

LATTICE-ORDERED INJECTIVE HULLS

BY

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ABSTRACT. It is well known that the injective hull of a lattice-ordered group (l -group) M can be given a lattice order in a unique way so that it becomes an l -group extension of M . This is not the case for an arbitrary f -module over a partially ordered ring (po-ring). The fact that it is the case for any l -group is used extensively to get deep theorems in the theory of l -groups. For instance, it is used in the proof of the Hahn-embedding theorem and in the characterization of $\aleph_\alpha$ -injective l -groups.

In this paper we give a necessary and sufficient condition on the injective hull of a torsion-free f -module M (over a directed essentially positive po-ring) for it to be made into an f -module extension of M (in a unique way). An f -module is called an i - f -module if its injective hull can be made into an f -module extension. The class of torsion-free i - f -modules is closed under the formation of products, sums, and Hahn products of strict f -modules. Also, an l -submodule and a torsion-free homomorphic image of a torsion-free i - f -module are i - f -modules.

Let R be an f -ring with zero right singular ideal whose Boolean algebra of polars is atomic. We show that R is a qf -ring (i.e., R_R is an i - f -module) if and only if each torsion-free R - f -module is an i - f -module. There are no injectives in the category of torsion-free R - f -modules, but there are $\aleph_\alpha$ -injectives. These may be characterized as the f -modules that are injective R -modules and $\aleph_\alpha$ -injective l -groups. In addition, each torsion-free f -module over R can be embedded in a Hahn product of l -simple $Q(R)$ - f -modules. We note, too, that a totally ordered domain has an i - f -module if and only if it is a right Ore domain.

1. Introduction. Our methods and characterization of torsion-free i - f -modules are modelled after Anderson's work on the maximal right quotient ring of an f -ring [1]. Throughout this paper $\mathbb{Z}$ and $\mathbb{Q}$ will denote the totally ordered rings of integers and rational numbers, respectively. If R is a po-ring, then R_* will denote the po-ring obtained by freely adjoining $\mathbb{Z}$ to R .

We begin by recalling the requisite module theory. All modules will be right modules. An R -module E is *injective* if for every pair of R -homomorphisms $f: K \rightarrow E$ and $g: K \rightarrow L$, where g is monic, there exists an R -homomorphism $h: L \rightarrow E$ such that $hg = f$. A submodule N of the R -module M is an *essential submodule* (and M is an *essential extension* of N) if $N \cap K \neq 0$ for every nonzero submodule K of M . Every module M has a maximal essential extension $E = E(M_R)$ which is unique up to an isomorphism over M . $E(M)$ is the smallest injective module containing M , and is called the *injective hull* of M . If F is an essential extension of M and

Presented to the Society, February 11, 1971; received by the editors February 10, 1971.
AMS 1970 subject classifications. Primary 06A70; Secondary 16A52, 06A60.

Key words and phrases. f -module, lattice-ordered injective hull, relative injectives, torsion-free f -module, Hahn embedding theorem.

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it is injective, then $F = E(M)$ ([9] and [10]).

Let M^∇ be the set of essential submodules of M . Then M^∇ is a dual ideal in the lattice of submodules of M . If N is a submodule of M and T is a subset of M , then $(N:T) = \{r \in R: Tr \subseteq N\}$ is a right ideal of R . If $N \in M^\nabla$ and $x \in M$, then $(N:x) \in R^\nabla$. For a subset T of M we will sometimes write $r(T)$ for $(0:T)$. In [17] Johnson has defined the *singular submodule* of M by $Z(M) = \{x \in M: r(x) \in R^\nabla\}$. M is called a *torsion-free R -module* if $Z(M) = 0$. Note that when R is a commutative integral domain, $Z(M)$ is just the torsion submodule of M .

More generally, if N is a submodule of M , then the *closure* of N in M is defined by $\text{Cl}_M N = \{x \in M: xD \subseteq N \text{ for some } D \in R^\nabla\}$. When no confusion is likely we will write $\text{Cl} N$ for $\text{Cl}_M N$. $\text{Cl} N$ is, of course, a submodule of M containing N . In fact, $\text{Cl} N/N = Z(M/N)$. N is said to be *closed* in M if $\text{Cl} N = N$. In general, $\text{Cl Cl} N$ is the smallest closed submodule of M containing N [13]. The intersection of a family of closed submodules of M is clearly closed. Thus $C_r(M)$, the set of closed submodules of M , is a complete lattice with greatest lower bound being intersection.

If $Z(M) = 0$, then $\text{Cl} N$ is the largest essential extension of N contained in M . In particular, when $Z(M) = 0$, every submodule of M has a unique injective hull contained in $E(M)$. If K is any essential extension of M (and $Z(M) = 0$), then the map $C_r(K) \rightarrow C_r(M)$, given by $N \rightarrow N \cap M$, is a lattice isomorphism. The inverse map sends N to its closure in K . For proof of these facts see [10].

An *f -module over the po-ring R* is a lattice-ordered R -module (l -module) that is embeddable in a product of a family of totally ordered R -modules. For the basic properties of f -modules see [20].

This paper is based on a portion of the author's dissertation written at the University of Illinois at Urbana-Champaign under the direction of Professor Elliot Weinberg.

2. Torsion-free f -modules. A po-ring R is called (*right*) *essentially positive* if $D \in R^\nabla$ implies $D^+ R_* \in R^\nabla$. Since $D^+ R_* = D^+ - D^+ + (D^+ - D^+)R$ is the right ideal of R generated by D^+ , a directed po-ring is essentially positive if and only if each of its essential right ideals contains a directed essential right ideal. In order to characterize the torsion-free i - f -modules over R it will be necessary to assume that R is directed and essentially positive. Some examples of essentially positive po-rings are

- (1) any totally ordered ring;
- (2) any right quotient ring of a torsion-free essentially positive po-ring, in its canonical order;
- (3) the n -by- n matrix ring over a totally ordered right Ore domain, ordered coordinatewise.

A word about (2) is in order. If R is a ring with $Z(R_R) = 0$, then a ring S is

a right quotient ring of R if S_R is an essential extension of R_R . It is known that $E(R_R)$ can be made into a ring extension of R , so $Q(R) = E(R_R)$ is the maximal right quotient ring of R ([17], [22]). Now suppose that R is an essentially positive po-ring, and let S be a right quotient ring of R . Then

$$S^+ = \{s \in S: sD^+ \subseteq R^+ \text{ for some } D \in R^\nabla\}$$

is a partial order of the ring S (Lemma 3.1 implies that S_R is a po-module extension of R_R ; but if $a, b \in S^+$, $aD^+ \subseteq R^+$, and $C = \{r \in R: br \in D\}$, then $abC^+ \subseteq aD^+ \subseteq R^+$; also see [1, Theorem 2.3]), called the canonical order of S . Note that $S^+ \cap R$ can contain R^+ properly, and, in fact, (S, S^+) is a po-ring extension of (R, R^+) if and only if R is *essentially semiclosed*, i.e. $rD^+ \subseteq R^+$ for $D \in R^\nabla$ implies $r \in R^+$. Let D be an essential right ideal of S . Then D_R is essential in S_R . (Suppose that $D \cap X = 0$ for some $X_R \subseteq S_R$. If $0 \neq d \in XS$, $d = \sum_{i=1}^n x_i s_i$ with $x_i s_i \neq 0$, then there is an element $t \in R$ such that $dt \neq 0$ and $s_i t \in R$. Thus $0 \neq dt \in D \cap X = 0$; so $D \cap XS = 0$ and $X = 0$.) Since $D \cap R \in R^\nabla$, $(D \cap R)^+ R_* \in R^\nabla$. But then $(D \cap R)^+ R_* S_* \in S^\nabla$, and since $D^+ S_* \supseteq (D \cap R)^+ R_* S_*$, $D^+ S_* \in S^\nabla$. Thus S is essentially positive.

Note, by Proposition 3.2, that a torsion-free right f -ring R (i.e., R_R is an f -module) is essentially semiclosed. Note also that any torsion-free f -ring is essentially positive [1, Lemma 2.1]. That (3) is true follows easily from (2). For if D is a totally ordered right Ore domain, then (R, R^+) is an essentially positive right f -ring with $Z(R_R) = 0$ and right quotient ring (D_n, D_n^+) , where $R = \{[a_{ij}] \in D_n: a_{ij} = 0 \text{ for } j > 1\}$, $R^+ = \{[a_{ij}]: a_{ij} \geq 0\}$, and $D_n^+ = \{[a_{ij}] \in D_n: a_{ij} \geq 0\}$. That (D_n, D_n^+) is essentially positive is actually useless for our purposes, since it has no nontrivial f -modules [20].

An l -module M is called *distributive* if the map induced on M by each $r \in R^+$ is a lattice homomorphism. If M_R is distributive, and $x \in M$ and $xR = 0$ implies $x = 0$, then M is an f -module [20].

Lemma 2.1. *Let K be a po-module over the essentially positive po-ring R , and let M be an essential submodule of K .*

- (a) *If N is a convex submodule of M , then $\text{Cl}_K N$ is a convex submodule of K .*
- (b) *If M is a distributive l -module and N is an l -submodule of M , then $\text{Cl}_M N$ is an l -submodule of M .*
- (c) *Suppose that K is a distributive l -module, and M is an l -submodule of K . If N is a prime submodule of M , then $\text{Cl}_K N$ is a prime submodule of K .*
- (d) *If M is archimedean and x and y are elements of K such that $nx \leq y$ for all $n \in \mathbb{Z}$, then $x \in Z(K)$. Thus K is archimedean if M is torsion-free and archimedean.*

Proof. Suppose that N is a convex submodule of M and $0 \leq x \leq y$ where $x \in K$

and $y \in \text{Cl}_K N$. Then there exists $D \in R^\nabla$ such that $yD \subseteq N$ and $xD \subseteq M$. If $d \in D^+$, then $0 \leq xd \leq yd$; so $xD^+ \subseteq N$. Thus $x \in \text{Cl}_K N$, and we have (a).

