

ON THE IDEAL STRUCTURE OF BANACH ALGEBRAS

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ABSTRACT. For Banach algebras A in a class which includes all group and function algebras, we show that the family of ideals of A with the same hull is typically quite large, containing ascending and descending chains of arbitrary length through any ideal in the family, and that typically a closed ideal of A whose hull meets the Šilov boundary of A cannot be countably generated algebraically.

Guided by the results of [6], we explore here some consequences of the Cohen factorization theorem [3] for the ideal structure of a commutative Banach algebra A . If we denote the maximal ideal space of A by $\mathfrak{M}_A$ and define the zero set of an ideal I of A to be $Z(I) = \bigcap \{Z(\hat{f}) = \hat{f}^{-1}(0) : f \in I\} \subset \mathfrak{M}_A$, we develop results which indicate that typically the class of ideals of A with the same zero set is quite large, containing ascending and descending chains of arbitrary length, and that typically a closed ideal of A whose zero set meets the Šilov boundary of A cannot be (algebraically) countably generated. For example, if G is a locally compact Abelian group, there is an ideal of $L^1(G)$ strictly between any two ideals $I \subsetneq J$ which have the same zero set whenever either I or J is closed (compare [17, 7.7.2]). Further, a closed ideal J of $L^1(G)$ can be countably generated only if $Z(J)$ is open-closed, and a maximal ideal can be countably generated only if G is finite (compare [11, 2.1], [5, Corollary 3]).

Algebraic applications of Cohen's theorem have been few to date (principally [4, 4.7]), but because the result transforms a purely analytic condition on A (existence of a bounded approximate identity) into a purely algebraic conclusion (factorization of elements), it provides a most appropriate tool for just such applications.

1. Chains of ideals. Let A be a commutative algebra⁽¹⁾ over a field F . An ideal of A will be a subspace over F closed under multiplication from A . If I and J are ideals, IJ will denote $\{fg : f \in I, g \in J\}$, not the ideal generated by this set. We begin with a purely algebraic remark.

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⁽¹⁾ All algebras discussed will be assumed commutative.

Lemma 1.1. *If $I \subset J$ are ideals of A and there are no ideals of A strictly between them, then either $AJ \subset I$ or $MJ \subset I$ where M is some maximal ideal of A containing I .*

Proof. Let M be the ideal $\{f \in A: fJ \subset I\}$. If $A = M$ we are done; if not, take any $g \in A \setminus M$. $kg \notin I$ for some $k \in J$, and $I + kgA + Fkg$ is an ideal of A between I and J which properly contains I . Thus $J = I + kgA + Fkg$, and for each $f \in A$ we may write $fk = i + kga + \alpha kg$, $i \in I$, $a \in A$, $\alpha \in F$; that is, $k(f - ga - \alpha g) \in I$. Since $k \notin I$, we also have $J = I + kA + Fk$, so that $(f - ga - \alpha g)J = (f - ga - \alpha g)I + k(f - ga - \alpha g)A + Fk(f - ga - \alpha g) \subset I$. This means $f - ga - \alpha g \in M$; that is, $A = M + gA + Fg$. Thus M is a maximal ideal of A .

Corollary 1.2. *Suppose $I \subsetneq J$ are ideals of A contained in the same family of maximal ideals of A . Then if either $J^2 = J$ or I is semiprime, there is an ideal of A strictly between I and J .*

Proof. If not, 1.1 implies $J^2 \subset I$, since every maximal ideal of A which contains I also contains J . But then if $J^2 = J$, $J = I$; if I is semiprime, $J = I$ also.

Corollary 1.3. *Suppose X is locally compact and $I \subsetneq J$ are ideals of $C_0(X)$ with the same zero set. If either I or J is semiprime, there is an ideal of $C_0(X)$ strictly between them.*

Proof. The maximal ideals of $C_0(X)$ correspond to the points of X , so that I and J are contained in the same maximal ideals. If J is semiprime, $J^2 = J$. For if $f \in J$, $|f|^2 = f\bar{f} \in J$, and hence $|f|$, $|f|^{1/2}$, $|f|^{1/4} \in J$ as well. But then $f = |f|^{1/2}(|f|^{1/4} \cdot |f|^{1/4} \text{ sign } f) \in J^2$. Apply 1.2.

Corollary 1.4. *If $I \subsetneq J$ are semiprime ideals of $C_0(X)$ with the same zero set, J/I has no maximal or minimal ideals.*

A subset B of a Banach algebra is said to have approximate identity if there is a net $\{e_\alpha\}$ on B , uniformly bounded in norm, for which $\|e_\alpha f - f\| \rightarrow 0$ for each $f \in B$. A principal tool in our work is the following.

Theorem 1.5. *If I and J are closed ideals in a Banach algebra and either has approximate identity, then $IJ = I \cap J$.*

Proof. If I has approximate identity, apply Cohen's theorem to obtain, for each $z \in I \cap J \subset I$, $x, y \in I$ such that $z = xy$ where y is chosen from the closed ideal generated by z . Since $I \cap J$ is closed, this means $y \in I \cap J \subset J$; hence $I \cap J = IJ$. Because $IJ = JI$, this also obtains when J has approximate identity.

Theorem 1.6. *Let A be a Banach algebra with approximate identity. Suppose $I \subsetneq J$ are ideals of A with $Z(I) = Z(J)$, and that for each $p \in Z(I)$,*

$I_p = \{f \in A: \hat{f}(p) = 0\}$ has approximate identity. Then if either I or J is closed in A , there is an ideal of A strictly between them.

Proof. 1.5 implies that $A^2 = A$, and from this it follows that every maximal ideal M of A is regular; that is, A/M has identity. For $A^2 \not\subset M$, so that $(A/M)^2 \neq 0$; since A/M is a simple commutative algebra, this means it is actually a field [15, 2.1.5, p. 45]. Thus every maximal ideal of A is closed and those which contain I and J are of the form I_p , $p \in Z(I)$.

Suppose there are no ideals between I and J . Then by 1.1, $AJ \subset I$ or $MJ \subset I$ for some maximal ideal M containing I (and hence J). If J is closed, then since M and A have approximate identity, 1.5 implies that $J = J \cap A = AJ$ and $J = J \cap M = MJ$, so that in either case above, $J = I$, a contradiction. If I is closed, then again because A and M have approximate identities, we see that $J \subset \overline{AJ} \subset I$ or $J \subset \overline{MJ} \subset I$, hence $I = J$.

Corollary 1.7. *With I and J as above, the algebra J/I does not have the ascending chain condition on ideals when J is closed, and does not have the descending chain condition when I is closed.*

Corollary 1.8 (compare [17, 7.7.2, p. 183]). *Let G be a locally compact Abelian group. If $I \subsetneq J$ are ideals of $L^1(G)$ with the same zero set, and if one of them is closed, then there is an ideal of $L^1(G)$ strictly between them.*

Proof. $L^1(G)$ has approximate identity, as does each of its maximal ideals [14, p. 151]. Apply 1.6.

