

A CHARACTERIZATION OF M -SPACES IN THE CLASS OF SEPARABLE SIMPLEX SPACES

BY

ALAN GLEIT⁽¹⁾

ABSTRACT. We show that a separable simplex space is an M -space iff the arbitrary intersection of closed ideals is always an ideal.

Edward Effros in [5, end of §3] was unable to determine if an arbitrary intersection of closed ideals in a simplex space was necessarily an ideal. That this was not true in general was shown by J. Bunce [2] and F. Perdrizet. Here we shall show that for separable simplex spaces this property is equivalent to the simplex space being a (Kakutani) M -space.

In §1 we extend some of the results of [7]. We show that certain subspaces of certain simplex spaces are again simplex spaces. In §2 we give the aforementioned characterization.

I should like to thank the referee for his comments. The much simplified proof of Theorem 1.2 and its extension to the nonstandard case are due to him.

0. Conventions. All vector spaces are assumed to have nonzero elements. The term measure will always denote a regular bounded Borel measure. We use $\delta(q)$ for the point measure at q .

1. An existence theorem. An ordered Banach space V with closed positive cone is a simplex space if its dual is a (Kakutani) L -space. If Y is a compact Hausdorff space, we let $C(Y)$ be the space of (real) continuous functions on Y with the natural pointwise order and the supremum norm. Obviously, $C(Y)$ is a simplex space. Its dual, $C^*(Y)$, is the space of all measures on Y . More generally, if X is a Borel subset of Y , we let $C^*(Y; X)$ be the space of all measures on Y whose total variation on X is zero. Then $C^*(Y; X)$ is an L -space and the extreme points of the positive part of its unit ball are $\{\delta(y) | y \in Y - X\} \cup \{0\}$ [7, Proposition 1.1].

If V is a simplex space, we let

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$$P_1(V) = \{f \in V^* \mid \|f\| \leq 1, f(x) \geq 0 \text{ for each } x \geq 0\}$$

and $EP_1(V)$ be its extreme points. We take

$$EP_1(V)^+ = EP_1(V) - \{0\}, \quad Z(V) = \text{weak}^* \text{ closure of } EP_1(V)^+.$$

Since $P_1(V)$ is a simplex, each $q \in P_1(V)$ is the resultant of a unique maximal probability measure μ_q . We take $\pi_q = \mu_q - \mu_q(\{0\})\delta(0)$. If V is separable, μ_q is supported by $EP_1(V)$ and π_q by $EP_1(V)^+$.

We may characterize V as the set of affine continuous functions on $Z(V)$ vanishing at zero. Hence

$$V = \{f \in C(Z(V)) \mid f(q) = \pi_q(f) \text{ for each } q \in Z(V) - EP_1(V)^+\}.$$

Let

$$X(V) = \{\delta(q) - \pi_q \mid q \in Z(V) - EP_1(V)^+\}.$$

We shall always assume that $X(V)$ is given the weak* topology relative to $C(Z(V))$.

We will be using the following well-known results repeatedly and include them for completeness. First, a map $x \rightarrow \eta_x$ of a topological space E into the measures on a locally compact space F is *weak* Borel measurable* iff $x \rightarrow \eta_x(f)$ is a Borel measurable function for each $f \in C(F)$. It is *bounded* if $\sup \|\eta_x\| < \infty$. We then have [1, V, §3, Proposition 2, Definition 3, Corollary to Proposition 12]

Lemma 1.1. *Let E and F be locally compact Hausdorff spaces. Suppose $x \rightarrow \eta_x$ is a weak* bounded Borel measurable map of E into the positive measures on F . Let μ be a positive measure on E .*

1. *Then there is a measure ν on E defined by*

$$\nu = \int \eta_x d\mu(x).$$

2. *Suppose f is a bounded universally measurable function on F . Then*

$$x \rightarrow \int f(y) d\eta_x(y)$$

is universally measurable and

$$\int f(y) d\nu = \int d\mu(x) \int f(y) d\eta_x(y).$$

Our first result identifies V^+ for V a simplex space. Throughout, we always consider V as a subset of $C(Z(V))$ and not as a subset of $C(P_1(V))$.

