, L , provided (I) and (II) are satisfied.

We will now see how the number of points varies as V is specialized. Thus, let A be a noetherian ring, and X a closed subscheme of $\mathbf{P}_A^n = \text{Proj } A[T_0, \dots, T_n]$; assume that $p: X \rightarrow \text{Spec}(A)$ is smooth and has absolutely irreducible fibres of dimension r . Assume that $n \geq 2r-1$ and that X does not meet the closed subscheme given by $T_0 = \dots = T_{2r-1} = 0$. We define N to be the normal bundle of X in $\mathbf{P}_A^n$; thus $N = (I/I^2)^*$, $I = \text{Ker}(\mathcal{O}_P \rightarrow \mathcal{O}_X)$.

We consider geometric points $\eta: \text{Spec}(k) \rightarrow \text{Spec}(A)$ ($k = \bar{k}$) such that the projection π_η of the geometric fibre $X_\eta \subset \mathbf{P}_k^n$ from the linear subspace $T_0 = \dots = T_{2r-1} = 0$ satisfies (I) and (II) above. Now, the bundle N_η induced on X_η by N is just the normal bundle of X_η in $\mathbf{P}_k^n$. Thus, by Theorem 2, the degree of the cycle $S_1(\pi_\eta)$ is $\deg(c_1((N \otimes \mathcal{O}_X(-1))_\eta))$. By Theorem 1, this is independent of η . Using Proposition 5, we obtain the following conclusion.

PROPOSITION 6. *Let A and $X \subset \mathbf{P}_A^n$ be as above. Assume that $\text{Spec}(A)$ is connected. For $i=1, 2$, let $\eta_i: \text{Spec}(k_i) \rightarrow \text{Spec}(A)$ be geometric points such that the corresponding projections π_{η_i} both satisfy (I) and (II). If $\text{char}(k_1)$ and $\text{char}(k_2)$ are both $\neq 2$ or both $= 2$, then $\#(\text{points in } S_1(\pi_{\eta_1})) = \#(\text{points in } S_1(\pi_{\eta_2}))$. If $\text{char}(k_1) = 2$ and $\text{char}(k_2) \neq 2$, then $\#(\text{points in } S_1(\pi_{\eta_1})) = \frac{1}{2}(\#(\text{points in } S_1(\pi_{\eta_2})))$.*

If $r \geq 2$, we substitute $\#(\text{pinch-points of } \pi_{\eta_i}(X_{\eta_i}))$ for $\#(\text{points in } S_1(\pi_{\eta_i}))$ to obtain a statement about the behavior under specialization of the number of pinch-points. If $r=1$, we obtain a similar statement about the behavior under specialization of the number of ramification points of the covering $V \rightarrow \mathbf{P}^1$.

EXAMPLE. Let $V^2 \subset \mathbf{P}^5$ be the Veronese surface, i.e. the image of $\mathbf{P}^2$ embedded by the complete linear system of conics. Explicitly, let points of $\mathbf{P}^5$ have homogeneous coordinates (y_{ij}) with $0 \leq i \leq j \leq 2$. Then $(x_0, x_1, x_2) \in \mathbf{P}^2$ is sent to the point of $\mathbf{P}^5$ with $y_{ij} = x_i x_j$. Let $\pi: V \rightarrow \mathbf{P}^3$ be induced by projection from the line $y_{01} = y_{02} = y_{12} = y_{00} + y_{11} + y_{22} = 0$. Thus, the composed map $\mathbf{P}^2 \rightarrow \mathbf{P}^3$ sends (x_0, x_1, x_2) to $(x_1 x_2, x_0 x_2, x_0 x_1, x_0^2 + x_1^2 + x_2^2)$, and the image is the surface $V' \subset \mathbf{P}^3$ whose equation is $t_0^2 t_1^2 + t_0^2 t_2^2 + t_1^2 t_2^2 - t_0 t_1 t_2 t_3 = 0$. The singular locus of V' consists of the three lines Λ_0, Λ_1 , and Λ_2 given respectively by $t_1 = t_2 = 0$, $t_0 = t_2 = 0$, and $t_0 = t_1 = 0$. If $\text{char}(k) \neq 2$, then V' has 6 pinch-points, two of which lie on each of the lines Λ_i ; if $\text{char}(k) = 2$, then V' has 3 pinch-points, one on each Λ_i .

To see this, note that $C_i = \pi^{-1}(\Lambda_i)$ is a plane conic for each i . The projection center meets the plane of C_i in a point which lies on two tangent lines of C_i if $\text{char}(k) \neq 2$, but on just one tangent line of C_i if $\text{char}(k) = 2$. (The thing to note is that if $\text{char}(k) = 2$, there is a point, not on the projection center, which lies on every tangent line of C_i .) It might also be noted that the plane conic provides the simplest example of the case $r=1$ of Proposition 6.

REFERENCES

1. Séminaire C. Chevalley 2e année: 1958. *Anneaux de Chow et applications*, Secrétariat mathématique, Paris, 1958. MR 22 #1572.
2. H. Fitting, *Die Determinantenideale eines Moduls*, Jber. Deutsch. Math.-Verein. **1936**, 195–228.

3. J. Fogarty and D. S. Rim, *Serre sequences and Chern classes*, J. Algebra **10** (1968), 436–447. MR **39** #2771.
4. A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958), 137–154. MR **22** #6818.
5. W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry*. Vol. 2, Cambridge Univ. Press, New York, 1952. MR **13**, 972.
6. A. Mattuck, *Secant bundles on symmetric products*, Amer. J. Math. **87** (1965), 779–797. MR **33** #7345.
7. D. Mumford, *Lectures on curves on an algebraic surface*, Ann. of Math. Studies, no. 59, Princeton Univ. Press, Princeton, N. J., 1966. MR **35** #187.
8. J. Roberts, *Generic projections of algebraic varieties*, Amer. J. Math. **93** (1971), 191–215.

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