The maximal ideal space of $L^1(G)$ is of course the dual group Γ , which we may also view as a subset of the maximal ideal space of the measure algebra $M(G)$. For $\gamma \in \Gamma$, let K', K be compact neighborhoods of γ in Γ with $K' \subset \text{int } K$, and choose a bounded net $\{j_\alpha\}$ on $L^1(G)$, indexed by the neighborhoods $\{U_\alpha\}$ of γ contained in K' , for which $\text{supp } \hat{j}_\alpha \subset K$, $\hat{j}_\alpha|_{U_\alpha} = 1$ and $\|j_\alpha * g\|_1 \rightarrow 0$ when $\hat{g}(\gamma) = 0$ (cf. [14, p. 151]). Then notice $\mu_\alpha = \delta_0 - j_\alpha \in M(G)$ is a bounded approximate identity for the maximal ideal $I_\gamma = \{\mu \in M(G): \hat{\mu}(\gamma) = 0\}$ (here δ_0 denotes point mass of 0). For if $h \in L^1(G)$ is chosen so that $\hat{h}|_K = 1$, then, for each $\mu \in I_\gamma$, $\|\mu_\alpha * \mu - \mu\| = \|j_\alpha * \mu\|_1 = \|j_\alpha * h * \mu\|_1 \rightarrow 0$. From 1.6 we therefore conclude

Corollary 1.9. *If $I \subsetneq J$ are ideals in $M(G)$ with $Z(I) = Z(J) \subset \Gamma$, there is an ideal of $M(G)$ strictly between them whenever either is closed.*

The assumption in 1.8 that I or J is closed cannot be deleted. For suppose there is an $f \in L^1(G)$ whose zero set $Z(\hat{f})$ is exactly a nonisolated point $p \in \Gamma$. Such an f will exist whenever Γ is nondiscrete and second countable. Set $I = f * I_p$ and $J = f * L^1(G)$. Certainly $Z(I) = p = Z(J)$ and $I \subsetneq J$. For if $g \in L^1(G) \setminus I_p$,

$f * g \in J \setminus I$, since otherwise $f * g = f * b$ for some $b \in I_p$, and taking transforms and dividing out $\hat{f}$, we obtain $\hat{g} = \hat{b}$ on $\Gamma \setminus \{p\}$; hence $g = b \in I_p$, a contradiction. However there is no ideal K of $L^1(G)$ strictly between I and J . For if K is any ideal of $L^1(G)$ contained in J and properly containing I , and if $k \in K \setminus I$ with $k = f * b$, $b \in L^1(G) \setminus I_p$, we have $L^1(G) = I_p + b * L^1(G) + Cb$, because I_p is a maximal ideal. Thus $J = f * L^1(G) = f * I_p + k * L^1(G) + Ck \subset K$.

If A is a function algebra on a compact space X , let ∂A denote its Choquet boundary (cf. [2]). Similar to 1.9 we have

Corollary 1.10. *Suppose $I \subsetneq J$ are ideals of a function algebra with $Z(I) = Z(J) \subset \partial A$. If either I or J is closed, there is an ideal strictly between them.*

Proof. For $p \in X$, the maximal ideal $I_p = \{f \in A; f(p) = 0\}$ has approximate identity iff $p \in \partial A$ [2, 1.6.3, 2.3.4]. We may therefore apply 1.6.

If A has unique representing measures on X , then $\partial A = X$, and 1.10 applies whenever $Z(I) = Z(J) \subset X$. 1.10 also provides new ideals even in function algebras where all the closed ideals are known. For example, in the disc algebra A , every nonzero closed ideal is of the form $FI_K = \{Fg; g \in A, g|_K = 0\}$ where F is an inner function and K is a closed set of zero measure on the circle T [13, p. 85]. If $p \in T = \partial A$, $I = (z - p)A$ is an ideal of A , proper and dense in I_p , and $Z(I) = p = Z(I_p)$. Thus although there are no closed ideals between I and I_p , 1.10 yields an infinite ascending chain of ideals of A between them. More generally if we set

$$u_\alpha(z) = (z - p) \exp(\alpha(z + p)/(z - p)),$$

then every closed ideal of A with zero set p is of the form $J_\alpha = \overline{u_\alpha A}$, $\alpha \geq 0$, [13, p. 88]. But $I_\alpha = u_\alpha A$ is then proper and dense in J_α , $Z(I_\alpha) = Z(J_\alpha)$, and we may obtain a chain of (nonclosed) ideals between them. This example also reveals the necessary role of the approximate identity in 1.6: if p is in the open unit disc, the family of ideals of A which have p as zero set is the discretely descending chain $(z - p)^n A$, $n = 1, 2, \dots$, [9, p. 235].

Actually there is an abstract reason why the disc algebra has a dense chain of ideals at every maximal ideal on its boundary.

Proposition 1.11. *If E is a nonopen peak set for a function algebra A , there are infinite ascending and descending chains of ideals of A whose zero sets are E and which are all densely contained in $I_E = \{f \in A; f|_E = 0\}$.*

Proof. Choose an $b \in A$ which peaks to 1 on E , and set $f = 1 - b$ and $I_n = f^n I_E$, $n = 0, 1, \dots$; $Z(I_n) = Z(f^n) = Z(f) = E$ and $I_{n+1} \subsetneq I_n \subset I_E$. For

if $I_{m+1} = I_m$ for some m , $f^{m+1} \in I_m = I_{m+1}$; hence $f^{m+1} = f^{m+1}g$, $g \in I_E$, so that $\chi_{E'} = g \in A$ and E is open-closed in X , a contradiction. Induction shows that each I_n is dense in $I_0 = I_E$. For assuming I_n is dense, $I_{n+1} = (1-b)I_n$ will be dense if $(1-b)I_E$ is. But for fixed $g \in I_E$, $(1-b^k)g = (1-b) \cdot (1+b+\dots+b^{k-1})g \in (1-b)I_E$ and $\|(1-b^k)g - g\|_\infty = \|b^k g\|_\infty \rightarrow 0$. Since I_E has approximate identity, 1.5 implies $I_E^2 = I_E$. 1.2 then yields an infinite ascending chain of ideals of A , $\{J_n\}$, between I_1 and $I_0 = I_E$.

For example, we obtain such ideals in the disc algebra whenever E is a nonempty closed set of measure zero on the circle [13, p. 81]. Likewise, if A is a logmodular or hypo-Dirichlet [1, 3.1] function algebra on a metrizable space X , then we obtain dense descending and ascending chains of primary ideals at every nonisolated point of X .

Chains of ideals may be constructed in other ways. If A is a Banach algebra and E is a closed subset of its maximal ideal space, set $F_E = \{f \in A: \hat{f} \text{ vanishes on a neighborhood of } E\}$, $K_E = \{f \in F_E: \hat{f} \in C_c(\mathfrak{M}_A)\}$ and $I_E = \{f \in A: \hat{f}|_E = 0\}$. Certainly $K_E \subset F_E \subset I_E$ are ideals of A ; if A is regular, $Z(K_E) = E = Z(I_E)$ and every ideal of A whose zero set is E lies between K_E and I_E [14, 25D, p. 84].

