

ON ANTIFLEXIBLE ALGEBRAS

BY

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ABSTRACT. In this paper we begin a classification of simple and semisimple totally antiflexible algebras (finite-dimensional) over splitting fields of char. $\neq 2, 3$. For such an algebra A , let P be the largest associative ideal in A^+ and let N^+ be the radical of P . We determine all simple and semisimple totally antiflexible algebras in which $N \cdot N = 0$. Defining A to be of type (m, n) if N^+ is nilpotent of class m with $\dim A = n$, we then characterize all simple nodal totally antiflexible algebras (over fields of char. $\neq 2, 3$) of types (n, n) and $(n-1, n)$ and give preliminary results for certain other types.

1. Introduction. A totally antiflexible algebra is a nonassociative algebra (finite-dimensional) satisfying

$$(1) \quad (x, y, z) = (z, y, x) \quad (\text{the antiflexible law})$$

and

$$(2) \quad (x, x, x) = 0,$$

where $(x, y, z) = (xy)z - x(yz)$. Throughout this paper we assume char. $\neq 2, 3$ and we define $x \cdot y = (xy + yx)/2$. The algebra A^+ is that formed from A with multiplication $x \cdot y$. Define $(x, y) = xy - yx$.

Define $x^1 = x$, $x^{k+1} = x^k x$ and $x^{-1} = x$, $x^{-k+1} = x^{-k} \cdot x$. It is known that a totally antiflexible algebra A with char. $\neq 0$ need not be power-associative [6]. However A^+ is known to be power-associative so $x^m \cdot x^n = x^{(m+n)}$ for all positive integers m, n . We will call y nilpotent or nil if, for some n , $y^n = 0$. If x in A implies $x = \alpha 1 + z$ for α in the base field and z nil and if the set of nil elements is not a subalgebra, we say that A is *nodal*.

2. Preliminaries. We will state some known results on the structure of simple and semisimple totally antiflexible algebras. We also need (see [1], [7])

Definition 2.1. A field K is said to be a splitting field for an algebra A if every primitive idempotent e of A_K is absolutely primitive and if every element in $(A_K)_e$ (1) for e primitive can be written as $ke + y$ with k in K and y nilpotent or $y = 0$.

Definition 2.2. Let A be an algebra over a field F of char. $\neq 2, 3$. The mapping $\phi: A \times A \rightarrow B$ for $B \subset A$ will be called an antiflexible map provided $B \subseteq \{x: xy = yx \text{ for all } y \text{ in } A\}$ and

Presented to the Society, February 19, 1971; received by the editors March 31, 1971.
AMS 1969 subject classifications. Primary 1760.

Key words and phrases. Simple algebras, antiflexible algebras.

- (3) ϕ is bilinear over F ,
- (4) $\phi(x, x) = 0$,
- (5) $\phi(x^2, x) = 0$,
- (6) $\phi(x, y) = 0$ if y is in B ,
- (7) $\phi((x, y), z) = 0$.

This ϕ in our definition is similar to maps used in [1], [4], [7]. For char. $\neq 2$, (4) is equivalent to

$$(8) \phi(x, y) = -\phi(y, x).$$

Also, for char. $\neq 3$, (5) is equivalent to

$$(9) \phi(x \cdot y, z) + \phi(y \cdot z, x) + \phi(z \cdot x, y) = 0.$$

For α, β in F and antiflexible maps ϕ_1, ϕ_2 define $\alpha\phi_1 + \beta\phi_2$ by

$$(\alpha\phi_1 + \beta\phi_2)(x, y) = \alpha\phi_1(x, y) + \beta\phi_2(x, y).$$

For char. $\neq 2, 3$, it is clear that $\alpha\phi_1 + \beta\phi_2$ is an antiflexible map.

Definition 2.3. Let A be an algebra over a field of char. $\neq 2, 3$ and let ϕ be an antiflexible map. Define $A(\phi)$ as the algebra formed from A with multiplication replaced by $x * y = xy + \phi(x, y)$.

It is known [4] that A is antiflexible if and only if $A(\phi)$ is. From this, the following lemma is obvious.

Lemma 2.1. Let A be an algebra over a field of char. $\neq 2, 3$ and let ϕ be an antiflexible map. Then A is totally antiflexible if and only if $A(\phi)$ is totally antiflexible. Also, if ψ is an antiflexible map on $A(\phi)$ then $A(\phi)(\psi) = A(\phi + \psi)$.