Theorem 1.2. *Let V be a simplex space. Let $Z = Z(V)$. Then $V^+ \subseteq C^*(Z)$ may be identified as follows:*

$$V^+ = \left\{ w \in C^*(Z) \mid \text{there is a measure } \nu \text{ on } Z \text{ such that} \right. \\ \left. w(f) = \int_Z (\delta(z) - \pi_z)(f) d\nu \text{ for each } f \in C(Z) \right\}.$$

Proof. Let $w \in V^+$. We may assume that $\|w\| \leq 1$. Writing $w = w^+ - w^-$, we obviously have $\|w^+\| \leq 1$, $\|w^-\| \leq 1$. Let

$$w_1 = w^+ + (1 - \|w^+\|)\delta(0), \quad w_2 = w^- + (1 - \|w^-\|)\delta(0).$$

Then w_1, w_2 are probability measures on $P_1(V)$. Let b be a continuous affine function on $P_1(V)$. Then $b = f + b(0)1$ for some $f \in V$. Since $w^\pm(1) = \|w^\pm\|$ and $w^+(f) - w^-(f) = w(f) = 0$ for $f \in V$, we have that $w_1(b) = w_2(b)$. Hence, w_1 and w_2 have the same resultant. Let n be the unique maximal measure dominating both w_1 and w_2 . Then [8, Theorem 30, p. 232]

$$n = \int_{P_1(V)} \mu_q dw_1(q) = \int_{P_1(V)} \mu_q dw_2(q).$$

Since w_1, w_2 are supported by $Z \cup \{0\}$ and recalling that $\pi_q = \mu_q - \mu_q(\{0\})\delta(0)$ we get

$$w_i - n = \int_Z (\delta(q) - \pi_q) dw_i(q) + c_i \delta(0), \quad i = 1, 2,$$

for suitable constants c_i . But then

$$w = \int_Z (\delta(q) - \pi_q) d(w_1 - w_2) + c \delta(0)$$

for a suitable constant c . Noting that $\pi_0 = 0$, we may write $\nu = w_1 - w_2 + c\delta(0)$ to get the required representation.

Conversely, any $w \in C^*(Z)$ which has the representation

$$w = \int_Z (\delta(q) - \pi_q) d\nu$$

for some measure ν obviously annihilates V .

Corollary 1.3. Let V be a simplex space. Let $Z = Z(V)$, $E = EP_1(V)^+$, and $X = X(V)$. Suppose E is a universally measurable subset of Z . Then $V^+ \subseteq C^*(Z)$ may be identified as follows:

$$V^+ = \text{linear span } (\overline{\text{co}}(X)) \\ = \left\{ w \in C^*(Z) \mid \text{there is a measure } \nu \text{ on } Z \text{ such} \right. \\ \left. \text{that } w = \int_{Z-E} (\delta(z) - \pi_z) d\nu \right\}.$$

Proof. For each $z \in E$, we have $\delta(z) = \pi_z$ and so the second representation is clear. The first follows from the second by approximating ν by atomic measures.

A simplex K is *standard* if it satisfies

1. The extreme points of K , denoted by $E(K)$, is a universally measurable subset of K .

2. Each maximal measure is supported by $E(K)$.

By Lemma 1.1 above, condition (2) is equivalent to

2'. The maximal measures representing each point in the closure of $E(K)$ are supported by $E(K)$.

We call a simplex space V a *standard simplex space* if $P_1(V)$ is a standard simplex. Note that all separable simplex spaces are standard.

We are now prepared to show that certain subspaces of standard simplex spaces are again simplex spaces.

Theorem 1.4. *Let V be a standard simplex space. Let $Z = Z(V)$ and $E = EP_1(V)^+$. Let $q_0 \in Z - E$. Suppose Y is a closed set in Z satisfying*

1. $Y \cap (Z - E) = \{q_0\}$.

2. $\pi_{q_0}(E - Y) \neq 0$.