For (b), suppose $x \in \text{Cl}_M N$ and $D \in R^\nabla$ such that $xD \subseteq N$. Then $x^+d = (xd)^+ \in N$ for all $d \in D^+$. Thus $x^+D^+ \subseteq N$, so $x^+ \in \text{Cl}_M N$.

For (c), suppose that N is a prime submodule of M . By (a) and (b), $\text{Cl}_K N$ is a convex l -submodule of K . Suppose that x and y are disjoint elements of K , and $x \notin \text{Cl}_K N$. There exists $D \in R^\nabla$ such that $xD \subseteq M$ and $yD \subseteq M$. Since $x \notin \text{Cl}_K N$, there exists $d \in D^+$ such that $xd \notin N$. If $e \in D^+$, then $xd \wedge ye = 0$; so $ye \in N$ since N is prime. Thus $yD^+ \subseteq N$, i.e. $y \in \text{Cl}_K N$.

Finally, if $D = (M : x) \cap (M : y)$ where $nx \leq y$ for all $n \in \mathbb{Z}$, then $nxd \leq yd$ for all $d \in D^+$. Since M is archimedean, $xd = 0$. Therefore, $xD^+ = 0$, and $x \in Z(K)$.

Corollary 2.2. *Let M be a distributive l -module over an essentially positive po-ring. Then the closed convex l -submodules of M form a complete sublattice of $C_r(M)$.*

Proof. If $\{N_\alpha : \alpha \in A\}$ is a family of closed convex l -submodules of M , then clearly $\bigcap \{N_\alpha : \alpha \in A\}$ is a closed convex l -submodule. By 2.1, so is $\text{Cl Cl } \sum N_\alpha$.

If M is an l -module over a directed po-ring, then whether or not M is an f -module depends only on $P(M)$, the Boolean algebra of polars of M [20]. In light of this fact, the following proposition is not surprising. $X^\perp$ (or $X^{\perp M}$) will denote the polar of a subset X of M .

Proposition 2.3. *An f -module M over an essentially positive directed po-ring R is torsion-free if and only if each of its (principal) polars is a closed submodule.*

Proof. Suppose that M is torsion-free, and let N be a polar of M . If $xD \subseteq N$ for some $D \in R^\nabla$, then $|xd| \wedge |y| = 0$ for all $d \in D^+$ and for all $y \in N^\perp$. So $(|x| \wedge |y|)D^+ = 0$, and hence $x \in N^{\perp\perp} = N$.

The converse is obvious.

Because of 2.2 and 2.3 one might suspect that $P(M)$ is a sublattice of $C_r(M)$ when M is torsion-free. However, if $R = \mathbb{Q}$, then each submodule of an l -module is closed, but the sum of two polars need not be a polar.

An f -module whose Boolean algebra of polars is atomic will be called *irredundant*. An irredundant f -module M over a directed po-ring is an irredundant subdirect product of totally ordered modules, i.e. $M \subseteq \prod M_\alpha$, where each M_α is totally ordered, $M \cap M_\alpha \neq 0$, and the set $\{M_\alpha\}$ consists of those homomorphic images of M whose kernels are the maximal polars of M ([14, p. 40] and [20]).

Corollary 2.4. *An irredundant f -module over an essentially positive directed po-ring is torsion-free if and only if it is a subdirect product of totally ordered torsion-free modules.*

Without the irredundancy hypothesis, 2.4 is false (see 2.7). By an *essential l-submodule* of an *l-module* we shall mean an *l-submodule* which is also an essential *R-submodule*.

Proposition 2.5. *Let K be a torsion-free f -module over the essentially positive directed po-ring R , and suppose that M is an essential l -submodule of K .*

(a) *If N is a convex l -submodule of M , then $\text{Cl}_K(N^{\perp M}) = [\text{Cl}_K(N)]^{\perp K}$.*

(b) *If N is a convex l -submodule of K , then $N^{\perp K} \cap M = (N \cap M)^{\perp M}$.*

Proof. (a) Let $K_1 = \text{Cl}_K(N^{\perp M})$ and $K_2 = \text{Cl}_K N$. Recall that N is an essential submodule of K_2 , since M is torsion-free. Thus $K_1 \cap K_2 = 0$ since $N \cap N^{\perp M} = 0$. By 2.1, K_1 and K_2 are convex l -submodules of K . Hence $K_1 \subseteq K_2^{\perp K}$. Let $x \in K_2^{\perp K}$ and let $D = (M : x)$. Then $xD \subseteq N^{\perp M} \subseteq K_1$. Thus, since D is an essential right ideal and K_1 is closed in K , $x \in K_1$. So $K_1 = K_2^{\perp K}$.

(b) It is clear that $N^{\perp K} \cap M \subseteq (N \cap M)^{\perp M}$. Let x be a positive element of $(N \cap M)^{\perp M}$, and let y be a positive element of N . Since N is an essential extension of $N \cap M$, $yD \subseteq N \cap M$ for some essential right ideal D . If $d \in D^+$, then $x \wedge yd = 0$, so $(x \wedge y)D^+ = 0$. Since M is torsion-free, $x \in N^{\perp K}$. Thus $N^{\perp K} \cap M = (N \cap M)^{\perp M}$.

Corollary 2.6. *Let K be a torsion-free f -module over the essentially positive directed po-ring R , and suppose that M is an essential l -submodule of K . Then the map $N \rightarrow \text{Cl}_K N$ is an isomorphism between the Boolean algebras of polars of M and K . Its inverse is the map $N \rightarrow N \cap M$.*

Proof. We already know that these correspondences are one-to-one between $C_r(M)$ and $C_r(K)$. By 2.5 they take polars to polars. In fact, if $N \in P(K)$, then $N = \text{Cl}_K(N \cap M) = \text{Cl}_K(N^{\perp K \perp K} \cap M) = \text{Cl}_K[(N^{\perp K} \cap M)^{\perp M}]$. Thus the map $N \rightarrow \text{Cl}_K N$ between $P(M)$ and $P(K)$ is an order isomorphism that is onto. Hence it is an isomorphism of Boolean algebras.

A *qf-ring* is an *f-ring* R whose maximal right quotient ring Q is an *f-ring* extension [1]. If R is a *qf-ring* with $Z(R) = 0$, then $Q = E(R_R)$ is a (strongly) regular self-injective ring.

Proposition 2.7. *Let R be a semiprime (right) qf-ring with maximal right quotient ring Q . The following are equivalent:*

- (a) R_R is a subdirect product of totally ordered torsion-free modules.
- (b) Q_R is a subdirect product of totally ordered torsion-free modules.
- (c) Q_Q is a subdirect product of totally ordered torsion-free modules.
- (d) The Boolean algebra of polars of Q is atomic.
- (e) The Boolean algebra of polars of R is atomic.
- (f) Q is the direct product of a family of totally ordered division rings.

Proof. (a) implies (b). Let $\{N_\alpha: \alpha \in A\}$ be a collection of closed prime submodules of R_R whose intersection is zero, and let $E_\alpha = E((N_\alpha)_R) \subseteq Q$. By 2.1 each E_α is a closed prime submodule of Q_R . Since $C_r(R)$ and $C_r(Q)$ are isomorphic via the correspondence $N \rightarrow E(N_R)$, $\bigcap \{E_\alpha: \alpha \in A\} = 0$.

That (b) implies (c) follows from the fact that every closed submodule of Q_R is a right ideal of Q [10, p. 70].

(c) implies (d). Let $\{E_\alpha: \alpha \in A\}$ be a collection of closed prime submodules of Q_Q whose intersection is zero. Since Q_Q is injective, $Q = E_\alpha \oplus F_\alpha$ as Q -modules. Since Q is a regular f -ring, each right ideal is an l -ideal, so the direct sum is a sum of f -rings. Thus each F_α is a totally ordered division ring. The projections onto the F_α induce an isomorphism of Q into the product of the F_α whose image contains the direct sum. Thus $P(Q)$ is atomic.

Finally, (d) and (e) are equivalent by 2.6, (e) implies (a) by 2.4, and (d) is equivalent to (f) by [10, p. 117].

It is, of course, not always the case that $P(R)$ is atomic. For instance, it is known that $C([0, 1])$, the f -ring of real-valued continuous functions defined on the unit interval, has no maximal polars. We do, however, have the following positive results:

Proposition 2.8. *Let R be a directed po-ring which has the property that a right ideal is essential if and only if it contains a positive regular element. If M is a torsion-free f -module over R , then every minimal prime submodule of M is closed. Thus M is a subdirect product of totally ordered torsion-free f -modules.*

Proof. First note that R is essentially positive. Suppose that N is a minimal prime submodule of M . By 2.1, ClN is a convex l -submodule of M . If $N \subsetneq ClN$, there exists a positive element $x \in (ClN) \setminus N$. By hypothesis $(N:x)$ contains a positive regular element d . Since N is a minimal prime subgroup [20], there exists $y \in M^+ \setminus N$ such that $y \wedge xd = 0$ [16]. Therefore $(y \wedge x)d = yd \wedge xd = 0$. Since M is torsion-free and dR is an essential right ideal of R , $x \wedge y = 0$. Thus $x \in N$, since N is prime.