We now summarize certain results in [1], [4], [7] by the following two theorems.

Theorem 2.1. If A is a simple not associative totally antiflexible algebra over a field F of char. $\neq 2, 3$ then A^+ is associative and $A = A_1 + \cdots + A_n$ where $A_i = A_{11}(e_i)$ for e_i primitive. Furthermore, $\phi(x, y) = \frac{1}{2}(x, y)$ is an antiflexible map and $A = A^+(\phi)$.

Theorem 2.2. If A is a semisimple totally antiflexible algebra over a field F of char. $\neq 2, 3$ then $A = C + D$ where $C = 0$ or C is an associative semisimple ideal with identity e and D^+ is associative. If $D \neq 0$ then $D = A_1 + \cdots + A_n$ where $A_i = A_{11}(e_i)$ for e_i primitive, $i \neq n$, and either $A_n = A_{11}(e_n)$ for e_n primitive or A_n is nil and $A_n = A_{00}(e + e_1 + \cdots + e_{n-1})$. Furthermore, if w, x in C and y, z in D define ϕ by $\phi(w + y, x + z) = \frac{1}{2}(y, z)$. Then ϕ is antiflexible and $A = (C \oplus D^+)(\phi)$.

We will thus be interested in those algebras from which simple or semisimple algebras can be constructed.

Definition 2.4. A totally antiflexible algebra will be called nearly simple (nearly semisimple) if there is an antiflexible map ϕ such that $A(\phi)$ is simple (semisimple).

We will now state some preliminary results on nearly simple and nearly semisimple algebras. Obviously, an associative semisimple algebra C is nearly semisimple.

Theorem 2.3. *Let A be a totally antiflexible algebra over a field F of char. $\neq 2, 3$ and let $A = C + D$ where C is a semisimple associative ideal with identity e and $D = A_1 + \cdots + A_n$ with $A_i = A_{11}(e_i)$ for e_i primitive, $i \neq n$, and either $A_n = A_{11}(e_n)$ for e_n primitive or A_n nil and $A_n = A_{00}(e + e_1 + \cdots + e_{n-1})$. Also, assume D^+ is associative. Then A is nearly semisimple if and only if $C \oplus D^+$ is nearly semisimple.*

Proof. To begin with, let w, x be in C and y, z in D . Define

$$\phi(w + y, x + z) = \frac{1}{2}(y, z).$$

We claim that ϕ is an antiflexible map. The proof is a routine verification of the conditions of Definition 2.2. Recall also that, in a totally antiflexible algebra, $A_{11}(f)A_{00}(f) = A_{00}(f)A_{11}(f) = 0$ for f an idempotent. Also, $A = (C \oplus D^+)(\phi)$. Now suppose A is nearly semisimple so $A(\psi)$ is semisimple for some ψ . Now $A(\psi) = (C \oplus D^+)(\phi + \psi)$ so $C \oplus D^+$ is nearly semisimple. Also, $C \oplus D^+ = A(-\phi)$. Now if $(C \oplus D^+)(\psi)$ is semisimple then $A(\psi - \phi)$ is semisimple.

In a similar way, we can prove

Theorem 2.4. *Let A be a totally antiflexible algebra over a field of char. $\neq 2, 3$ and assume A^+ is associative. Then A is nearly simple (nearly semisimple) if and only if A^+ is nearly simple (nearly semisimple).*

Proof. The only additional fact we need is the fact that $\phi(x, y) = \frac{1}{2}(x, y)$ is an antiflexible map on A . It is known [3, p. 474] that if A^+ is associative and A is antiflexible then $((w, x), y) = 0$. It is then easy to verify the fact that ϕ is an antiflexible map.

Theorem 2.5. *Let A satisfy the hypotheses of Theorem 2.3 and let $Z = \text{center of } C$. Then A is nearly semisimple if and only if $Z \oplus D^+$ is nearly semisimple.*

Proof. For any antiflexible map ϕ on A , $\{\phi(x, y)\} \subseteq \{x: xy = yx \text{ for all } y \text{ in } A\}$. Hence, $\{\phi(x, y)\} \cap C \subseteq Z$. The proof is then routine.

We remark that $Z \oplus D^+$ is the largest associative ideal in A^+ .

The above results reduce the problem of finding all simple (semisimple) algebras to the following two problems:

- I. Find all nearly simple (nearly semisimple) associative commutative algebras.
- II. Given a nearly simple (nearly semisimple) associative commutative algebra A , find all simple (semisimple) algebras that can be constructed, using antiflexible maps, from A .