Theorem 1.12 (compare [6, 4.5]). *If J is an ideal of $L^1(G)$ strictly between K_E and I_E , there are ideals $\underline{J}$ and $\overline{J}$ of $L^1(G)$ for which*

$$K_E \subset \underline{J} \subset J \subset \overline{J} \subset I_E$$

with all inclusions proper.

Proof. Since I_E is a closed ideal of $L^1(G)$ and $Z(J) = Z(I_E)$, the existence of $\overline{J}$ follows from 1.8. Since K_E is semiprime, 1.2 yields $\underline{J}$.

By repeated use of 1.12 we see that through any such J we may thread infinite ascending and descending chains of ideals of $L^1(G)$ whose zero sets are E . Notice this happens whether or not E is a set of spectral synthesis (in which case, of course, every ideal in the chain will be dense in I_E). In particular, infinite ascending and descending chains between K_E and I_E will exist whenever $K_E \neq I_E$ (compare [6, 1.11, 1.12]). If G is second countable, every closed set in Γ is of the form $\hat{f}^{-1}(0)$ for some $f \in L^1(G)$. So that if Γ is also connected, we have $K_E \neq I_E$ (and hence chains of ideals) for every nonempty proper closed set $E \subset \Gamma$. Of course, in any group this is also true whenever E is a set of non-synthesis. For $E = \emptyset$, we obtain infinite ascending and descending chains of dense ideals in $L^1(G)$ whenever G is second countable and nondiscrete. In view of 1.9, similar results hold for $M(G)$.

2. Finitely generated ideals. Motivated by [6, 2.6] and [5, Corollary 3], we

combine 1.5 and a theorem due to Gleason [10, 2.1] to show here that closed ideals in Banach algebras can rarely be generated algebraically by a finite number of their elements (2.3).

Lemma 2.1. *If A is a regular Banach algebra with identity and J is a closed ideal of A , then the algebra $J + C$ is also regular.*

Proof. Standard arguments [8, p. 12] show that $J + C$ is a Banach algebra, and that restriction $r: \mathfrak{M}_A \rightarrow \mathfrak{M}_{J+C}$ is an identification which collapses $Z(J)$ to a point. Thus if F is a closed set in $\mathfrak{M}_{J+C}$ not containing a point $p \in \mathfrak{M}_{J+C}$, either $r^{-1}(F) \cup Z(J)$ and $r^{-1}(p)$ or $r^{-1}(F)$ and $Z(J) = r^{-1}(p)$ are disjoint compact subsets of $\mathfrak{M}_A$. In the first case, normality of $\mathfrak{M}_A$ and regularity of A yield a $g \in A$ whose transform vanishes on a neighborhood of $r^{-1}(F) \cup Z(J)$ and is identically 1 on $r^{-1}(p)$ [14, 25C, p. 84]. $g \in K_{Z(J)} \subset J + C$ [14, 25D] then separates p and F in $\mathfrak{M}_{J+C}$. In the second case, we select a $g \in A$ with transform vanishing on a neighborhood of $Z(J)$ which is 1 on $r^{-1}(F)$; again $g \in J + C$ serves to separate F and p .

Theorem 2.2. *If A is a regular Banach algebra with identity and J is a closed, finitely generated ideal of A with approximate identity, then $Z(J)$ is open-closed in $\mathfrak{M}_A$.*

Proof. J is the maximal ideal in $J + C$ which corresponds to the point $r(Z(J))$ in $\mathfrak{M}_{J+C}$. Because J has approximate identity, it is finitely generated as an ideal of $J + C$. For if $J = f_1 A + \cdots + f_n A$, $f_i \in J$, 1.5 yields $g_i, b_i \in J$ for which $f_i = g_i b_i$, so that $J = g_1 b_1 A + \cdots + g_n b_n A = g_1 (J + C) + \cdots + g_n (J + C)$. Because $J + C$ is regular (2.1), its Šilov boundary is $\mathfrak{M}_{J+C}$, and Gleason's theorem [11, 2.1] combined with the maximum modulus principle for analytic varieties implies that $r(Z(J))$ is isolated in $\mathfrak{M}_{J+C}$. Thus $Z(J) = r^{-1}(r(Z(J)))$ is open-closed in $\mathfrak{M}_A$.

Corollary 2.3. *If $\mathfrak{M}_A$ is connected, 0 and A are the only closed ideals of A with approximate identity which can be finitely generated.*

Actually, 2.2 holds whether or not A has identity.

Theorem 2.4. *If A is a regular Banach algebra and J is a closed, finitely generated ideal of A with approximate identity, then $Z(J)$ is open-closed in $\mathfrak{M}_A$.*

Proof. Set $\tilde{A} = A \times C$ and make $\tilde{A}$ into a Banach algebra with identity $(0, 1)$ via $(f, \alpha)(g, \beta) = (fg + \alpha g + \beta f, \alpha\beta)$ and $\|(f, \alpha)\| = \|f\| + |\alpha|$. The embedding $b \rightarrow \tilde{b}$ [$\tilde{b}(x, \alpha) = b(x) + \alpha$] is a homeomorphism $\mathfrak{M}_A \rightarrow \mathfrak{M}_{\tilde{A}}$, and $\mathfrak{M}_{\tilde{A}} \setminus \mathfrak{M}_A$ is a single point ∞ corresponding to the maximal ideal $A = A \times 0$. Thus $\mathfrak{M}_{\tilde{A}}$ is the one-point compactification of $\mathfrak{M}_A$, so that $\tilde{A}$ is a regular algebra also. For if F

is a closed set in $\mathfrak{M}_A^\sim$ not containing ∞ , F is a compact subset of $\mathfrak{M}_A$, and there is an $f \in A$ with $\tilde{b}(f, 0) = b(f) = 1$ for all $\tilde{b} \in F$ [14, 25C]. Since $\infty(f, 0) = 0$, $(f, 0)$ separates ∞ and F . If F is a closed set of $\mathfrak{M}_A^\sim$ not containing $p \in \mathfrak{M}_A$, we choose $f \in A$ with $\hat{f}(p) = 1$ and $\text{supp } \hat{f} \subset \mathfrak{M}_A$ disjoint from $F \cap \mathfrak{M}_A$. Then $(f, 0)$ separates p and F in $\mathfrak{M}_A^\sim$. Certainly $\tilde{J} = J \times 0$ is a closed, finitely generated ideal of $\tilde{A}$ with approximate identity. So $\{\infty\} \cup Z(J) = Z(\tilde{J})$ is open-closed in $\mathfrak{M}_A^\sim$ (2.2); hence $Z(J) = Z(\tilde{J}) \cap \mathfrak{M}_A$ is open(-closed) in $\mathfrak{M}_A$.