A semiprime right Goldie ring can be characterized as a ring R that has a classical right quotient ring which is semisimple and artinian ([12] and [10]). R satisfies and, in fact, is characterized by the following condition: A right ideal I of R is essential if and only if it contains a regular element. If R is also an f -ring, then, since it has no nilpotent elements, it is of the following form: There is a family of totally ordered right Ore domains $\{R_i: i = 1, \dots, n\}$ with totally ordered quotient division rings $\{D_i: i = 1, \dots, n\}$ such that R is (isomorphic to) an l -subring of $D_1 \oplus \dots \oplus D_n$ containing $R_1 \oplus \dots \oplus R_n$ (see [10] and [1]).

The ring R of 2.8 is a semiprime right Goldie ring. It need not be an f -ring, however, but, for example, need only be directed and have the property that the

square of every element is positive. We suspect that R cannot have any nilpotent elements.

A module M over the semiprime ring R is called *l -torsion-free* [18] if $NJ = 0$ for $0 \neq N_R \subseteq M$ and some ideal J of R implies $JK = 0$ for some nonzero ideal K . Note that if M is torsion-free, then it is *l -torsion-free*. For if $NJ = 0$, then J_R essential in R implies $J = 0$, while $J \cap K = 0$ for $0 \neq K_R \subseteq R$ implies $KJ \subseteq J \cap K = 0$, and hence $JK = 0$.

Theorem 2.9. *Let R be a torsion-free right qf -ring. Then every torsion-free f -module over R is a subdirect product of totally ordered torsion-free modules if and only if the Boolean algebra of polars of R is atomic.*

Proof. Suppose that $P(R)$ is atomic. By 2.7, $Q(R) = \prod D_\alpha$, where D_α ($\alpha \in A$) is a totally ordered division ring. Let $R_\alpha = \text{image}(R \rightarrow D_\alpha)$. Then R is an irredundant subdirect product of the R_α : $R \subseteq \prod R_\alpha \subseteq \prod D_\alpha$. Since $Q(\prod R_\alpha) = \prod D_\alpha$, $D_\alpha = Q(R_\alpha)$ [22, 2.2]. But then R_α is a right Ore domain [1, 5.2].

Let M be a torsion-free f -module over R , let $N_\alpha = \{x \in M: x(R \cap R_\alpha) = 0\}$, and let $P_\alpha = \text{kernel}(R \rightarrow R_\alpha)$. By [18, 3.7], $MP_\alpha \subseteq N_\alpha$, $M_\alpha = M/N_\alpha$ is an *l -torsion-free R -module* (R_α -module), and M is a subdirect product of the M_α (as R -modules). It is easily seen that N_α is a convex *l -submodule* of M , so the subdirect product is one of f -modules. Clearly, M_α is an f -module over R_α .

Let E_α be the R_α -injective hull of M_α . Then E_α is the R -injective hull of M_α and $E(M_R) = E = \prod E_\alpha$ [18, 4.1]. Thus each E_α is a torsion-free R -module. But $R_\alpha = R/P_\alpha$ and $M_\alpha P_\alpha = 0$, so M_α is a torsion-free R_α -module [13, Lemma 3.4].

Let N_α be a minimal prime R -submodule of M_α . Then N_α is a minimal prime R_α -submodule of M_α , and so, by 2.8, is a closed R_α -submodule. Suppose that $xT \subseteq N_\alpha$ for some $x \in M_\alpha$ and some essential right ideal T of R . If $T \subseteq P_\alpha$, then $P_\alpha \in R^\nabla$. This contradicts the fact that R_α is a torsion-free R -module (2.4). Thus $T \not\subseteq P_\alpha$, and, since every nonzero right ideal of R_α is essential, $x \in N_\alpha$. So N_α is a closed R -module. Therefore M_α , and hence M , is a subdirect product of totally ordered torsion-free R -modules.

The converse is given by the equivalence of (a) and (e) of 2.7.

Proposition 2.10. *Let M be an essential l -submodule of the f -module K_R . If R is directed, or if it is essentially positive and M is torsion-free, then any weak order unit of M is a weak order unit of K . Thus, K is totally ordered if and only if M is.*

Proof. Suppose that $x \in K$ with $x^\perp \neq 0$. If R is directed, then $x^\perp$ is a submodule of K , so $x^\perp \cap M \neq 0$. Suppose M is torsion-free and R is essentially positive, and let $0 \neq y \in x^\perp$. There exists $d \in (M: y)^+$ such that $yd \neq 0$. So $yd \in x^\perp \cap M$. Thus, in either case, $x^\perp \cap M \neq 0$, and a weak order unit of M is a weak

order unit of K . If M is totally ordered and x is a nonzero element of K , then any positive nonzero element of $x^\perp \cap M$ is a weak order unit of M . So $x^\perp \cap M = 0$. If R is directed this says that $x^\perp = 0$, since $x^\perp$ is a submodule of K . If R is essentially positive, M is torsion-free, and $0 \neq y \in x^\perp$, then $0 \neq yd \in x^\perp \cap M$ for some $d \in (M:y)^+$. Thus $x^\perp = 0$. So K is totally ordered in both cases.

An element g in the f -module M (over the directed po-ring R) is called *basic* [4] if $C_R(g)$, the convex l -submodule generated by g , is totally ordered. This is equivalent to saying that $C(g)$ ($= C_Z(g)$) is totally ordered. A subset X of M is a *basis* of M if X is a maximal set of disjoint elements and each element of X is basic.

Proposition 2.11. *Let R be a directed po-ring, and let M be an essential l -submodule of the f -module K_R . Then M has a basis of cardinality u if and only if K has a basis of cardinality u .*

Proof. If x and y are two elements of K such that $0 \neq x \in y^\perp$, and if $r \in R_*$ is such that $0 \neq xr \in M$, then $xr \in y^\perp$, since every polar of K is a submodule. Let $X = \{x_\alpha : \alpha \in A\}$ be a maximal set of disjoint elements of K . There is a subset $\{r_\alpha : \alpha \in A\}$ of R_* such that $Y = \{x_\alpha r_\alpha : \alpha \in A\}$ is a set of disjoint elements of M . If X is now a basis of K , then, clearly, each element of Y is basic. Suppose that y is a nonzero element of M such that $y \wedge |x_\alpha r_\alpha| = 0$ for all $\alpha \in A$. Let α be an element in A such that $y \wedge x_\alpha > 0$. Then $y \wedge x_\alpha \wedge |x_\alpha r_\alpha| = 0$, contradicting the fact that x_α is basic. So Y is a basis of M .

On the other hand, if $Y = \{y_\alpha : \alpha \in A\}$ is a maximal set of disjoint elements of M , and if $0 \neq x \in K$ is such that $x \wedge y_\alpha = 0$ for all $\alpha \in A$, then $Y \cup \{|xr|\}$ is a disjoint set in M for some $r \in R_*$. Thus x does not exist. So Y is a maximal set of disjoint elements of K . Since the convex l -submodule of K generated by an element of M is totally ordered exactly when the convex l -submodule of M generated by it is totally ordered, Y is a basis of M if and only if it is a basis of K .

For an f -module M_R , $\Gamma_R(M)$ will denote the rooted po-set of R -values of M [20]. It is known [7] that an f -module M has a finite basis with n elements if and only if $\Gamma_Z(M)$ has exactly n roots. Since there is a one-to-one correspondence between the roots of Γ_R and the minimal prime submodules of M , and since the sets of minimal prime subgroups and submodules coincide, M has a basis containing n elements exactly when Γ_R has n roots. This fact gives the following corollary.

Corollary 2.12. *Let M be an i - f -module over the directed po-ring R . Then $\Gamma_R(M)$ has exactly n roots if and only if $\Gamma_R(E)$ has exactly n roots.*

Let M be an l -group, and let $d(M)$ be its Z -injective hull. There is a lattice isomorphism between the lattice of convex l -subgroups of M , $\mathcal{Q}(M)$, and the lattice of convex l -subgroups of $d(M)$ given by: $N \in \mathcal{Q}(M)$ corresponds to the convex

l -subgroup of $d(M)$ generated by N , and $N \in \mathcal{L}(d(M))$ corresponds to $N \cap M$. With respect to this correspondence, the value sets of M and $d(M)$ are isomorphic. That the R -value sets of M and $E(M_R)$ are not always isomorphic for an (totally ordered torsion-free) i - f -module M_R is shown by the following example.

Example 2.13. Let $R = \mathbb{Q}[[x]]$ be the formal power series ring with coefficients in $\mathbb{Q}$ and exponents in $\mathbb{Z}^+$. Order R lexicographically with the constant term dominating. Thus $R = \{\sum_{i=0}^{\infty} a_i x^i: a_i \in \mathbb{Q}\}$ and $R^+ = \{\sum_{i=n}^{\infty} a_i x^i: a_n > 0\} \cup \{0\}$. The units of R are the elements with nonzero constant term. Every element of R is of the form $x^k u$, u a unit or zero. The quotient field F of R is $\{\sum_{i=-n}^{\infty} a_i x^i: n \geq 0, a_i \in \mathbb{Q}\}$. It is a totally ordered field if its positive cone is defined by $F^+ = \{\sum_{i=-n}^{\infty} a_i x^i: a_{-n} > 0\}$.

Let $M = R_R$. Then $E(M_R) = F_R$, so M is an i - f -module. The convex l -submodules of M are

$$R \supseteq xR \supseteq x^2R \supseteq \dots,$$

and the convex l -submodules of E are

$$F \supseteq \dots \supseteq x^{-2}R \supseteq x^{-1}R \supseteq R \supseteq xR \supseteq x^2R \supseteq \dots,$$

Thus $\Gamma_R(M) \cong \mathbb{Z}^+$ and $\Gamma_R(E) \cong \mathbb{Z}$.