Then $A = \{f \in V \mid f(y) = \pi_{q_0}(f) \text{ for each } y \in Y\}$ is a nontrivial simplex space with the relative order and norm. Further, A^ is isometrically order isomorphic to $C^*(Z; (Z - E) \cup Y)$ and so $EP_1(A)^+ = E - Y$.*

Proof. We divide the proof into several stages. Throughout, we take π_0 to be π_{q_0} . Let $\alpha = \pi_0(Y)$. Then

$$(1.1) \quad 0 < \pi_0(E - Y) \leq 1 - \alpha.$$

A. First, let us consider the space D defined by

$$D = \{f \in C(Z) \mid f(y) = \pi_0(f) \text{ for each } y \in Y\}.$$

Let $X(D) = \{\delta(y) - \pi_0 \mid y \in Y\}$. As $y \rightarrow \delta(y)$ is continuous and Y is compact, Y is homeomorphic to $X(D)$. Let m be a measure on $X(D)$. Then there is a measure λ on Y induced by m . So, for $f \in C(Z)$,

$$(1.2) \quad \int_{X(D)} x(f) dm(x) = \int f(y) d\lambda - \pi_0(f)\lambda(Y).$$

Let

$$F = \left\{ w \in C^*(Z) \mid \begin{array}{l} \text{there exists a measure } \lambda \text{ on } Y \text{ such that} \\ w(f) = \int f(y) d\lambda - \pi_0(f)\lambda(Y) \text{ for each } f \in C(Z) \end{array} \right\}.$$

Approximating λ by atomic measures clearly yields

$$F = \text{linear span}(\overline{\text{co}}(X(D))).$$

Clearly, the weak* closure of F is $D^\perp$. To show that F is weak* closed it suffices to show it is norm closed [3, V, 5.9].

In order to show F is norm closed, we consider a $w \in F$. Then there is a measure λ on Y such that

$$w(f) = \int f(y) d\lambda - \pi_0(f)\lambda(Y)$$

for each $f \in C(Z)$. By Lemma 1.1, for each Borel set $B \subseteq Z$,

$$w(B) = \lambda(B \cap Y) - \pi_0(B)\lambda(Y).$$

For each Borel set $A \subseteq Y$ we easily get

$$\lambda(A) = w(A) + \pi_0(A)w(Y)/(1 - \alpha)$$

and so w uniquely determines λ . Further

$$|\lambda|(Y) \leq |w|(Y) + \pi_0(Y)|w(Y)|/(1 - \alpha)$$

and so

$$\|\lambda\| = |\lambda|(Y) \leq \|w\|(1 + \alpha/(1 - \alpha)).$$

It should now be clear that F is norm closed. Hence $F = D^+$.

B. The determination of A^+ . It is clear that $A = D \cap V$ and so

$$A^+ = \text{weak}^* \text{ closure } (D^+ + V^+).$$

We claim that $D^+ + V^+$ is already weak* closed. From the representations for D^+ in A and V^+ in Corollary 1.3,

$$D^+ + V^+ = \text{linear span } (\overline{\text{co}}(X(V) \cup X(D)))$$

and so again we need only demonstrate that $D^+ + V^+$ is norm closed. We let

$$(1.3) \quad W = \left\{ w \in C^*(Z) \mid \begin{array}{l} \text{there exists a measure } \nu \text{ on } Z \text{ and a} \\ \text{measure } \lambda \text{ on } Y \text{ such that, for each } f \in C(Z), \\ w(f) = \int_{Z-E} (\delta(z) - \pi_z)(f) d\nu + \int f(y) d\lambda - \pi_0(f)\lambda(Y) \end{array} \right\}.$$

Then, from the representations for D^+ in A and V^+ in Corollary 1.3, $W = D^+ + V^+$. Let $w \in W$ be determined by measures ν on Z and λ on Y . Then, using Lemma 1.1, for each Borel set $B \subseteq Z$,

$$(1.4)' \quad w(B) = \nu(B \cap (Z - E)) - \int_{Z-E} \pi_z(B) d\nu + \lambda(B \cap Y) - \pi_0(B)\lambda(Y).$$