For example in the regular algebra $L^1(G)$, the closed ideal I_E will have approximate identity whenever E is a closed subgroup in the dual group Γ . We conclude from 2.4 that, for instance, I_Z is not finitely generated in $L^1(R)$. Actually the only finitely generated, closed ideal in $L^1(R)$ is the zero ideal (cf. §3). This does not follow from 2.4 however. For the approximate identity assumption here is quite restrictive. If we let $\mathfrak{R}(\Gamma)$ denote the ring of subsets of Γ generated by the cosets of all subgroups of Γ , we have

Lemma 2.5. *An ideal J in $L^1(G)$ has approximate identity iff $Z(J) \in \mathfrak{R}(\Gamma)$.*

Proof. Suppose J has an approximate identity $\{e_\alpha\}$ bounded by M . For each $\gamma \in \Gamma \setminus Z(J)$, there is an $f \in J$ whose Fourier transform is 1 at γ . Since $\hat{e}_\alpha(\gamma)\hat{f}(\gamma) \rightarrow \hat{f}(\gamma)$, we see that $\hat{e}_\alpha \rightarrow \chi_{Z(J)\gamma}$ pointwise on Γ . If we view each e_α as a measure on $\bar{G}$, the Bohr compactification of G , then as a net in the weakly compact ball of measures $\{\mu \in M(\bar{G}): \|\mu\| \leq M\}$, $\{e_\alpha\}$ must have a subnet $\{e_{\alpha\beta}\}$ converging weakly to some $\mu \in M(\bar{G})$. Since each $\gamma \in \Gamma$ extends to a continuous function on $\bar{G}$, $\hat{e}_{\alpha\beta}(\gamma) \rightarrow \hat{\mu}(\gamma)$ pointwise on Γ . Thus $\hat{\mu} = \chi_{Z(J)\gamma}$ and μ is an idempotent measure on $\bar{G}$. By Cohen's theorem [17, 3.1.3, p. 60], $Z(J) \in \mathfrak{R}(\Gamma)$.

Conversely, the arguments of [18, 1.7, 2.6] imply that if $Z(J) \in \mathfrak{R}(\Gamma)$, there is a constant K so that given $\epsilon > 0$ and $f_1, \dots, f_n \in I_E$, there is an $e \in F_E$ for which

$$(\#) \quad \|e\|_1 \leq K \text{ and } \|e * f_i - f_i\|_1 < \epsilon, \quad i = 1, 2, \dots, n.$$

But if $\epsilon > 0$ and $f_1, \dots, f_n \in J$ are given, set $\delta = \min\{1, \epsilon/(2 \max \|f_i\|_1)\}$, choose e as above for $\epsilon/2$, pick $u \in L^1(G)$ with $\|u\|_1 = 1$ and $\|e - u * e\|_1 < \delta/2$ [17, 1.1.8], and then find $v \in L^1(G)$ such that $\hat{v}$ has compact support and $\|u - v\|_1 < \delta/(2\|e\|_1 + 1)$ [17, 2.6.6]. Then

$$\|e * v - e\|_1 < \delta, \quad \|e * v\|_1 \leq \|e\|_1(\|u - v\|_1 + \|u\|_1) \leq 2\|e\|_1 \leq 2K,$$

and for each i ,

$$\|e * v * f_i - f_i\|_1 \leq \|f_i\|_1 \|e * v - e\|_1 + \|e * f_i - e\|_1 < \epsilon.$$

But $e * v \in K_E \subset J$, and we conclude that (#) holds with $2K$ in place of K and e chosen from J . Thus J has approximate identity.

In particular, $Z(J)$ must be a set of synthesis if J is to have approximate identity [18, 2.6]. Further, if J is a proper nonzero ideal and Γ is connected, $Z(J)$ must be contained in a finite union of cosets of proper closed subgroups of G ; in fact for $G = R$, $Z(J)$ must differ from some finite union of arithmetic progressions by at most a finite set [18, 1.5, 1.6].

Furthermore it is not clear that 2.4 holds if the approximate identity assumption on J is deleted—the principal difficulty being that ideals J finitely generated over A need not be finitely generated over $J + C$. Consider, for example, a semisimple Banach algebra A with 1 and a nonisolated point p in its Šilov boundary Γ_A . Suppose there is an $f \in A$ with $Z(\hat{f}) = \{p\}$. Such f will exist if A is regular and selfadjoint, and $\mathfrak{M}_A - \{p\}$ is σ -compact. Let $J = fA$ and suppose $J = g_1(J + C) + \cdots + g_n(J + C)$ for some $g_i = fb_i \in J$. For each $g \in A$,

$$fg = \sum_{i=1}^n f^2 b_i k_i + \alpha_i f b_i, \quad k_i \in A, \alpha_i \in C.$$

Taking transforms, dividing out $\hat{f}$ and using the assumptions on p and A , we conclude that each $g \in A$ can be written $g = g' + \sum_{i=1}^n \alpha_i b_i$, $g' \in J$, $\alpha_i \in C$; that is, $\{b_i + J\}$ spans the vector space A/J . In particular, the subspace I_p/J is finite dimensional with some basis $\{s_1 + J, \dots, s_k + J\}$. We conclude that $\{f, s_1, \dots, s_k\}$ generates I_p as an ideal of A . But this contradicts Gleason's theorem since p is nonisolated in Γ_A .

On the positive side, this method yields

Theorem 2.6. *If J is a finitely generated, closed ideal of finite codimension in a Banach algebra A , then $Z(J) \cap \Gamma_A$ is open-closed in Γ_A .*

Proof. Suppose first that the result is valid for every A with identity. For $\tilde{A} = A \times C$ and $\tilde{J} = J \times 0$, there is a vector space isomorphism $\tilde{A}/\tilde{A} \oplus A/J \approx \tilde{A}/\tilde{J}$. Hence $\dim \tilde{A}/\tilde{J} = \dim \tilde{A}/\tilde{A} + \dim A/J = 1 + \dim A/J < \infty$. Since $\tilde{J}$ is a finitely generated, closed ideal in $\tilde{A}$, $Z(\tilde{J}) \cap \Gamma_{\tilde{A}}$ is open in $\Gamma_{\tilde{A}}$. But the canonical embedding $\mathfrak{M}_A \rightarrow \mathfrak{M}_{\tilde{A}}$ carries Γ_A onto a subset of $\Gamma_{\tilde{A}}$ so that $Z(J) \cap \Gamma_A = Z(\tilde{J}) \cap \Gamma_A$ is open in Γ_A .

Now suppose A has identity. For each $p \in Z(J) \cap \Gamma_A$, I_p/J is a finite dimensional vector space with some basis $\{s_1 + J, \dots, s_n + J\}$. But if $f_1, \dots, f_m$ generate J as an ideal over A , this means $I_p = f_1 A + \cdots + f_m A + s_1 A + \cdots + s_n A$, a finitely generated ideal of A . By Gleason's theorem, $\{p\}$ is open in Γ_A ; hence $Z(J) \cap \Gamma_A$ is open-closed in Γ_A .