A nearly simple algebra possesses an identity element and the adjunction of an identity element to a nearly semisimple algebra does not destroy its being nearly semisimple. Hence, throughout the rest of this paper, we will assume that each algebra considered has an identity element.

3. Conditions on $\phi(x, y)$.

Theorem 3.1. *Let P be an associative commutative algebra over a field of char. $\neq 2, 3$ and let ϕ be a bilinear map from $P \times P \rightarrow B \subset P$ such that $\phi(P, B) = 0$. Then ϕ is an antiflexible map if and only if, for every n ; $y_1, \dots, y_n$,*

$$(10) \quad \sum_{j=1}^n \phi \left(\prod_{i \neq j} y_i, y_j \right) = 0.$$

Proof. If ϕ satisfies (10) then it must satisfy (4) and (5). Also $(x, y) = 0$ in P so (7) is satisfied and ϕ is an antiflexible map. Conversely, let ϕ be an antiflexible map. Then, for $n = 1, 2$; ϕ satisfies (10). Assume (10) for $n \leq k$ and let $y_1, \dots, y_{k+1}$ be given. For $z = \prod_{i=1}^{k-1} y_i$, we have from (9) (since P is commutative)

$$(11) \quad \phi(y_{k+1}y_k, z) + \phi(y_kz, y_{k+1}) + \phi(zy_{k+1}, y_k) = 0.$$

But, using (10) with $n = k$ yields

$$(12) \quad \phi(y_{k+1}y_k, z) = -\phi(z, y_{k+1}y_k) = \sum_{j=1}^{k-1} \phi \left(\prod_{i \neq j} y_i, y_j \right).$$

Putting (12) in (11) yields (10) with $n = k + 1$ and we are done.

Except where otherwise stated, we will assume that A is a totally antiflexible algebra with identity element over a splitting field K of char. $\neq 2, 3$ and that A^+ is associative. Hence, $A = A_1 + \dots + A_n$ with $A_i = A_{11}(e_i)$ for e_i primitive and $A_i A_j = 0$ if $i \neq j$. For, since A^+ is associative then $A_{10}(e) + A_{01}(e) = 0$ for any idempotent e (see also [5], [7]). In addition, since K is a splitting field, each element in A_i has the form $\alpha e_i + z$ for α in K and z nil. Thus, A has a basis consisting of primitive idempotents and nil elements. We define the following sets:

$$(13) \quad N = \{x: x \text{ is nil}\},$$

$$(14) \quad N_i = N_{i-1} \cdot N \text{ with } N_1 = N,$$

$$(15) \quad N'_i = N_i - N_{i+1} \text{ (quotient or difference algebra),}$$

$$(16) \quad M_i = \{x: x \cdot N \subseteq M_{i-1}\} \text{ with } M_0 = 0.$$

Define $T_x: y \rightarrow y \cdot x$ and note that, since there is an identity element 1 in A and A^+ is associative, $x \rightarrow T_x$ is an isomorphism of A^+ with $\{T_x\}$. Thus, if

$\dim A = n$, we can think of A^+ or of one of its subalgebras as an algebra of commutative $n \times n$ matrices.

For some m , $N_m = 0$ with $N_{m-1} \neq 0$. We say that A (or N) is of type (m, n) if A^+ (or N^+) is isomorphic to an algebra of commutative $n \times n$ matrices for $n = \dim A$ with $N_m = 0 \neq N_{m-1}$. The algebra A (or N) is said to be of class m .

Definition 3.1. The algebra A (or N , the radical of A^+) is of type $(m, n, d_1, \dots, d_q)$ if A (or N) is of type (m, n) , $\dim N'_i = d_i$ for $1 \leq i \leq q$ and $\dim N'_i = 1$ for $q < i \leq m-1$.

Note that if $N_i = N_{i+1}$ then $N_i = N_j$ for all $j \geq i$. Hence, either $N_i = 0$ or $\dim N'_i \geq 1$.

Lemma 3.1. The following hold for x in M_i , y in N_j and z in N_{j+1} with $j \geq i \geq 1$:

(a) $x \cdot y = 0$.

(b) If ϕ is an antiflexible map, $\phi(x, z) = 0$.