In particular, for Borel $A \subseteq Z - E - \{q_0\}$,

$$(1.4)'' \quad \nu(A) = w(A).$$

Also, $\nu(\{q_0\}) + \lambda(\{q_0\}) = w(\{q_0\})$. We may, by transferring an atom if necessary, take $\lambda(\{q_0\}) = 0$. Hence

$$(1.4)''' \quad \nu(\{q_0\}) = w(\{q_0\}), \quad \lambda(\{q_0\}) = 0.$$

Finally, for Borel $C \subseteq Y - \{q_0\}$,

$$(1.4)''' \quad \lambda(C) = w(C) + \int_{Z-E} \pi_z(C) d\nu + \pi_0(C)\lambda(Y)$$

with $\lambda(Y)$ found by consistency. With the choice (1.4)''', $\nu|(Z-E) = w|(Z-E)$ and so we may rewrite W as

$$(1.5) \quad W = \left\{ w \in C^*(Z) \mid \begin{array}{l} \text{there exists a measure } \lambda \text{ on } Y \text{ without} \\ \text{an atom at } q_0 \text{ such that, for each } f \in C(Z), \\ w(f) = \int_{Z-E} (\delta(z) - \pi_z)(f) dw + \int f(y) d\lambda - \pi_0(f)\lambda(Y) \end{array} \right\}.$$

In this representation, we note that if $w \in C^*(Z)$ is determined by the measure λ on Y , then again

$$\|\lambda\| \leq \|w\| (1 + \alpha/(1 - \alpha)).$$

So clearly W is norm closed. Therefore $W = A^+$.

C. *The determination of A^* .* The dual of A is $C^*(Z)/A^+$. To complete the proof we need only show that $C^*(Z)/A^+$ is isometric order isomorphic to $C^*(Z; (Z-E) \cup Y)$. We claim that each class of A^* contains one and only one member of $C^*(Z; (Z-E) \cup Y)$. Indeed, let $m \in A^*$ and suppose $n_1, n_2 \in m$ each were in $C^*(Z; (Z-E) \cup Y)$. Then $n_1 - n_2 = w \in W \cap C^*(Z; (Z-E) \cup Y)$. Hence, there is a measure λ on Y such that for each Borel $B \subseteq Z$ (using (1.4)' and the representation (1.5) for W)

$$(1.6) \quad w(B) = w(B \cap (Z-E)) - \int_{Z-E} \pi_z(B) dw + \lambda(B \cap Y) - \pi_0(B)\lambda(Y).$$

If w vanishes on $Z-E$, then $w(B) = \lambda(B \cap Y) - \pi_0(B)\lambda(Y)$. Using $w(Y) = 0$ we get $\lambda(Y) = 0$. But then, since w vanishes on Y , $\lambda = 0$ and so $w \equiv 0$. Thus $n_1 = n_2$.

On the other hand, let $n \in m$. We define a measure $\lambda(n)$ on Y by

$$(1.7) \quad \begin{aligned} \lambda(n)(\{q_0\}) &= 0, \\ \lambda(n)(A) &= \int_{Z-E} \pi_z(A) dn + n(A) + \pi_0(A)\lambda(n)(Y) \end{aligned}$$

for each Borel $A \subseteq Y - \{q_0\}$, where $\lambda(n)(Y)$ is found by consistency. Let $w(n)$ be the element of W determined by n and $\lambda(n)$ by (1.3). Then $n - w(n) \in m$. Using (1.4)'' to (1.4)''' and (1.7) one easily verifies that $w(n)(B) = n(B)$ for each Borel set $B \subseteq (Z-E) \cup Y$. Hence, $n - w(n) \in C^*(Z; (Z-E) \cup Y)$ and the claim is established.

The element $n - w(n)$ depends only on the class m and not on the particular representative n . We may therefore define a map $\phi: A^* \rightarrow C^*(Z; (Z-E) \cup Y)$ by $\phi(m) = n - w(n)$ for each $m \in A^*$ and any representative $n \in m$. Obviously, ϕ is a linear, one-to-one map of A^* onto $C^*(Z; (Z-E) \cup Y)$. It is positive.

Indeed, let m be positive and so n is positive. We must show that $(n - w(n))(B) \geq 0$ for each Borel $B \subseteq Z$. This is trivial for $B \subseteq (Z - E) \cup Y$ so assume $B \subseteq E - Y$. But then (1.4)' yields

$$(1.8) \quad w(n)(B) = - \int_{Z-E} \pi_z(B) dn - \pi_0(B) \lambda(n)(Y)$$

and so $n - w(n)$ is indeed positive.