3. i - f -modules. In this section we show that part of Anderson's characterization of a unital q -ring [1] characterizes the torsion-free i - f -modules over an essentially positive directed po-ring. Using this characterization we show that a totally ordered domain has torsion-free i - f -modules if and only if it is a right Ore domain. We also show that every torsion-free f -module over a torsion-free irredundant f -ring R is an i - f -module if and only if R is a q -ring. In addition, we examine some properties of the class of torsion-free i - f -modules, and we show that the i - f -property is a local property over a right noetherian ring.

Let M be a po-module over the po-ring R , and let N be an R -module containing M . Define

$$N^+ = \{x \in N: xD^+ \subseteq M^+ \text{ for some } D \in R^\nabla\}.$$

Notice that if R contains an essential right ideal D for which $D^+ = 0$, then $N^+ = N$. When $R = \mathbb{Z}$, N^+ consists of those elements x of N such that $nx \in M^+$ for some positive integer n .

Lemma 3.1.

(a) $N^+ + N^+ \subseteq N^+.$

(b) $N^+ R^+ \subseteq N^+.$

(c) If R is essentially positive, then $N^+ \cap -N^+ = Z(N_R).$

(d) If R is essentially positive and if M is a distributive l -module, then $M^+ \subseteq N^+ \cap M = \{x \in M: x^- \in Z(M_R)\}.$

Proof. The first statement is an immediate consequence of the fact that R^∇ is closed under intersection.

For (b), take $x \in N^+$, $a \in R^+$, and $D \in R^\nabla$ such that $xD^+ \subseteq M^+$. Let $I = (D : a)$. Then $I \in R^\nabla$, and $(xa)I^+ = x(aI^+) \subseteq xD^+ \subseteq M^+$. Hence $xa \in N^+$.

If $x \in Z(N)$, then $xD = 0$ for some $D \in R^\nabla$; so, clearly, $x, -x \in N^+$. Conversely, if $x, -x \in N^+$, then $xD^+, -xF^+ \subseteq M^+$ for some $D, F \in R^\nabla$. Therefore, $x(D \cap F)^+ \subseteq M^+ \cap -M^+ = 0$, $x \in Z(N)$, and $N^+ \cap -N^+ = Z(N)$.

Finally, if $x \in M$ with $x^- \in Z(M)$, then $xD^+ = (x^+ - x^-)D^+ = x^+D^+ \subseteq M^+$ for some $D \in R^\nabla$. Thus $x \in N^+ \cap M$. On the other hand, if $x \in N^+ \cap M$ and $xD^+ \subseteq M^+$ for some $D \in R^\nabla$, then $x^-d = (xd)^- = 0$ for all $d \in D^+$. So $x^- \in Z(M)$.

Proposition 3.2. *Let M be a distributive l -module over the essentially positive po-ring R , and let N be a torsion-free R -module containing M . Then $N^+ = \{x \in N : xD^+ \subseteq M^+ \text{ for some } D \in R^\nabla\}$ is a partial order on N , and (N, N^+) is a po-module extension of (M, M^+) . If N is an essential extension of M , then*

(a) N is semiclosed.

(b) *The greatest lower bound (least upper bound) of two elements of M is also their greatest lower bound (least upper bound) in N .*

(c) N^+ is the largest partial order P of N for which (N, P) is a po-module extension of (M, M^+) .

(d) $N^+ = \{x \in N : x(N : x)^+ \subseteq M^+\}$.

Proof. By 3.1, (N, N^+) is a po-module extension of (M, M^+) . Suppose that N is an essential extension of M . Let $x \in M$ and $y \in N$ with $y \geq x, 0$. Since N is an essential extension of M , $(M : y) \in R^\nabla$. If $d \in (M : y)^+$, then $yd \geq (xd)^+ = x^+d$. Therefore, $(y - x^+)(M : y)^+ \subseteq M^+$, so $y - x^+ \in N^+$. Thus x^+ is the least upper bound of x and 0 in N , and we have (b). A similar argument gives (a).

Suppose that P is a partial order on N for which $P \cap M = M^+$ and $PR^+ \subseteq P$. If $y \in P$, then $y(M : y)^+ \subseteq PR^+ \cap M \subseteq P \cap M = M^+$. Thus $y \in N^+$ and $P \subseteq N^+$. Therefore (c) is true.

The last statement is obvious.

Proposition 3.3. *Let M be a distributive torsion-free l -module over the essentially positive po-ring R , and let N_R be an essential extension of M_R . If (N, P) is a distributive l -module extension of (M, M^+) , then $P = N^+$.*

Proof. By 3.2(c) we only have to show that $N^+ \subseteq P$. Take $x \in N^+$ and let $D = (M : x) \cap (M : x^+)$. (All lattice operations are with respect to P .) If $x \notin P$, then $x^- \neq 0$, and $(xd)^- = x^-d \neq 0$ for some $d \in D^+$, i.e. $xd \in M \cap N^+ = M^+$ and $(xd)^- \neq 0$. Thus $x \in P$, and $N^+ \subseteq P$.

Notice that 3.3 says that N^+ is the only partial order P of N for which (N, P) can be a distributive l -module extension of (M, M^+) .

Theorem 3.4. *Let M be a torsion-free right f -module over the essentially positive directed po-ring R . Then the injective hull E of M is an f -module extension of M if and only if for all $x \in E$ and, for all $d_1, d_2 \in R^+$ for which $xd_i \in M$,*

$$(xd_1)^+ \wedge (xd_2)^- = 0.$$

When this is the case the lattice order of E is uniquely determined by that of M .

Proof. First note that this condition is equivalent to: If ϕ is a homomorphism of M onto a totally ordered R -module and $x \in E$, then $\phi(x(M:x)^+) \subseteq \phi(M)^+$ or $-\phi(x(M:x)^+) \subseteq \phi(M)^+$. For if this latter condition is satisfied and $d_1, d_2 \in (M:x)^+$, then

$$\phi[(xd_1)^+ \wedge (xd_2)^-] = \phi(xd_1)^+ \wedge \phi(xd_2)^- = 0,$$

for arbitrary ϕ . Therefore, $(xd_1)^+ \wedge (xd_2)^- = 0$, since M is a subdirect product of totally ordered R -modules. Conversely, suppose that the condition in the theorem is satisfied and $\phi(xd_1) > 0$, $\phi(xd_2) < 0$. Then $\phi[(xd_1)^+ \wedge (xd_2)^-] = \phi(xd_1)^+ \wedge \phi(xd_2)^- = \phi(xd_1) \wedge -\phi(xd_2) > 0$, which contradicts the fact that $(xd_1)^+ \wedge (xd_2)^- = 0$.

If E is an f -module extension of M , then clearly $(xd_1)^+ \wedge (xd_2)^- = x^+d_1 \wedge x^-d_2 = 0$. Conversely, suppose that the condition holds. By 3.2, (E, E^+) is a po-module extension of (M, M^+) . Let $x \in E$, and consider the correspondence $b: (M:x)^+R \rightarrow E$ given by $\sum_{i=1}^n d_i r_i \rightarrow \sum_{i=1}^n (xd_i)^+ r_i$. Suppose that $\sum d_i r_i = 0$. Let $\phi: M \rightarrow \phi(M)$ be any homomorphism onto a totally ordered R -module. If $\phi(x(M:r)^+) \subseteq \phi(M)^+$, then

$$0 = \phi\left(\sum x d_i r_i\right) = \sum \phi(x d_i)^+ r_i = \phi\left[\sum (x d_i)^+ r_i\right],$$

and similarly, if $-\phi(x(M:x)^+) \subseteq \phi(M)^+$, then

$$\phi\left[\sum (x d_i)^+ r_i\right] = \sum \phi(x d_i)^+ r_i = 0.$$

Since M is a subdirect product of totally ordered R -modules, $\sum (x d_i)^+ r_i = 0$, and thus b is a well-defined function. Clearly, it is an R -homomorphism. Since E is an injective R -module, b can be extended to an R -homomorphism g defined on R_* .

Let $y = g(1)$. Then $b(dr) = g(dr) = ydr$ for all $d \in (M:x)^+$ and $r \in R$, so $ydr = (xd)^+r$. Since $Z(E) = 0$, $yd = (xd)^+$ for all $d \in (M:x)^+$. We claim that $y = x^+$. Since $y(M:x)^+ \subseteq M^+$, y is in E^+ . Also, $(y-x)d = (xd)^+ - xd \geq 0$ for all $d \in (M:x)^+$, so $y-x \in E^+$. Suppose that $z \in E$ with $z \geq \{0, x\}$. If $0 \leq d \in (M:x) \cap (M:z)$, then $zd \geq (xd)^+ = yd$, so $z-y \in E^+$. Therefore $y = x^+$, and E is an l -module. Note that $x^+d = (xd)^+$ for all $d \in (M:x)^+$.

All that remains is to show that E is an f -module. Take $x \in E$, $a \in R^+$, and let $C = \{b \in R : ab \in (M : x)\}$. Then $C \in R^\nabla$, and if $b \in C^+$, $(x^+a)b = (xab)^+ = (xa)^+b$. Thus $[x^+a - (xa)^+]C^+ = 0$, and hence $x^+a = (xa)^+$. Therefore E is a distributive l -module, and, hence it is an f -module.

Note that without the assumption that R is directed, Theorem 3.4 becomes

E is a distributive l -module extension of M if and only if $(xd_1)^+ \wedge (xd_2)^- = 0$ for all $x \in E$ and for all $d_1, d_2 \in R^+$ such that $xd_i \in M$.

Corollary 3.5. *Let M be a torsion-free f -module over the po-ring R . Then M is an i - f -module if either*

- (a) *R is commutative, essentially positive, and directed, or*
- (b) *R is an f -ring and a semiprime right Goldie ring.*

Proof. (a) If $x \in E(M)$ and $d_1, d_2, d \in (M : x)^+$, then

$$[(xd_1)^+ \wedge (xd_2)^-]d = (xd_1d)^+ \wedge (xd_2d)^- = (xd)^+d_1 \wedge (xd)^-d_2 = 0.$$

Thus $[(xd_1)^+ \wedge (xd_2)^-](M : x)^+ = 0$, so $(xd_1)^+ \wedge (xd_2)^- = 0$.