Thus in a regular algebra with connected maximal ideal space, no proper closed ideal of finite codimension can be finitely generated. For example, in the algebra $C^*[0, 1]$ of C^* functions on $[0, 1]$ with the C^* norm, no closed ideal I

whose zero set is finite can be finitely generated. For if $Z(I)$ is finite and we set $E_i = \bigcap \{Z(f^{(i)}): f \in I\}$, then it follows easily from Whitney's theorem [20, Theorem 1, p. 636] that $I = \{f \in C^r[0, 1]: f^{(i)}|_{E_i} = 0\}$. Because $E_0 \supset E_1 \supset \dots \supset E_r$, if we set $r = \max \{i: p \in E_i\}$ and $I_p^r = \{f \in C^r[0, 1]: f^{(i)}(p) = 0 \text{ for } 0 \leq i \leq r\}$ for each $p \in E_0$, we have $I = \bigcap \{I_p^r: p \in E_0\}$. But this yields a vector space isomorphism

$$C^r[0, 1]/I \approx \bigoplus_{p \in E_0} C^r[0, 1]/I_p^r,$$

and since E_0 is finite and each summand is finite dimensional (cf. [9, p. 206]), I has finite codimension. Actually, recent results of Roth [16], established with more direct methods, imply that 0 and $C^r[0, 1]$ are the only closed ideals in this algebra that can be finitely generated. In view of this, one might conjecture that 2.6 holds for a closed, finitely generated ideal of any codimension. In the next section we show that at least for group and function algebras, this is indeed the case.

3. Countably generated ideals in group and function algebras. Besides improving 2.6 for these algebras (3.1, 3.9), we provide, for compactly generated LCA groups G , a complete description of the countably generated, closed ideals of $L^1(G)$ (3.7).

Theorem 3.1. *Let G be a locally compact Abelian group with dual Γ . If J is a closed, countably generated ideal of $L^1(G)$, then $Z(J)$ is open-closed in Γ .*

Proof. Suppose instead that $\partial Z(J) \neq \emptyset$. If $\eta_0 \in \partial Z(J)$, the map $f \rightarrow \bar{\eta}_0 f$ [$\bar{\eta}_0 f(x) = \overline{\eta_0(x)} f(x)$] is a Banach algebra automorphism of $L^1(G)$ which carries J onto a closed, countably generated ideal whose zero set is $Z(J) - \eta_0$. We may therefore assume $0 \in \partial Z(J)$.

J is the smallest ideal of $L^1(G)$ containing some sequence $\{w_n\}$. Since $0 \in Z(J)$ and I_0 has approximate identity, $J = I_0 * J$ (1.5). If we choose $b_i \in I_0$ so that $\|b_i\|_1 \leq 1/2^i$ and $w_i \in b_i * J$, then J is the sum of its subspaces $\{b_i * J\}$. Let $\bigoplus_1^\infty J$ denote the direct sum of countably many copies of J , and J_n the subspace of sequences whose entries following the n th one are all zero. J_n is a Banach space with the norm $\|\{f_i\}\| = \max_{1 \leq i \leq n} \{\|f_i\|_1\}$, and with the final topology induced by the inclusions $J_n \hookrightarrow \bigoplus_1^\infty J$, $\bigoplus_1^\infty J$ is the strict inductive limit of the J_n 's. The mapping $T: \bigoplus_1^\infty J \rightarrow J$ given by $T(\{f_i\}) = \sum_i b_i * f_i$ is linear, continuous and surjective. A theorem of Dieudonné and Schwartz [7, Theorem 1, p. 72] implies T is open. In particular, if U denotes the set of all sequences in $\bigoplus_1^\infty J$ whose entries have norm no larger than 1, there is a $\delta > 0$ so that $B(0, \delta) \subset T(U)$. It follows that there is a constant $M > 0$ such that given $g \in J$, there are $g_1, \dots, g_p \in J$ with

$$(*) \quad g = \sum_{i=1}^p b_i * g_i \quad \text{and} \quad \|g_i\|_1 \leq M \|g\|_1, \quad i = 1, 2, \dots, p.$$

Since $\|b_i\|_1 \leq 1/2^i$, there is an n so large that $\sum_{i=n+1}^{\infty} M \|b_i\|_1 < 1/2$. Set $\epsilon = 1/2 M n$, $\delta = \epsilon / (2 \max_{1 \leq i \leq n} \{1 + \|b_i\|_1\})$ and find a compact set $K \subset G$ so large that the integral of each $\widehat{b_i}$ over the complement of K is less than δ . $W = \{\gamma \in \Gamma: |1 - (x, \gamma)| < \delta, x \in K\}$ is an open neighborhood of 0 in Γ . Since $0 \in \partial Z(J)$, we may choose a $\gamma_0 \in W \cap Z(J)^c$, and then a compact neighborhood V of 0 so that $V \subset W$ and $\gamma_0 + V - V \subset W \cap Z(J)^c$. Choose $s, t \in L^2(G)$ whose Plancherel transforms are X_V and $X_{\gamma_0 - V}$ respectively, and define $\widehat{k}(x) = s(x)t(x)/m(V)$ where m denotes Haar measure on Γ . Then [17, 2.6.1] $\widehat{k}$ is 0 off the compact set $\gamma_0 + V - V$, so that $\widehat{k} \in K_{Z(J)} \subset J$, and $1 = \widehat{k}(\gamma_0) \leq \|\widehat{k}\|_1 \leq \{m(\gamma_0 - V)/m(V)\}^{1/2} = 1$, so that $\|\widehat{k}\|_1 = 1 = \widehat{k}(\gamma_0)$. Further, since $\gamma_0 - V$ and V are contained in W , and the transform of each b_i vanishes at 0, the computations of [17, p. 50] yield $\|b_i * \widehat{k}\|_1 < \epsilon$ for $i = 1, 2, \dots, n$.