Proof. The proof of (a) is by induction on i . By definition, $M_1 \cdot N_j = 0$. Suppose $M_{i-1} \cdot N_k = 0$ for $k \geq i-1$ and choose x in M_i , y in N_{j-1} and z in N where $j \geq i \geq 1$. Then $x \cdot (y \cdot z) = (x \cdot z) \cdot y = 0$ for $x \cdot z$ is in M_{i-1} and $M_{i-1} \cdot N_{j-1} = 0$. Therefore, $M_i \cdot N_j = 0$. If ϕ is an antiflexible map on A , we can regard ϕ as an antiflexible map on $P = A^+$. Hence, (a) and Theorem 3.1 imply (b).

The results of the following theorem are found in [1], [7].

Theorem 3.2. Let A be a totally antiflexible algebra over a field of char. $\neq 2, 3$. Then A is simple (semisimple) if and only if $(I, A) \not\subseteq I$ where I is any ideal (nil ideal) of A^+ .

Theorem 3.3. Let A be a totally antiflexible algebra over a splitting field of char. $\neq 2, 3$ with A^+ associative. Then A is semisimple if and only if

(17) for every nonzero x in M_1 there is a y in N with $(x, y) \neq 0$,

(18) no nil element in $\{(x, y)\}$ generates a proper nil ideal.

Proof. First, suppose A is semisimple and note that (18) is trivially satisfied. Now, let $J = \{x \text{ in } M_1 : (x, y) = 0 \text{ for all } y \text{ in } N\}$. The algebra A has a basis of idempotents and nil elements. We have $JN = NJ = J \cdot N = 0$. Since A^+ is associative, if e is an idempotent, $A = A_{11}(e) + A_{00}(e)$. If x is in J and y is in N then $x = x_1 + x_0$ and $y = y_1 + y_0$ for x_1, y_1 in $A_{11}(e)$ and x_0, y_0 in $A_{00}(e)$. The product $xy = 0$ so $0 = xy = x_1y_1 + x_0y_0$ and $x_1y_1 = x_0y_0 = 0$. Hence, $(ex)y = (xe)y = x_1y = x_1y_1 = 0$. Similarly, $y(ex) = y(xe) = 0$ so $(ex, y) = (xe, y) = 0$. Since $J^2 = 0$ then $(ex)^2 = (e \cdot x)^2 = (e \cdot e) \cdot (x \cdot x) = 0$. Thus, $ex = xe$ in J and J is a nil ideal of A . We conclude that $J = 0$ so (17) is satisfied.

Conversely, suppose (17) and (18) are satisfied in A and let J be a proper nil ideal of A with $x \neq 0$, x in J . We first show $J \cap M_1 \neq 0$. For, if x is not in M_1 then, since $M_0 \subset M_1 \subset \cdots \subset M_{n-1} = N$ where $N^n = 0$, we have an integer i with x in M_i but not in M_{i-1} . There must be an element y in N such that $x \cdot y$ is in M_1 and since $x \cdot y$ is in J we have $M_1 \cap J \neq 0$. Now let u be nonzero in $M_1 \cap J$. By (17) there is a v in N with $z = (u, v) \neq 0$. Clearly, z is in J . If z is not nil we contradict the assumption that J is nil. If z is nil and I is the ideal generated by z then I is not nil by (18). However $I \subseteq J$ so J is not nil. We have proved A semisimple.

Define $H(A)$ by

$$(19) \quad H(A) = \{x: (x, y) = 0 \text{ for all } y \text{ in } A\}.$$

In all known examples of semisimple totally antiflexible algebras (see [3], [4], [6]), $H(A) \cap N = 0$. In many of these, $N \cdot N = 0$.

Corollary 3.1. *Let A be a totally antiflexible algebra over a field of char. $\neq 2, 3$ with A^+ associative. If either $H(A) \cap N = 0$ or $N \cdot N = 0$ then A is semisimple if and only if A satisfies (17).*

Proof. Since $((x, y), z) = 0$ for all x, y, z then $\{(x, y)\} \subseteq H(A)$ and the condition $H(A) \cap N = 0$ implies (18). Suppose $N \cdot N = 0$ and A satisfies (17). Observe that $N = M_1$ so (17) implies that if x is nil then x is not in $H(A)$. Hence, $H(A) \cap N = 0$ and we are done.