(b) Let $x \in E(M)$, $d_1, d_2 \in (M : x)^+$, and $d \in (M : x)^+$ with d regular. There are elements $a, b, c, e \in R^+$, with a and c regular, such that $d_1da = db$ and $d_2dc = de$. Therefore,

$$(xd_1da)^+ \wedge (xd_2dc)^- = (xdb)^+ \wedge (xde)^- = (xd)^+b \wedge (xd)^-e = 0;$$

so $(xd_1)^+da \wedge (xd_2)^-dc = 0$. Let s and t be positive regular elements of R such that $as = ct$. Then

$$[(xd_1)^+ \wedge (xd_2)^-]das = (xd_1)^+das \wedge (xd_2)^-dct = 0.$$

But das is regular. Therefore $dasR$ is an essential right ideal of R , and $(xd_1)^+ \wedge (xd_2)^- = 0$.

Note that any commutative, semiprime, directed po-ring, in which the square of every element is positive, is an example of a po-ring satisfying the conditions of 3.5(a). An archimedean semiprime f -ring is, of course, such a po-ring.

The equivalence of (1) and (2) in the next corollary is a generalization of the fact that a semiprime f -ring R with the maximum condition on polars (i.e., $P(R)$ is finite) is a right qf -ring if and only if it is a right Goldie ring [1, Theorem 6.1].

Corollary 3.6. *The following statements are equivalent for an irredundant torsion-free f -ring R .*

- (1) *R is a qf -ring.*
- (2) *Each component of the irredundant representation of R is a totally ordered right Ore domain.*
- (3) *Every torsion-free f -module over R is an i - f -module.*

Proof. That (1) implies (2) has already been observed in the proof of 2.9, and that (3) implies (1) follows from the fact that $Q (= E(R))$ is an f -module extension of R_R if and only if it is an f -ring extension of R .

Let M be a torsion-free f -module over R . Assuming (2), we have (see the proof of 2.9): $R \subseteq \prod R_\alpha$ and $M \subseteq \prod M_\alpha \subseteq \prod E_\alpha = E(M)$, where R_α is a totally ordered right Ore domain, and E_α is the R_α -(R -)injective hull of the torsion-free R_α -(R -) f -module M_α . By 3.5, each E_α is an f -module extension of M_α (over R_α , hence over R). Thus $E(M)$ is an f -module extension of M , and (2) implies (3).

An interesting example of a torsion-free i - f -module may be obtained as follows. Let R be a semiprime right qf -ring with maximal right quotient ring Q . Suppose that I is an l -submodule of Q_R , and let $I' = E(I_R) \subseteq Q$. Then $I' = eQ$ for some idempotent e of Q . Consequently, $S = \text{Hom}_R(I_R, I_R) \cong \{q \in Qe: qI \subseteq I\}$, and $T = \text{Hom}_R(I'_R, I'_R) \cong eQ$. Thus S and T can be made into f -rings in a natural way. Now S_S is an essential submodule of T_S [10, p. 97], and ${}_S I$ is a torsion-free f -module [21, Theorem 3.25]. If ${}_S S$ is essential in ${}_S T$, then ${}_S I$ is an i - f -module.

There are i - f -modules that are not torsion-free. For instance, over a quasi-Frobenius f -ring R each unital module can be made into an f -module. If R is totally ordered, but not a division ring, then it has no torsion-free modules [21, p. 114]. It is not hard to see that if M is any i - f -module over a directed essentially positive po-ring, then $M/Z_2(M)$ is a torsion-free i - f -module, where $Z_2(M) = \text{Cl Cl } 0$ is the torsion submodule of M .

We now present an example of a totally ordered, archimedean, torsion-free f -module, over a unital essentially positive l -ring, that is not an i - f -module, but whose injective hull is an l -module extension.

Example 3.7. Let $R = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Q}\}$, and let $R^+ = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Q}^+\}$. Then R is an essentially positive unital l -ring. Let $M = \{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} : a \in \mathbb{Q}\}$, and let $M^+ = \{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} : a \in \mathbb{Q}^+\}$. Then M_R is a simple R -module and a totally ordered torsion-free f -module. But $E(M_R) = \{\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} : a, b \in \mathbb{Q}\}$ cannot be made into an f -module extension of M (use 3.4).

More generally, if R is any essentially positive sub-po-ring of the canonically ordered matrix ring D_n ($n > 1$, D a totally ordered division ring), such that $Q(R) = D_n$, then 3.12 implies that R has no torsion-free i - f -module (D_n has no non-trivial f -modules). Any l -subring R of D_n with $Q(R) = D_n$ is essentially positive.

Proposition 3.8. Let M be a torsion-free i - f -module over an essentially positive po-ring R .

- (a) Every l -submodule of M is an i - f -module.
- (b) If N is a closed convex l -submodule of M , then M/N is an i - f -module.

Proof. Let $E = E(M_R)$. If N is an l -submodule of M , then $E(N) = \text{Cl}_E N$ is an l -submodule of E by 2.1(b). Thus N is an i - f -module. If N is a closed convex

l -submodule of M , then $E_1 = E(N)$ is a convex l -submodule of E by 2.1, and $N = E_1 \cap M$. Therefore, $M/N \rightarrow E/E_1$ is a monomorphism of f -modules. Since $E = E_1 \oplus E_2$ as R -modules, E/E_1 is an injective R -module. But M/N is an l -submodule of E/E_1 , so M/N is an i - f -module by (a). In fact, $E/E_1 = E(M/N)$.

Using the notation of (b), note that E_2 has two partial orders. One comes from E/E_1 , and one is inherited from E . (E_2 is not uniquely determined by N , though E_1 is.) Let P be the partial order of E_2 coming from E/E_1 . Then $P = \{x \in E_2 : x + E_1 \geq 0\}$ contains $E_2 \cap E^+$. The following example shows that, in general, this containment is proper. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$, $E = \mathbb{Q} \oplus \mathbb{Q}$, $N = \mathbb{Z} \oplus 0$, $E_1 = \mathbb{Q} \oplus 0$, and $E_2 = (1, -1)\mathbb{Q}$. Then $P = (1, -1)\mathbb{Q}^+$ and $E_2 \cap E^+ = \{0\}$. Thus (E_2, P) is not a sublattice of E . If, however, $E = \text{lex } E_1$, i.e. $M = \text{lex } N$ (see 3.14), then $P = E_2 \cap E^+$. For then, $x \in P$ and $x + E_1 \geq 0$ implies $x \geq 0$. This is the case if M is totally ordered.

It is easy to see that a finite direct sum of i - f -modules (over any po-ring) is an i - f -module.

Corollary 3.9. *Let R be an essentially positive po-ring, and let $\{M_\alpha : \alpha \in A\}$ be a collection of torsion-free f -modules. The following are equivalent.*

- (a) M_α is an i - f -module for each $\alpha \in A$.
- (b) $\prod M_\alpha$ is an i - f -module.
- (c) $\sum \bigoplus M_\alpha$ is an i - f -module.

Proof. (b) implies (c) and (c) implies (a) by 3.8(a). Let $E_\alpha = E(M_\alpha)$. Then (a) implies that $\prod E_\alpha$ is an i - f -module, so $\prod M_\alpha$ is an i - f -module, again by 3.8(a).

Corollary 3.10. *The inverse limit of torsion-free i - f -modules over an essentially positive po-ring is an i - f -module.*

Proof. Let $(\{M_\alpha : \alpha \in A\}, \{f_{\alpha\beta} : \alpha \geq \beta\})$ be an inverse limit system of f -modules. Then $M = \{(m_\alpha) \in \prod M_\alpha : f_{\alpha\beta} m_\alpha = m_\beta \text{ whenever } \alpha \geq \beta\}$ is the inverse limit of this system, and M is an l -submodule of $\prod M_\alpha$. Thus M is an i - f -module if each M_α is, by 3.9 and 3.8.

Corollary 3.11. *Let M be a torsion-free f -module over an essentially positive directed po-ring R . Then M is an i - f -module if and only if every torsion-free homomorphic image of M is an i - f -module.*

Proof. This follows immediately from 3.8(b).

Proposition 3.12. *Let R be an essentially positive po-ring with $Z(R) = 0$, and let S be a directed (in its canonical order) right quotient ring of R . Suppose that M is a torsion-free f -module over R . Then M is an i - f -module if and only if it is contained in a torsion-free i - f -module over S . In fact, $E(M_R)$ is an f -module over S and an injective S -module.*

Proof. Let $E = E(M_R)$ be an f -module extension of M . By [23, Theorem 3 and Proposition 10], E is a Q -module, where Q is the maximal right quotient ring of R . If $x \wedge y = 0$ in E , $q \in Q^+$, and $D \in R^\nabla$ with $qD \subseteq R$, then $xqD^+ \subseteq E^+$, so $xq \in E^+$. Also, if $d \in D^+$, then $(xq \wedge y)d = xqd \wedge yd = 0$, so $xq \wedge y = 0$. Thus E is a distributive l -module over Q . Since S is directed (and E_S is torsion-free) E_S is an f -module. But any S -essential extension of E is also an R -essential extension of E , so E_S is injective.

Now suppose that M is an R -submodule of a torsion-free injective f -module K_S . Since $E(K_R)$ is an S -module, it is an S -essential extension of K , so K is R -injective. By 3.8, M is an i - f -module.