Let $\alpha_i = |\widehat{b_i}(\gamma_0)|/\widehat{b_i}(\gamma_0)$ when $\widehat{b_i}(\gamma_0) \neq 0$, 1 otherwise, and let $t_i = \alpha_i b_i$. Since α_i has modulus 1, the series $\sum_{i=1}^{\infty} t_i * \widehat{k}$ is absolutely convergent and defines an element g in the closed ideal J . (*) yields $f_1, \dots, f_p \in J$ with $g = \sum_{i=1}^p b_i * f_i$, and setting $g_i = f_i/\alpha_i$ we obtain $g = \sum_{i=1}^p t_i * g_i$ and $\|g_i\|_1 \leq M \|g\|_1$. In particular,

$$\begin{aligned} \sum_{i=1}^{\infty} |\widehat{b_i}(\gamma_0)| &= \sum_{i=1}^{\infty} \widehat{t_i}(\gamma_0) \widehat{k}(\gamma_0) = \widehat{g}(\gamma_0) \\ &= \sum_{i=1}^p |\widehat{b_i}(\gamma_0)| \widehat{g_i}(\gamma_0) \leq \sum_{i=1}^p |\widehat{b_i}(\gamma_0)| |\widehat{g_i}(\gamma_0)|. \end{aligned}$$

But then if $|\widehat{g_i}(\gamma_0)| < 1$ for each $i = 1, 2, \dots, p$, this inequality forces $\widehat{b_i}(\gamma_0) = 0$ for all i . Since J is the vector space sum of the $b_i * J$'s, this implies $\gamma_0 \in Z(J)$, a contradiction. We conclude that $|\widehat{g_j}(\gamma_0)| \geq 1$ for some j . Thus

$$\begin{aligned} 1 &\leq \|\widehat{g_j}\|_{\infty} \leq \|g_j\|_1 \leq M \|g\|_1 \\ &\leq \sum_{i=1}^n M |\alpha_i| \|b_i * \widehat{k}\|_1 + \sum_{i=n+1}^{\infty} M |\alpha_i| \|b_i\|_1 \|\widehat{k}\|_1 < M n \epsilon + 1/2 = 1. \end{aligned}$$

From this contradiction we conclude $Z(J)$ is open-closed.

Corollary 3.2. *If Γ is connected, no proper nonzero closed ideal of $L^1(G)$ can be countably generated.*

Corollary 3.3. *If $E \subset \Gamma$ is a set of nonsynthesis, then no closed ideal of $L^1(G)$ whose zero set is E can be countably generated.*

Proof. If J were such an ideal, $Z(J) = E$ would be open-closed; hence $F_E = I_E$, and by definition, E would be a set of synthesis.

Malliavin's theorem [17, 7.6.1] yields closed sets E of nonsynthesis in every nondiscrete LCA group, and Helson's result [17, 7.7.2] then yields ascending and descending chains of closed ideals in $L^1(G)$ whose zero sets are E . Whereas 3.3 implies that none of these can be generated algebraically by even an infinite set, they may be the *closure* of even principal ideals (cf. [17, 7.6.3, p. 174]).

The method of 3.1 can be used to yield more explicit information about the countably generated ideals in group algebras. We begin with

Theorem 3.4. $L^1(G)$ is a countably generated ideal of itself iff G is discrete.

Proof. Certainly if G is discrete, $L^1(G) = M(G)$ is generated by its identity. Suppose on the other hand that G is nondiscrete, but $L^1(G)$ is the smallest ideal of itself containing some sequence $\{w_n\}$. Since $L^1(G)$ has approximate identity, we may choose $b_i \in L^1(G)$ so that $\|b_i\|_1 \leq 1/2^i$ and $w_i \in b_i * L^1(G)$ (1.5). Thus $L^1(G)$ is the vector space sum of the $b_i * L^1(G)$'s, and, as before, the Dieudonné-Schwartz theorem yields a constant $M > 0$ such that given $g \in L^1(G)$ there are $g_1, \dots, g_p \in L^1(G)$ satisfying (*). Choose n so that $\sum_{i=n+1}^{\infty} M \|b_i\|_1 < 1/2$, set $\epsilon = 1/2Mn$ and choose a compact set $C \subset \Gamma$ so large that, off C , $|\hat{b}_i| < \epsilon/2$ for each $i = 1, 2, \dots, n$. Since Γ is not compact, there is some γ_0 in the complement of C . Because the b_i 's generate $L^1(G)$, $\hat{b}_i(\gamma_0) \neq 0$ for at least one i .

Letting $\alpha_i = |\hat{b}_i(\gamma_0)|/\hat{b}_i(\gamma_0)$ when $\hat{b}_i(\gamma_0) \neq 0$, 1 otherwise, and setting $s_i(x) = \alpha_i \overline{\gamma_0(x)} \hat{b}_i(x)$, we observe that for each $g \in L^1(G)$ there are $g_1, \dots, g_p \in L^1(G)$ such that $g = \sum_{i=1}^p s_i * g_i$ and $\|g_i\|_1 \leq M \|g\|_1$. For if $g'_1, \dots, g'_p \in L^1(G)$ are chosen to satisfy (*) for $\gamma_0 g$, then

$$g = \sum_{i=1}^p \bar{\gamma}_0(\alpha_i \hat{b}_i * g'_i / \alpha_i) = \sum_{i=1}^p s_i * \bar{\gamma}_0 g'_i / \alpha_i,$$

and, setting $g_i = \bar{\gamma}_0 g'_i / \alpha_i$, we have $\|g_i\|_1 = \|\bar{\gamma}_0 g'_i\|_1 / |\alpha_i| \leq M \|g\|_1$.

Let $\delta = \epsilon / (4 \max_{1 \leq i \leq n} \{1 + \|s_i\|_1\})$ and find a compact set $K \subset G$ so that the integral of each $|s_i|$ ($1 \leq i \leq n$) over K^c is less than δ . Let $W = \{\gamma \in \Gamma: |1 - (x, \gamma)| < \delta, x \in K\}$ and choose a compact neighborhood V of 0 in Γ such that $V - V \subset W$. Let $s, t \in L^2(G)$ be functions whose Plancherel transforms are X_V and X_{-V} respectively, and set $k(x) = s(x)t(x)/(m(V))$. $\hat{k}(0) = 1 = \|k\|_1$ [17, 2.6.1], and we have $\|s_i * k\|_1 < \epsilon$ for $i = 1, \dots, n$. For

$$s_i * k(x) = \int_G s_i(y) \{k(x-y) - k(x)\} dy + k(x) \hat{s}_i(0),$$

so that

$$\|s_i * k\|_1 \leq \int_G |s_i(y)| \|k_y - k\|_1 dy + |\hat{s}_i(0)| \|k\|_1.$$

Since $-V$, $V \subset W$ and $\|k\|_1 = 1$, the computations of [17, p. 50] show that the right-hand side is less than $\epsilon/2 + |\hat{s}_i(0)| = \epsilon/2 + |\hat{b}_i(\gamma_0)| < \epsilon$.

Let $g = \sum_{i=1}^{\infty} s_i * k$ and choose $g_1, \dots, g_p \in L^1(G)$ so that $\|g_i\|_1 \leq M \|g\|_1$ and $g = \sum_{i=1}^p s_i * g_i$. Since

$$\begin{aligned} \sum_{i=1}^{\infty} |\hat{b}_i(\gamma_0)| &= \sum_{i=1}^{\infty} \hat{s}_i(0) \hat{k}(0) \\ &= \sum_{i=1}^p |\hat{b}_i(\gamma_0)| \hat{g}_i(0) \leq \sum_{i=1}^p |\hat{b}_i(\gamma_0)| |\hat{g}_i(0)|, \end{aligned}$$

and not all the $\hat{b}_i(\gamma_0)$'s are zero, $|\hat{g}_j(0)| \geq 1$ for at least one j . Hence

$$1 \leq \|g_j\|_1 \leq \sum_{i=1}^n M \|s_i * k\|_1 + \sum_{i=n+1}^{\infty} M |\alpha_i| \|\bar{\gamma}_0 b_i\|_1 \|k\|_1 < Mn\epsilon + 1/2 = 1.$$

This contradiction completes the proof.