Lemma 3.2. *If R is a nodal algebra over a field F with R^+ power-associative and if $J \neq R$ is an ideal of R then J is nil or zero. Thus R is simple if and only if R is semisimple.*

Proof. Let x be a nonnil member of J and write $x = \alpha \cdot 1 + z$ with z nil and $\alpha \neq 0$, α in F . Define $u = -(1/\alpha)z$ and define n as an integer with $u^n = 0$. Then $1 = (1 - u) \cdot (1 + u + \cdots + u^{n-1}) = (1/\alpha)x \cdot (1 + u + \cdots + u^{n-1})$ is in J so $J = R$.

We shall construct two nodal algebras A, B with $A^+ = B^+$ in which $H(A) = H(B)$ contains nil elements. The algebra B satisfies (17) but not (18) and A is simple but $H(A) \cap N \neq 0$. Let P be the associative commutative algebra generated by $1, w, x, y, z$ subject only to the conditions that $w^2 = x^2 = y^2 = z^2 = 0$ and $N \cdot N \cdot N = 0$ where N is generated by w, x, y, z and 1 is the identity element of P . Thus, P has a basis $1, w, x, y, z, w \cdot x, w \cdot y, w \cdot z, x \cdot y, x \cdot z, y \cdot z$.

Let $\phi(x, y)$ be defined on this basis by $\phi(z \cdot y, x) = -\phi(x, z \cdot y) = \phi(z \cdot w, y) = -\phi(y, z \cdot w) = -\phi(z \cdot x, y) = \phi(y, z \cdot x) = -\phi(z \cdot y, w) = \phi(w, z \cdot y) = 1, \phi(x \cdot y, w) = -\phi(w, x \cdot y) = \phi(y \cdot w, x) = -\phi(x, y \cdot w) = z, \phi(w \cdot x, y) = -\phi(y, w \cdot x) = -2z$ and $\phi(u, v) = 0$ where (u, v) is any other pair of basis elements. Extend ϕ bilinearly

to all of $P \times P$. Now, define $\psi(u, v)$ by $\psi(u, v) = \alpha \cdot 1 + \beta z$ if $\phi(u, v) = \beta \cdot 1 + \alpha z$. Assume $\text{char.} \neq 2, 3$ and let $A = P(\phi)$, $B = P(\psi)$. Since $\phi(N, N) = \psi(N, N) \not\subseteq N$, A and B are nodal.

It is verified that ϕ and ψ are antiflexible maps by routinely checking (8) and (9). In addition, $H(A) = H(B) = \{\alpha 1 + \beta z : \alpha, \beta \text{ in } F\}$ so $H(A) \cap N = H(B) \cap N = \{\beta z : \beta \text{ in } F\}$. In both A and B , (17) holds. Routinely, we can show that A is simple while z , $z \cdot x$, $z \cdot y$ and $z \cdot w$ span a nil ideal of B .

Theorem 3.4. *Let A be a totally antiflexible algebra over a splitting field F of char. $\neq 2, 3$ with A^+ associative. Then A is simple if and only if*

(20) *for every x in M_1 there is a y in N with $(x, y) \neq 0$,*

(21) *no element of $\{e(x, y)\}$ generates a proper ideal where e is a primitive idempotent,*

(22) *for each primitive idempotent e in A , $\{e(x, y)\}$ is not nil.*

Proof. If A is simple then (20) is true from Theorem 3.3 and (21) is obvious. If e is primitive with $\{e(x, y)\}$ nil, recall the fact that $A = A_{11}(e) + A_{00}(e)$ and write $C = (N \cap A_{11}(e)) + A_{00}(e)$. It is routine to check $C \cdot A \subseteq C$. Also, e is in $H(A)$. If u is in (C, A) then $u = eu + u_0$ with u_0 in $A_{00}(e)$ and eu in $A_{11}(e)$. Since $\{e(x, y)\}$ is nil, eu is in N so $(C, A) \subseteq C$ and C is a proper ideal of A .

Conversely, suppose A satisfies (20), (21) and (22). Let J be a proper ideal and suppose $x \neq 0$, x in J . Since $A = A_{11}(e_1) + \dots + A_{11}(e_n)$ for e_i primitive then $x = x_1 + \dots + x_n$ for x_i in $A_{11}(e_i)$. We have some $x_i \neq 0$ so $y = x_i = e_i x$ is in J . Either y is nil or y is not nil. If y is not nil then $y = \alpha e_i + z$ with α in F , $\alpha \neq 0$ and z nil. Write $u = -(1/\alpha)z$ and note that for some n , $e_i = (1/\alpha)(e_i - u) \cdot (e_i + u + \dots + u^{n-1})$ is in J . Now, for arbitrary u, v , $e_i(u, v)$ is in J . Since $\{e_i(u, v)\}$ is not nil, some $z = e_i(u, v) \neq 0$, and by (21), z in J must generate A so $A = J$.