Theorem 3.13. *A totally ordered domain has a nonzero torsion-free i - f -module if and only if it is a right Ore domain.*

Proof. Let M be a torsion-free i - f -module over the totally ordered domain R , and let $0 < x \in M$. Then $R \rightarrow xR$ is an f -module homomorphism, so $R/r(x) \cong xR$. Since M is torsion-free, $r(x)$ is not an essential right ideal of R . Let $0 \neq J$ be a right ideal such that $J \cap r(x) = 0$, and take $0 < y \in J$. Then yR is isomorphic to an f -submodule of M , so it is an i - f -module. But $yR \cong R$ as f -modules, so R_R is an i - f -module. Therefore, $E(R_R)$ satisfies the condition of Theorem 3.4, and so R is a right Ore domain, by [1, 3.1 and 5.2].

Theorem 3.13 is not true for an arbitrary f -ring with $Z(R) = 0$, i.e. a torsion-free f -ring can have a torsion-free i - f -module without being a qf -ring. In particular, let R be an irredundant semiprime f -ring, and let M be a torsion-free f -module over R . Let $R \subseteq \prod R_\alpha$ and $M \subseteq \prod M_\alpha$ be the decompositions of R and M . Then M is an i - f -module if and only if $M_\alpha = 0$ for each α for which R_α is not a right Ore domain.

Proposition 3.14 (see [4, p. 239]). *Let K be a torsion-free f -module over an essentially positive po-ring, and let M be an essential l -submodule of K . Suppose that N is a convex l -submodule of M , and let $K_1 = \text{Cl}_K N$.*

(a) *If $M = \text{lex } N$, then $K = \text{lex } K_1$.*

(b) *If $K = \text{lex } K_1$, then $M = \text{lex } \text{Cl}_M N$.*

(c) *Suppose that $K = E(M)$ and $K = \text{lex } K_1$. If K_2 is any R -complement of K_1 in K , then K_2 is a totally ordered submodule of K , and*

$$K^+ = \{y + z \in K_1 \oplus K_2 : z > 0, \text{ or } z = 0 \text{ and } y \geq 0\}.$$

Proof. (a) Suppose that $y \in K^+ \setminus K_1$ and $x \in K_1$. Then $(y - x)^+ \neq 0$. Assume that $(y - x)^- \neq 0$, also. There exists $0 \leq d \in (M : y) \cap (M : x) = D$ such that $(y - x)^- d \neq 0$. So $(y - x)^+ D^+ \subseteq N$, since N contains all the nonunits of M [5, p. 111]. Therefore $(y - x)^+ \in K_1$. Similarly, $(y - x)^- \in K_1$. Thus $y - x \in K_1$ and $y \in K_1$. So $(y - x)^- = 0$ and $y \geq x$. Since K_1 is prime in K by 2.1(c), $K = \text{lex } K_1$.

(b) If $x \in M^+ \setminus \text{Cl}_M N$, then $x \in K^+ \setminus K_1$, since $K_1 \cap M = \text{Cl}_M N$. Therefore $x > \text{Cl}_M N$. Clearly, $\text{Cl}_M N$ is prime in M .

(c) The remarks following 3.8 show that K_2 is totally ordered. Since $K = \text{lex } K_1$, we clearly have $K^+ = \{y + z \in K_1 \oplus K_2 : z > 0 \text{ or } z = 0 \text{ and } y \geq 0\}$.

We show next that the i - f -property is a local property.

Theorem 3.15. *Suppose that R is a right noetherian, essentially positive, directed po-ring, and let M be a subdirect product of totally ordered torsion-free f -modules. Then M is an i - f -module if and only if $C_R(g)$ is an i - f -module for each $g \in M$.*

Proof. Suppose that M is a subdirect product of the family $\{M_\alpha : \alpha \in A\}$ of totally ordered torsion-free f -modules. We may assume that $M_\alpha = M/N_\alpha$ for each $\alpha \in A$. Therefore, if $x + N_\alpha \in M_\alpha$, then $C_R(x + N_\alpha) = (C_R(x) + N_\alpha)/N_\alpha$. If $C_R(x)$ is an i - f -module, then so is $(C_R(x) + N_\alpha)/N_\alpha$, by 3.11. Now suppose that M is an i - f -module locally, i.e. $C_R(g)$ is an i - f -module for each $g \in M$. The preceding remarks show that each M_α is also an i - f -module locally. If each M_α is an i - f -module, then M is an i - f -module by 3.9 and 3.8. So, without loss of generality, we may assume that M is totally ordered.

Let $x \in E(M) = E$ and let $\{N_\alpha : \alpha \in A\} = \{C_R(xd) : d \in (M : x)^+\}$. If $E_\alpha = E(N_\alpha) \subseteq E$, then E_α is an f -module extension of N_α by hypothesis. Since M is totally ordered, $\{N_\alpha : \alpha \in A\}$ is totally ordered by inclusion, and so is the family $\{E_\alpha : \alpha \in A\}$. Thus $N = \bigcup N_\alpha$ is a submodule of $E_1 = \bigcup E_\alpha$. Since R is noetherian, and since E_1 is the direct limit of the E_α , E_1 is an injective R -module ([3, p. 17] and [10, p. 53]). Clearly, E_1 is an f -module extension of N with positive cone $\bigcup E_\alpha^+$, and it is the injective hull of N .

Now $x(M : x)^+ \subseteq N \subseteq E_1$; so $x \in E_1$, since E_1 is closed in E . If $d_1, d_2 \in R^+$, then $(xd_1)^+ \wedge (xd_2)^- = x^+ d_1 \wedge x^- d_2 = 0$. But then M is an i - f -module, by 3.4.

The converse follows from 3.8.

Corollary 3.16. *Let R be a right noetherian, essentially positive, directed po-ring, and let M be a subdirect product of totally ordered torsion-free R -modules. If M is a finitely-valued f -module, then M is an i - f -module if and only if $C_R(g)$ is an i - f -module for every special element g of M .*

Proof. If $g \in M$, then $C_R(g) = C_R(g_1) \oplus \dots \oplus C_R(g_n)$, where each g_i is special [20]. Now use 3.14.

Note that the proof of 3.15 is valid if, instead of assuming that R is noetherian, one assumes that M has the maximum condition on convex l -submodules.

We consider next the Hahn product of strict i - f -modules. Let Γ be a rooted po-set, and suppose that for each $p \in \Gamma$, M_p is an f -module over the po-ring R . Suppose further that M_p is totally ordered if p is not a minimal element of Γ .

For $v \in \prod M_p$ (R -module product) define the *support* (*supp*) of $v = \{p \in \Gamma : v(p) \neq 0\}$. It is well known ([7] or [24]) that

$$V(\Gamma, M_p) = \{v \in \prod M_p : \text{supp } v \text{ has the maximum condition}\}$$

is an l -group when provided with the positive cone

$$V^+ = \{v \in V : v(p) > 0 \text{ whenever } p \text{ is a maximal element of } \text{supp } v\} \cup \{0\}.$$

An f -module M is called *strict* if it satisfies the following condition: $(x, r) \in M^+ \times R^+$ and $xr = 0$ implies $x = 0$ or $r = 0$. Note that if R has a strict f -module, then R is a po-domain, i.e. $a, b \in R^+$ and $ab = 0$ implies $a = 0$ or $b = 0$.

Proposition 3.17. *Suppose that R is directed and each M_p is strict. Then $V = V(\Gamma, M_p)$ is an f -module over R .*

Proof. Since $(vr)(p) = v(p)r$ for all $v \in \prod M_p$, $r \in R$, and $p \in \Gamma$, $\text{supp } vr \subseteq \text{supp } v$. Thus V is an R -module. If $v \in V^+$, p is a maximal element of $\text{supp } v$, and $0 < r \in R^+$, then $(vr)(p) = v(p)r > 0$, since M_p is strict. So V is an l -module over R . If P is a maximal chain of Γ , then $V_P = \{v \in V : \text{supp } v \subseteq P\} \cong V(P, M_p)$ as l -modules, and V is a subdirect product of the V_P . Thus V is an f -module provided each V_P is an f -module. So we may assume that Γ is totally ordered.

Let $0 \neq v \in V$, $0 < r \in R^+$, and let q be the maximal element of $\text{supp } v$. Note that $\text{supp } v = \text{supp } vr$, and $v(q) < 0$ if and only if $(vr)(q) < 0$, since M_q is strict. Therefore, $v^+r = (vr)^+$. So V is a distributive l -module over R . Hence it is an f -module.

Proposition 3.18. *Let R be an essentially positive directed po-ring, and let M_p be an i - f -module for each p in the rooted po-set Γ . Suppose further that M_p is totally ordered if p is not a minimal element of Γ . If each $E_p = E(M_p)$ is strict, then $V(\Gamma, E_p) = E(V(\Gamma, E_p))$ and $V(\Gamma, M_p)$ is an i - f -module.*

Proof. Since $\prod E_p$ is an injective module, $V(\Gamma, E_p) \subseteq \prod E_p$ has an injective hull $E(V(\Gamma, E_p)) = E$ contained in $\prod E_p$. Let $w \in E$ and $0 \neq d \in R^+$ such that $wd \in V(\Gamma, E_p)$. Suppose that $p_1 < p_2 < \dots$ is a chain in $\text{supp } w$. Since p_i is not a minimal element of Γ for $i > 1$, $w(p_i) > 0$ or $w(p_i) < 0$. Thus $p_i \in \text{supp } wd$ for $i > 1$. This is impossible, since $\text{supp } wd$ has the maximum condition. So $w \in V(\Gamma, E_p)$ and $E(V(\Gamma, E_p)) = V(\Gamma, E_p)$. Since $V(\Gamma, M_p)$ is an l -submodule of $V(\Gamma, E_p)$, $V(\Gamma, M_p)$ is an i - f -module.