Corollary 3.5. *If Γ is a noncompact connected group, 0 is the only closed, countably generated ideal of $L^1(G)$.*

However, nontrivial countably generated ideals in $L^1(G)$ can exist.

Proposition 3.6. *Let G and H be LCA groups with duals Γ and Ω respectively. Suppose G is compact and Ω is connected. Then (i) if H is nondiscrete, 0 is the only closed, countably generated ideal of $L^1(G \times H)$, whereas (ii) if H is discrete, a closed ideal J in $L^1(G \times H)$ is finitely generated iff $Z(J) = E \times \Omega$ where E is a cofinite subset of Γ .*

Proof. Let J be a closed, countably generated ideal of $L^1(G \times H)$. Then $Z(J)$ is an open-closed $\mathcal{S}$ -set in $\Gamma \times \Omega$ (3.1), so $Z(J) = E \times \Omega$ and $J = I_{E \times \Omega}$, $E \subset \Gamma$. As before there is a sequence $\{b_n\}$ on $L^1(G \times H)$ with $\|b_n\|_1 \leq 1/2^n$, so that J is the sum of its subspaces $\{b_n * J\}$; in fact, there is a constant $M > 0$ so that given $g \in J$, there are $g_1, \dots, g_p \in J$ satisfying (*). Find $n > 0$ so that $\sum_{i=n+1}^{\infty} M \|b_i\|_1 < 1/2$, let $\epsilon = 1/2 Mn$ and choose $C \subset \Gamma \times \Omega$ compact so that $|\hat{b}_i(\gamma, \eta)| < \epsilon/2$ off C for each $i = 1, 2, \dots, n$.

(i) If $E = \Gamma$, we are done. If $\gamma_0 \in E^c$, then since Ω is not compact, $(\gamma_0, \eta_0) \notin C$ for some $\eta_0 \in \Omega$. Let $\alpha_i = |\hat{b}_i(\gamma_0, \eta_0)| / \overline{\hat{b}_i(\gamma_0, \eta_0)}$ if $\hat{b}_i(\gamma_0, \eta_0) \neq 0$, 1 otherwise. Define $s_i \in L^1(G \times H)$ by $s_i(x, y) = \alpha_i \overline{\eta_0(y)} b_i(x, y)$ and observe that, as before, given $g \in J$ there are $g_1, \dots, g_p \in J = I_{E \times \Omega}$ such that $g = \sum_{i=1}^p s_i * g_i$ and $\|g_i\|_1 \leq M \|g\|_1$. If we define $A_i \in L^1(H)$ by

$$A_i(w) = \int_G s_i(x, w) \overline{\gamma_0(x)} dx,$$

then since $|\hat{A}_i(0)| = |\hat{s}_i(\gamma_0, 0)| = |\hat{b}_i(\gamma_0, \eta_0)| < \epsilon/2$ for $i = 1, 2, \dots, n$, we may construct $k \in L^1(H)$ so that $\hat{k}(0) = 1 = \|k\|_1$ and $\|A_i * k\|_1 < \epsilon$ for $1 \leq i \leq n$ (cf. 3.4).

Since the transform of $\gamma_0 k: (x, y) \rightarrow \gamma_0(x) k(y)$ vanishes on $E \times \Omega$, $g = \sum_{i=1}^{\infty} s_i * \gamma_0 k \in J$, and hence for some $g_i \in J$ with $\|g_i\|_1 \leq M \|g\|_1$, we have $g = \sum_{i=1}^p s_i * g_i$. But then

$$\sum_{i=1}^{\infty} |\hat{b}_i(\gamma_0, \eta_0)| = \sum_{i=1}^{\infty} \hat{s}_i(\gamma_0, 0) \widehat{\gamma_0 k}(\gamma_0, 0) \leq \sum_{i=1}^p |\hat{b}_i(\gamma_0, \eta_0)| |\hat{g}_i(\gamma_0, 0)|,$$

and because at least one $\hat{b}_i(\gamma_0, \eta_0)$ is nonzero, this means $|\hat{g}_j(\gamma_0, 0)| \geq 1$ for some j . Since $s_i * \gamma_0 k(x, y) = \gamma_0(x) A_i * k(y)$, $\|s_i * \gamma_0 k\|_1 = \|A_i * k\|_1$, and we reach the following absurdity:

$$\begin{aligned} 1 &\leq \|g_j\|_1 \leq M \|g\|_1 \\ &\leq \sum_{i=1}^n M \|A_i * k\|_1 + \sum_{i=n+1}^{\infty} M |\alpha_i| \|b_i\|_1 \|\gamma_0 k\|_1 < Mn\epsilon + \frac{1}{2} = 1. \end{aligned}$$

(ii) Since Γ is discrete, the compact set C is contained in a product $F \times \Omega$, $F \subset \Gamma$ finite. But then if the complement of E in Γ is infinite, there is some $\gamma_0 \notin F \cup E$. Taking any $\eta_0 \in \Omega$, we have $(\gamma_0, \eta_0) \notin C \cup Z(J)$, and with the argument used above we obtain the required contradiction.

Of course, if the complement of E is a finite set $\{\gamma_1, \dots, \gamma_n\}$, then since $L^1(H)$ has identity δ_0 , $g(x, y) = \sum_{i=1}^n \gamma_i(x) \delta_0(y)$ is an $L^1(G \times H)$ function whose transform is the characteristic function of the complement of $E \times \Omega$. Since $\partial E \times \Omega = \emptyset$, $J = I_{E \times \Omega}$ is the only closed ideal of $L^1(G \times H)$ with zero set $E \times \Omega$, and clearly g generates this ideal.

Theorem 3.7. *Let G be a compactly generated LCA group. A nonzero closed ideal J of $L^1(G)$ is countably generated iff G is $Z^n \times K$, K a compact group, and $J = I_{T^n \times E}$ where E is a cofinite subset of the dual of K .*

Proof. G is a group of the form $R^m \times Z^n \times K$, K compact [12, II 9.8, p. 90]. If $L^1(G)$ has a nonzero closed, countably generated ideal J , then 3.6(i) applied to $H = R^m \times Z^n$ shows that $m = 0$. So G is $Z^n \times K$, and 3.6(ii) applied to $H = Z^n$ yields $Z(J) = T^n \times E$, a set of synthesis; whence $J = I_{T^n \times E}$. The converse also follows from 3.6(ii).