Last, we need show that ϕ is an isometry to complete the proof. Let $n \in m \in A^*$. Then

$$\|\phi(m)\| = |\phi(m)|(E - Y) = |n - w(n)|(E - Y) \leq |n|(E - Y) + |w(n)|(E - Y).$$

For any Borel $B \subseteq E - Y$, using (1.7) and (1.8), we get

$$\begin{aligned} w(n)(B) &= - \int_{Z-E} \pi_z(B) dn - \pi_0(B) \left(\int_{Z-E} \pi_z(Y) dn + n(Y) \right) / (1 - \alpha) \\ &= - \int_{Z-E} \left(\pi_z(B) + \pi_z(Y) \frac{\pi_0(B)}{1 - \alpha} \right) dn - n(Y) \frac{\pi_0(B)}{1 - \alpha}. \end{aligned}$$

Then

$$\begin{aligned} |w(n)|(E - Y) &\leq \int_{Z-E} \left(\pi_z(E - Y) + \pi_z(Y) \frac{\pi_0(E - Y)}{1 - \alpha} \right) d|n| + |n|(Y) \frac{\pi_0(E - Y)}{1 - \alpha} \\ &\leq \int_{Z-E} (\pi_z(E - Y) + \pi_z(Y)) d|n| + |n|(Y) \quad \text{by (1.1)} \\ &\leq |n|(Z - E) + |n|(Y). \end{aligned}$$

Thus, we finally get $\|\phi(m)\| \leq |n|(E - Y) + |n|(Z - E) + |n|(Y) = |n|(Z) = \|n\|$.

Since $\|m\| = \inf_{n \in m} \|n\|$, we have $\|m\| = \|\phi(m)\|$.

The same proof also establishes the following [cf. 7, Theorem 1.2].

Corollary 1.5. *Let V, Z, E be as in Theorem 1.4. Let X be a closed proper subset of $Z - E$. Let $x \rightarrow \mu_x$ be a weak* continuous map of X into $P_1(C(Z))$. Let $X = X_1 \cup X_2$ where $X_2 = \{x \in X \mid \mu_x = \delta(x)\}$. We assume*

1. *For each $x \in X \cap (Z - E)$, $\mu_x = \pi_x$.*
2. *For each $x \in X_1$, $\mu_x(X_1 \cup (Z - E)) = 0$.*
3. *$X_1 \cup (Z - E) \neq Z$.*

Then $A = \{f \in V \mid f(x) = \mu_x(f) \text{ for each } x \in X\}$ is a nontrivial simplex space with the relative norm and order. Further, A^ is isometrically order isomorphic to $C^*(Z; X_1 \cup (Z - E))$.*

2. The characterization. A subset F of a convex compact set K is called a *face* if it is convex and satisfies the following condition: if $\alpha x + (1 - \alpha)y \in F$ with $x, y \in K$ and $0 < \alpha < 1$, then $x, y \in F$. The following extension theorem is a well-known consequence of the Edwards separation theorem [4].

Lemma 2.1. *Let F be a closed face of a simplex K . Suppose f_1, f_2 are continuous*

affine functions on K . Let $\bar{g}$ be a continuous affine function on F satisfying $\alpha \geq \bar{g} \geq f_1, f_2$ for some $\alpha \in \mathbf{R}$. Then there exists a continuous affine extension g of $\bar{g}$ to K which satisfies $\alpha \geq g \geq f_1, f_2$.

Lemma 2.2. *Let V be a simplex space. Let $q \in Z(V) - EP_1(V)$. Suppose there exists $p \in \text{supp } \pi_q \cap EP_1(V)^+$ and a net $\{x_\beta\} \subseteq EP_1(V)^+ - \{p\}$ which converges weak* to q . Suppose, further, there is an element $f \in V$ such that*

1. $x_\beta(f) = 0 = q(f)$ for all β .
2. $p(f) > 0$.