If R is commutative, then E is strict provided M is, but we do not know if this is true for any R . The po-ring R in 3.18 need not be a domain. Diem [8] has given the following example of a commutative l -domain that is not a domain: $R = \mathbb{Q}a \oplus \mathbb{Q}b$ as l -groups and $a^2 = b^2 = ab = ba = a$. We remark that the essential ideals of R are those ideals that contain $\mathbb{Q}a + \mathbb{Z}qb$ for some $0 \neq q \in \mathbb{Q}$. Thus R

is essentially positive. As a ring $R = Qa \oplus Q(a - b)$, so Qa is a strict i - f -module over R .

We close this section with a remark about extending homomorphisms.

Proposition 3.19. *Let N and K be distributive l -modules over the essentially positive po-ring R , and suppose that M is an essential l -submodule of K . If $f \in \text{Hom}_R(M, N)$ is an l -homomorphism and if N is torsion-free, then any $g \in \text{Hom}_R(K, N)$ extending f is an l -homomorphism, and g is unique.*

Proof. Since N is torsion-free and M is essential in K , g is unique. If $x \in K$ and $d \in (M : x)^+$, then

$$[g(x^+)]d = g(x^+d) = g[(xd)^+] = f[(xd)^+] = [f(xd)]^+ = [g(xd)]^+ = [(gx)^+]d.$$

So $g(x^+) = (gx)^+$.

Corollary 3.20. *Let R be an essentially positive po-ring, and let M be an i - f -module over R . Then any l -homomorphism from M into a torsion-free injective f -module N has a unique extension to an l -homomorphism from $E(M)$ to N .*

4. Relative injective f -modules. Ribenboim [19] has observed that there are no injectives in the category of unital po-modules over a po-domain. Consequently, he defined and studied a certain type of relative injectivity. Let $\mathcal{C}$ be a category whose objects are sets, and let $\aleph_\alpha$ be an infinite cardinal number. An object E of $\mathcal{C}$ is called $\aleph_\alpha$ -*injective* if whenever C is a subobject of A in $\mathcal{C}$ and $\text{card}(A) < \aleph_\alpha$, then every morphism $C \rightarrow E$ can be extended to a morphism $A \rightarrow E$.

Weinberg [25] has recently given a characterization of the $\aleph_\alpha$ -injectives in the category of l -groups. In this section we show that his characterization (and his methods) holds in the category of torsion-free f -modules over an irredundant semi-prime right qf -ring.

Definition 4.1. An l -group M is an almost- η_α -group if, for each pair of subsets X and Y of M such that $X < Y$ and $\text{card}(X \cup Y) < \aleph_\alpha$, there exists an a in M such that $X \leq a \leq Y$.

Definition 4.2 (Weinberg [25]). An element y in an l -group M is said to *split* b from a if $y \geq a^+$, $y \wedge a^- = 0$, and $(b - y)^+ \wedge y = 0$. M is *self-splitting* if each ordered pair of elements of M is split by some element in M .

The cardinal number $\aleph_\alpha$ is called *regular* provided $\bigcup \{X_i : i \in I\}$ has cardinality less than $\aleph_\alpha$ whenever I and each X_i have cardinality less than $\aleph_\alpha$.

Theorem 4.3 (Weinberg [25]). *Let $\aleph_\alpha$ be a regular cardinal number, and let M be an l -group. The following are equivalent.*

- (a) M is $\aleph_\alpha$ -injective.
- (b) M is a divisible, self-splitting, almost- η_α -group in which any two pairwise disjoint subsets of cardinality less than $\aleph_\alpha$ have disjoint upper bounds.

Now let M_R be an f -module. We will show, with not too many restrictions on R , that Weinberg's four conditions are necessary for M to be an $\aleph_\alpha$ -injective f -module. The converse is much harder and we only have it for the case that R is an irredundant semiprime qf -ring.

Lemma 4.4. *Let M be a torsion-free i - f -module over the directed po-ring R , let $\mathcal{C}$ be the category of torsion-free f -modules over R . If M is $\aleph_\alpha$ -injective in $\mathcal{C}$, then so is $E = E(M_R)$.*

Proof. Suppose that $N \in \mathcal{C}$ with $\text{card}(N) < \aleph_\alpha$. Let N' be an l -submodule of N , and let f be a homomorphism from N' into E . Then $N'' = f^{-1}(M)$ is an essential l -submodule of N' . We have the following diagram:

$$\begin{array}{ccccc} N'' & \subseteq & N' & \subseteq & N \\ \downarrow f & & \downarrow f & \nearrow g & \\ M & \subseteq & E & & \end{array}$$

where g comes from the fact that M is $\aleph_\alpha$ -injective. Since $N'' \subseteq \ker(g - f)$, since N'' is essential in N' , and since M is torsion-free, $N' \subseteq \ker(g - f)$. Thus $f(N') \subseteq M$, and g is an extension of f .

Corollary 4.5. *Let M be a torsion-free i - f -module over the directed po-ring R . If $\aleph_\alpha > \text{card}(R)$ and if M is $\aleph_\alpha$ -injective in the category of torsion-free f -modules over R , then $M = E(M_R)$.*

Proof. Take $x \in E$ and let $N_1 = xR_*$. It is known that the sublattice of E generated by N_1 is a subgroup, and is given by

$$L(N_1) = \left\{ \bigvee_{j=1}^m \bigwedge_{i=1}^n s_{ij} : s_{ij} \in N_1 \right\}.$$

If $r \in R^+$, then $L(N_1)r \subseteq L(N_1)$, since E is an f -module. Thus $L(N_1)$ is an l -submodule of E , since R is directed. Clearly, $\text{card}(L(N_1)) < \aleph_\alpha$. In the proof of 4.4, let $N' = N = L(N_1)$, $N'' = L(N_1) \cap M$, and let f be the inclusion map from N' to E . Then $f(L(N_1)) \subseteq M$, so $x \in M$.

Lemma 4.6. *Let R be a directed po-domain which has a strict totally ordered module K . Then any f -module M can be embedded in an f -module which contains, for each a in M , an element y that splits b from a for every b in M .*

Proof. M can be embedded in a product of a family of totally ordered modules $\{M_i : i \in I\}$. Let $N_i = M \oplus K$ as R -modules with positive cone defined by $N_i^+ = \{(m, k) : k > 0, \text{ or } k = 0 \text{ and } m \geq 0\}$. Define $y \in \prod N_i$ by $y(i) = 0$ if $a^+(i) = 0$, and

$y(i) = (0, k_0)$ if $a^+(i) > 0$, where k_0 is a fixed nonzero positive element of K .

Proposition 4.7. *Let R be a directed po-domain which has a strict totally ordered module M , and let $\aleph_\alpha$ be a cardinal number greater than $\text{card}(R)$. If M is an $\aleph_\alpha$ -injective f -module, then M is self-splitting and each pair of pairwise disjoint subsets of M of cardinality less than $\aleph_\alpha$ have disjoint upper bounds.*

Proof. Let $a, b \in M$, and let A be the l -submodule of M generated by a and b . By 4.6, A is imbeddable in an f -module C containing an element y which splits b from a . Let B be the l -submodule of C generated by A and y . Then $\text{card}(B) < \aleph_\alpha$, and the injection of A into M can be extended to a homomorphism ϕ from B into M , since M is $\aleph_\alpha$ -injective. Clearly, $\phi(y)$ splits b from a . Thus M is self-splitting.

Let A_1 and A_2 be pairwise disjoint subsets of M such that $\text{card}(A_1) + \text{card}(A_2) < \aleph_\alpha$. Embed M in a product of totally ordered R -modules $M \rightarrow \prod M_i$. As in the proof of 4.6, let N_i be the lexicographic extension of M_i by K . Define t_j in $N = \prod N_i$, for $j = 1, 2$, by $t_j(i) = (0, k_0)$ if $\pi_i(A_j) \neq 0$ and $t_j(i) = 0$ if $\pi_i(A_j) = 0$. Then $t_1 \wedge t_2 = 0$ and $t_j \geq A_j$. Let L be the l -submodule of N generated by $A_1 \cup A_2$, and let P be the l -submodule of N generated by L and $\{t_1, t_2\}$. Then $\text{card}(P) < \aleph_\alpha$, and the injection of L into M can be extended to a homomorphism ϕ from P into M . Thus $\phi(t_1) \wedge \phi(t_2) = 0$, and $\phi(t_j) \geq A_j$.

A totally ordered set T is said to be *dense* if for all $a < b$ in T there exists $x \in T$ such that $a < x < b$. If M is a totally ordered module over a po-ring R , then $Q \otimes_Z M$ is a dense totally ordered R -module containing M .

Lemma 4.8. *Let M be a totally ordered module over the po-ring R , and suppose that X and Y are subsets of M such that $X < Y$. Then M can be embedded in a totally ordered module N containing an element a satisfying $X < a < Y$. Moreover, if R is right noetherian and M is torsion-free, then such an N may be found which is torsion-free.*

Proof. By the preceding remark we may assume that M is dense. Suppose that no element of M lies between X and Y . Then either X is nonempty and has no last element or Y is nonempty and has no first element. Suppose, for example, that X is nonempty and has no last element. For $x \in X$, let $A_x = \{z \in X : z > x\}$. Since X has no last element, $\{A_x : x \in X\}$ has the finite intersection property. Therefore, there is an ultrafilter $\mathcal{G}$ on X containing $\{A_x : x \in X\}$.

For each $b \in M^X$ let $Z(b) = \{x \in X : b(x) = 0\}$. Let $I = \{f \in M^X : Z(f) \in \mathcal{G}\}$. Then I is a prime submodule of M^X . Embed M in M^X/I via the map $m \rightarrow \langle m \rangle + I$, where $\langle m \rangle(x) = m$ for all $x \in X$. If $e \in M^X$ is the identity on X , i.e. $e(x) = x$ for all $x \in X$, then $\langle X \rangle + I < e < \langle Y \rangle + I$.