In fact, the proof of 3.6 reveals that for a compactly generated group G , a closed ideal in $L^1(G)$ will either be principal or else require uncountably many generators. Thus, in $L^1(T)$, say, no maximal ideal I_{γ_0} can be generated by the characters different from γ_0 . More generally, we have

Corollary 3.8. *For any LCA group G , $L^1(G)$ has a countably generated maximal ideal if and only if G is finite.*

Proof. If M is such an ideal, M is regular (cf. 1.6) so that $Z(M)$ is an isolated point in Γ (3.1). By duality G is compact, so that 3.7 implies that the complement of the singleton $Z(M)$ is finite. Thus Γ , and hence G , is finite. Of course if G is finite, every ideal of $L^1(G)$ is principal.

Simplification of these methods yields an improvement of 2.6 for the class of function algebras.

Theorem 3.9. *If J is a closed, countably generated ideal in a function algebra A , then $Z(J) \cap \Gamma_A$ is open-closed in Γ_A .*

Proof. J is the sum of spaces $\{b_i A\}$ where $b_i \in J$ has norm less than $1/2^i$. As before, the Dieudonné-Schwartz theorem applied to the map $\bigoplus_1^\infty A \rightarrow J$ given by $\{g_i\} \rightarrow \sum_i b_i g_i$ yields a constant $M > 0$ so that, given $g \in J$, there are $g_1, \dots, g_p \in A$ with $g = \sum_{i=1}^p g_i b_i$ and $\|g_i\|_\infty \leq M \|g\|_\infty$. Choose n so that

$$\sum_{i=n+1}^\infty M \|\hat{b}_i\|_\infty < 1/2.$$

If $\epsilon = 1/2Mn$, $U = \bigcap_{i=1}^n \hat{b}_i^{-1}(B(0, \epsilon))$ is an open neighborhood of $Z(J)$ in $\mathfrak{M}_A$. If $Z(J) \cap \Gamma_A$ is not open in Γ_A , $V = \Gamma_A \cap U - Z(J) = U \cap \Gamma_A - Z(J)$ is a nonempty open set in Γ_A . Since ∂A is dense in Γ_A , we may choose some $x_0 \in \partial A \cap V$. x_0 is a strong boundary point for $\hat{A}$ [2, 2.3.4]: there is a $k \in A$ with $k(x_0) = 1 = \|\hat{k}\|_\infty$ and $|\hat{k}| < 1$ off U . If k assumes its maximum modulus on the compact set $M_A - U$ at a , choose m so large that $|k(a)|^m < \epsilon / (1 + \max_{1 \leq i \leq n} \|\hat{b}_i\|_\infty)$. Notice $\|\hat{k}^m \hat{b}_i\|_\infty < \epsilon$ for $i = 1, \dots, n$. For if $b \in U$, $|\hat{k}(b)^m \hat{b}_i(b)| \leq \|\hat{k}\|_\infty^m |\hat{b}_i(b)| < \epsilon$, and if $b \notin U$, $|\hat{k}(b)^m \hat{b}_i(b)| \leq \|\hat{b}_i\|_\infty |\hat{k}(a)|^m < \epsilon$.

Let $s_i = \alpha_i b_i$ where $\alpha_i = |b_i(x_0)|/b_i(x_0)$ if $b_i(x_0) \neq 0$, 1 otherwise. Since α_i has modulus 1, we may choose $g_1, \dots, g_p \in J$ so that $\sum_{i=1}^p g_i s_i = \sum_{i=1}^\infty k^m s_i = g \in J$ and $\|g_i\|_\infty \leq M \|g\|_\infty$. Then

$$\sum_{i=1}^\infty |b_i(x_0)| = \sum_{i=1}^\infty k(x_0)^m s_i(x_0) = \sum_{i=1}^p g_i(x_0) s_i(x_0) \leq \sum_{i=1}^p |g_i(x_0)| |b_i(x_0)|.$$

Since $x_0 \notin Z(J)$, we conclude that $|g_j(x_0)| \geq 1$ for at least one j , and finally that

$$\begin{aligned} 1 &\leq \|g_j\|_\infty \leq M \|\hat{g}\|_\infty \\ &\leq \sum_{i=1}^n M |\alpha_i| \|\hat{k}^m \hat{b}_i\|_\infty + \sum_{i=n+1}^\infty M |\alpha_i| \|\hat{b}_i\|_\infty \|\hat{k}\|_\infty^m < Mn\epsilon + 1/2 = 1. \end{aligned}$$

This contradiction completes the proof.

3.9 has an obvious corollary when Γ_A is connected. In particular we have

Corollary 3.10. *In the disc algebra, a nonzero closed ideal is countably generated iff its zero set lies in the open unit disc.*

Proof. Plainly 3.9 makes the zero set condition necessary. On the other hand, if the zero set of a nonzero ideal J is contained in the open disc D , then $Z(J)$ is finite. For otherwise $Z(J)$ would have a cluster point $p \in Z(J) \subset D$, and the identity principle for analytic functions would force $J = 0$. If $Z(J)$ is empty, J is generated by 1. If $Z(J) = \{p_1, \dots, p_n\}$, a compactness argument (similar to [9, p. 235]) shows that J is the principal ideal generated by $\prod_{i=1}^n (z - p_i)^{k_i}$ where

$$k_i = \inf \{m: \text{some } f \in J \text{ has a zero of order } m \text{ at } p_i\}.$$

3.9 has application to other familiar examples. Consider for instance a Swiss cheese space K formed by deleting from the closed unit disc Δ countably many open discs $\{\Delta_j\}$, with centers $\{b_j\}$ and radii $\{r_j\}$ whose closures are pairwise disjoint, in such a way that K has no interior in the plane. As the intersection of a descending chain of compact, connected sets in the plane, K is connected [10, 16.14, p. 246]. The closure $R(K)$ in $C(K)$ of the rational functions with poles off K is a function algebra on its maximal ideal space K whose Šilov boundary is $\partial K = K$ [8, p. 27]. Thus 3.9 implies

Corollary 3.11. *If K is a Swiss cheese, 0 and $R(K)$ are the only countably generated, closed ideals of $R(K)$.*

If the radii $\{r_j\}$ are chosen so that $\sum r_j < \infty$, $R(K) \neq C(K)$ and the result is nontrivial (cf. [5, Corollary 3, p. 177]). Further, Sidney observes [19, p. 148] that if $0 \in K$ and $\sum r_j / (|b_j| - r_j)^m < \infty$ for $m = 1, 2, \dots$, then the closed powers $\overline{I}_0^n$ form a strictly decreasing chain of closed, primary ideals in $R(K)$. In view of 3.11, none of them can be countably generated.

We can also use 3.9 to obtain purely algebraic information.

Corollary 3.12. *If A is a regular function algebra, then a closed ideal of A which is countably generated is actually principal.*

Proof. If J is such an ideal, $Z(J)$ is open-closed in $\mathfrak{M}_A = \Gamma_A = X$. By Šilov's idempotent theorem [8, p. 88], there is some $e \in A$ vanishing on $Z(J)$ which is identically 1 elsewhere. By regularity, $e \in J$ and hence $eA = J$.

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