Then there exists a collection of closed maximal ideals I_β such that $\bigcap I_\beta$ is not an ideal. If the net is a sequence, then the collection of ideals is countable.

Proof. Let $F_\beta = \{\alpha x_\beta \mid 0 \leq \alpha \leq 1\}$. Then F_β is a maximal face containing zero of $P_1(V)$. Let I_β be the annihilator of F_β within V . Then I_β is a closed maximal ideal [5, Corollary 3.2]. Since $x_\beta(f) = 0$, $f \in I_\beta$ and so $f \in \bigcap I_\beta$.

Suppose $\bigcap I_\beta$ is an ideal. Then there would be a $v \in V^+$, $v \in \bigcap I_\beta$ and $v \geq f$. Thus, $v \in I_\beta$ for each β and so $x_\beta(v) = 0$. Therefore $q(v) = \lim x_\beta(v) = 0$. Since $v \in V^+$, it is zero on the smallest closed face containing q and so $p(v) = 0$. However, this contradicts the assumptions that $p(f) > 0$ and $v \geq f$.

We are now prepared for our characterization.

Theorem 2.3. *Let V be a separable simplex space. Then the following are equivalent:*

1. V is an M -space.
2. The intersection of an arbitrary collection of closed (maximal) ideals is always an ideal.
3. The intersection of a countable collection of closed (maximal) ideals is always an ideal.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

Not (1) $\Rightarrow$ not (3). Assume V is not an M -space. Let $Z = Z(V)$ and $E = EP_1(V)^+$. Then there is an element $q \in Z - E$ such that $\text{supp } \pi_q$ has at least two points ([10, Theorem 2], [6, Corollary 2.6]). So choose distinct points $p_1, p_2 \in E \cap \text{supp } \pi_q$. Since q is in the closure of E , there is a sequence $\{x_n\} \subseteq E - \{p_1, p_2\}$ such that $\lim x_n = q$.

Let A be defined by $A = \{v \in V \mid x_n(v) = q(v) = \pi_q(v), n = 1, 2, \dots\}$. Then A is a nontrivial simplex space with $EP_1(A)^+ = E - \{x_n \mid n = 1, 2, \dots\}$ by Theorem 1.4. Hence, p_1 and p_2 are in $EP_1(A)^+$. Let $F = \{\alpha p_1 + \beta p_2 \mid 0 \leq \alpha, 0 \leq \beta, \alpha + \beta \leq 1\}$. Then F is a closed face of the simplex $P_1(A)$. On F , define continuous affine functions $\bar{g}_1$ and $\bar{g}_2$ by

$$\bar{g}_1(\alpha p_1 + \beta p_2) = \alpha, \quad \bar{g}_2(\alpha p_1 + \beta p_2) = \beta.$$

Then $\bar{g}_i \geq 0$, $i = 1, 2$, on F and so there exist elements $g_1, g_2 \in A$ such that $g_i \geq 0$ and $g_i|_F = \bar{g}_i$ by Lemma 2.1. Since $p_1(g_1) = 1$ and $p_2(g_2) = 1$, we must have $q(g_1) > 0$ and $q(g_2) > 0$. Let

$$f = g_1 - (q(g_1)/q(g_2))g_2.$$

Then obviously $f \in A \subseteq V$, $q(f) = 0$ and $p_1(f) > 0$. Since $f \in A$, we also have $x_n(f) = 0$, $n = 1, 2, \dots$. Hence, Lemma 2.2 implies that (3) is not true.

We note that Theorem 1.4 and Lemma 2.2 allow us to conclude more than just Theorem 2.3. Let V be a standard simplex space and suppose $q \in Z(V) - EP_1(V)^+$ does not lie in the rays of $P_1(V)$. Suppose we could find a net $\{x_\beta\} \subseteq EP_1(V)^+$ converging to q such that $Y = \{x_\beta\}^-$ satisfies the hypotheses of Theorem 1.4. We could then conclude the existence of a collection of closed ideals whose intersection is not an ideal.

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DEPARTMENT OF MATHEMATICS, CARNEGIE-MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA 15213

Current address: Department of Mathematics, University of Massachusetts, Amherst, Massachusetts 01002