Suppose that M is torsion-free and R is right noetherian. Let $f \in M^X$ and $D \in R^\nabla$ such that $fD \subseteq I$. Since R is noetherian, $D = d_1 R_* + \dots + d_n R_*$. Clearly,

$Z(f) \subseteq \bigcap \{Z(f d_i): i = 1, \dots, n\}$. Conversely, if $(f d_i)(x) = 0$ for $1 \leq i \leq n$, then $f(x)D = 0$. Thus $f(x) = 0$, since M is torsion-free. So $Z(f) = \bigcap \{Z(f d_i): i = 1, \dots, n\}$. Hence $Z(f) \in \mathfrak{E}$ and $f \in I$.

Proposition 4.9. *Let M be an $\aleph_\alpha$ -injective f -module over the directed po-ring R , and suppose $\aleph_\alpha > \text{card}(R)$. Then M is an almost- η_α -module.*

Proof. Suppose that X and Y are subsets of M , each of which has cardinality less than $\aleph_\alpha$, and $X < Y$. Assume M is embedded in the product $\prod M_j$, where each M_j is totally ordered. Let N_j be a totally ordered module containing M_j and an element t_j such that $\pi_j(X) \leq t_j \leq \pi_j(Y)$. Let $t \in \prod N_j = K$ be defined by $t(j) = t_j$. Finally, let A be the l -submodule of M generated by $X \cup Y$, and let B be the l -submodule of K generated by A and t . Then $\text{card}(B) < \aleph_\alpha$, and the injection of A into M can be extended to B . Thus the image of t in M lies between X and Y .

Suppose that R is right noetherian (and directed), and M is a subdirect product of totally ordered torsion-free R -modules. If M is $\aleph_\alpha$ -injective in the category of torsion-free f -modules over R , then M is an almost- η_α -module. The proof is the same as that of 4.9. Of course, if M is a torsion-free f -module over an essentially positive po-ring R , then M is $\aleph_\alpha$ -injective in the category of R - f -modules if and only if it is $\aleph_\alpha$ -injective in the category of torsion-free R - f -modules. For then the torsion submodule, $\text{Cl Cl}_A 0 = Z_2(A)$, of each f -module A is a convex l -submodule; so an f -module homomorphism $A \rightarrow M$ induces a torsion-free f -module homomorphism $A/Z_2(A) \rightarrow M$.

For the remainder of this section $\aleph_\alpha$ will be a regular cardinal number.

Theorem 4.11. *Let R be a totally ordered right Ore domain, and let M be a torsion-free f -module over R . If $\aleph_\alpha > \text{card}(R)$, then M is $\aleph_\alpha$ -injective in the category of torsion-free f -modules over R if and only if it is injective in the category of R -modules and $\aleph_\alpha$ -injective in the category of l -groups.*

Proof. If M is $\aleph_\alpha$ -injective, then 4.5, 4.7, 4.9, and 4.3 imply that M_R is injective and that M is an $\aleph_\alpha$ -injective l -group.

Conversely, if M_R is torsion-free and injective, then M is a vector lattice over D , where D is the totally ordered right quotient division ring of R (see 3.12). Thus we may assume that $R = D$. Now copy the proof of 4.3 given in [24] for the case that $R = Q$.

Corollary 4.12. *Let R be an irredundant semiprime right qf-ring, and let M be a torsion-free f -module over R . Suppose that $\aleph_\alpha > \text{card}(R)$. Then M is $\aleph_\alpha$ -injective in the category of (torsion-free) f -modules over R if and only if M is an injective R -module and an $\aleph_\alpha$ -injective l -group.*

Proof. Let $R \subseteq \prod R_\lambda \subseteq \prod D_\lambda = Q$ be the decomposition of R (see 2.7). If M is either $\aleph_\alpha$ -injective or R -injective, then $M = \prod E_\lambda$, where E_λ is an f -module over

$R(R_\lambda)$ and an injective R - (R_λ) -module (3.6, 4.5, and the proof of 2.9). By the usual argument, it is easily seen that M is $\aleph_\alpha$ -injective if and only if each E_λ is $\aleph_\alpha$ -injective.

Now E_λ is an $\aleph_\alpha$ -injective f -module over R if and only if it is an $\aleph_\alpha$ -injective f -module over R_λ . For suppose that E_λ is $\aleph_\alpha$ -injective with respect to R . Let A be an f -submodule of the R_λ - f -module B ($\text{card}(B) < \aleph_\alpha$), and let $f: A \rightarrow E$ be a map in the category of R_λ - f -modules. Since R_λ is a homomorphic image of R , A and B are naturally f -modules over R , and then f is an R -homomorphism. Let $g: B \rightarrow E_\lambda$ be an R -extension of f . Then g is clearly an R_λ -extension of f ; so E_λ is $\aleph_\alpha$ -injective over R_λ .

On the other hand, suppose that E_λ is $\aleph_\alpha$ -injective over R_λ . Let A be an f -submodule of the torsion-free R - f -module B ($\text{card}(B) < \aleph_\alpha$), and let $f: A \rightarrow E$ be a map in the category of R - f -modules. By 3.20, we may assume that A and B are injective R -modules. Thus A and B (and E_λ) are Q -modules (see 3.12), and f is a Q -map. Let $g: B \rightarrow E_\lambda$ be the R_λ -extension of f . Then g is, in fact, a Q -extension of f , hence an R -extension. So E_λ is $\aleph_\alpha$ -injective over R .

In summary, we have M is R - $\aleph_\alpha$ -injective if and only if $M = \prod E_\lambda$, where E_λ is R_λ - $\aleph_\alpha$ -injective. But E_λ is R_λ - $\aleph_\alpha$ -injective if and only if it is $\mathbb{Z}$ - $\aleph_\alpha$ -injective, by Theorem 4.11. Thus M is R - $\aleph_\alpha$ -injective exactly when it is an injective R -module and an $\aleph_\alpha$ -injective l -group.

Note that there are no injectives in the category of torsion-free R - f -modules. For any nonzero torsion-free injective R - f -module gives rise to a nonzero torsion-free injective R_λ - f -module for some λ , and there are none. There are quasi-injectives, however. In particular, if R is any semiprime qf -ring and f is a homomorphism from the l -submodule M of Q_R into Q_R , then f may be extended to $E(M) \subseteq Q$, by 3.20, and thus it can be extended to Q , since $E(M)$ is a summand of Q . So Q_R is a quasi-injective R - f -module.

The next corollary is an immediate consequence of Weinberg's theorem (Theorem 4.3) and 4.12.

Corollary 4.13. *The following statements are equivalent for a torsion-free f -module M over an irredundant semiprime right qf -ring R ($\text{card}(R) < \aleph_\alpha$).*

- (a) M is $\aleph_\alpha$ -injective.
- (b) M is an injective R -module, and a self-splitting almost- η_α -module in which any two pairwise disjoint subsets of cardinality less than $\aleph_\alpha$ have disjoint upper bounds.

Finally, we remark that there are enough $\aleph_\alpha$ -injectives for embedding purposes when R is an irredundant semiprime right qf -ring. The proof is the same as that given by Weinberg in [25] for $R = \mathbb{Z}$. We can reduce to the case where R is a totally ordered division ring. Then the idea is to enlarge the given vector lattice M ,

successively, via vector lattices in which every pair of elements of M is split, every pair of disjoint subsets of M of small cardinality have disjoint upper bounds, and in which there is an element between every pair of subsets of M of small cardinality, one of which is smaller than the other. By repeating this procedure inductively one eventually gets a vector lattice having the properties of 4.3(b).

5. Remarks on an Hahn embedding theorem for f -modules. Let M be a torsion-free f -module over the irredundant semiprime right qf -ring R , and let $R \subseteq \prod R_\lambda \subseteq \prod D_\lambda = Q$ and $M \subseteq \prod M_\lambda \subseteq \prod E_\lambda = E$ be the representations of R and M , respectively. Let Γ_λ be the D_λ -value set of E_λ . For each lower submodule $M_\alpha \in \Gamma_\lambda$, let M^α be the convex l -submodule of E_λ that covers it. Since the proof of the Hahn embedding theorem for l -groups [7] is valid for a vector lattice over a totally ordered division ring, E_λ is D_λ -value embedded in the Hahn product $V_\lambda = V(\Gamma_\lambda, M^\alpha/M_\alpha)$. Thus the f -module M_R is embedded in the product of the V_λ . Let Γ be the cardinal sum of the Γ_λ . Then the map $(v_\lambda) \rightarrow \bar{v}$, $\bar{v}(\alpha) = v_\lambda(\alpha)$ embeds the Q - f -module $\prod V_\lambda$ onto the Hahn product $V(\Gamma, M^\alpha/M_\alpha) = V$. (V is a Hahn product as a Q - f -module, i.e. $V^+Q^+ \subseteq V^+$: Suppose $0 < v \in V$ and $0 < q = (q_\lambda) \in Q$. Let α be a maximal element in the support of vq , $\alpha \in \Gamma_\mu$. Then α is a maximal element in the support of v , and $(vq)(\alpha) = v(\alpha)(q_\lambda) = v(\alpha)q_\mu > 0$.)

If M has only a finite number of nonzero components, in particular, if R is a semiprime right Goldie f -ring, then Γ is the Q -value set of $E(M)$, but in general Γ is only contained in the latter. If a component R_λ of R is archimedean, in particular, if R is archimedean, then the M^α/M_α for $\alpha \in \Gamma_\lambda$ are D_λ -submodules of the reals. In this case Γ_λ is isomorphic to the value set of M_λ .

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