Now, suppose y is nil. If y is in M_1 let $u = y$; if not there is a z with $u = y \cdot z$ in M_1 : In either case, u is in $J \cap M_1$. Note also that u is in $A_{11}(e_i)$. There is a v such that $(u, v) \neq 0$. From [4], we know that (u, v) is in some $A_{11}(e_j)$ so $e_j(u, v) \neq 0$ and $e_j(u, v)$ is in J so $e_j(u, v)$ generates A . Hence $J = A$ and we have proved A simple.

4. Algebras with $N \cdot N = 0$.

Lemma 4.1. *If A is a semisimple algebra over a splitting field of char. $\neq 2, 3$ with A^+ associative and $N \cdot N = 0$ then $\{(x, y)\} \cap N \subseteq H(A) \cap N = 0$.*

Proof. We need only note that if z is in $H(A) \cap N$ then $\{\alpha z\}$ is a nil ideal.

We will first be interested in those associative commutative algebras which give rise to nodal simple algebras. The following definition is thus convenient.

Definition 4.1. An algebra A will be called nearly nodal if $A = F \cdot 1 + N$ where F is the base field, 1 is the identity of A and N is the set of nil elements of A .

Note that a nearly nodal algebra is nodal if and only if $N^2 \not\subseteq N$.

Theorem 4.1. Let P be a nearly nodal associative commutative algebra over a field of char. $\neq 2, 3$ with $N \cdot N = 0$. Let $\{x_i\}_{i=1}^n$ be a basis for N . If ϕ is an antiflexible map then $P(\phi)$ is simple if and only if there is a nonsingular matrix $X = ((x_{ij}))$ with $\phi(x_i, x_j) = x_{ij} \cdot 1$.

Proof. Suppose $P(\phi)$ is simple. Then $H(P) \cap N = 0$ so $\{\phi(x, y)\} = \{\alpha \cdot 1\} = H(P)$. Hence $\phi(x_i, x_j) = x_{ij} \cdot 1$. Now y in N can be written $y = \sum_{i=1}^n \alpha_i x_i$. By the bilinearity of ϕ , $\phi(y, x_j) = \sum_{i=1}^n \alpha_i x_{ij} \cdot 1$. Hence, X can be regarded as a linear mapping from N into $V_n(F)$ (space of n -tuples over F). By Lemma 3.2 and Corollary 3.1, $P(\phi)$ is simple if and only if $y \neq 0$ implies $X(y) \neq 0$. This says that X is nonsingular. Conversely, if X is nonsingular then for each y there is an x_j with $\phi(y, x_j) = \alpha \cdot 1$, $\alpha \neq 0$. Hence, there can be no ideals in $P(\phi)$.

Definition 4.2. If ϕ is an antiflexible map from $A \times A \rightarrow F \cdot 1$ and if $X = ((x_{ij}))$ such that, for a basis $\{x_i\}_{i=1}^n$ of N , $\phi(x_i, x_j) = x_{ij} \cdot 1$ then X is said to represent ϕ relative to the basis $\{x_i\}_{i=1}^n$.

Lemma 4.2. Let ϕ be an antiflexible map from $A \times A \rightarrow F \cdot 1$. Two matrices X and Y represent ϕ relative to different bases if and only if they are congruent.

Proof. The proof follows from observing that ϕ can be regarded as a bilinear form and then using standard linear algebra results (see [2, pp. 177–180]).

Theorem 4.2. A nodal antiflexible algebra over a field of char. $\neq 2, 3$ and $N \cdot N = 0$ is simple if and only if, for some basis $\{x_i\}_{i=1}^n$ of N , $\phi(x, y) = \frac{1}{2}(x, y)$ is represented by the matrix

$$x = \begin{vmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_k \end{vmatrix}$$

where $x_i = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$.

Proof. We know that $\phi(x, y)$ is skew-symmetric. It is a well known fact (see Exercise 9, p. 186 in [2]) that any skew-symmetric matrix C is congruent to a matrix having the following diagonal block form:

$$\begin{vmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_t & 0 \end{vmatrix}, \quad C_i = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}.$$

Our result then follows from Theorem 4.1.

Theorem 4.3. *If P is a nearly nodal associative commutative with $N \cdot N = 0$ algebra over a field of char. $\neq 2, 3$ then P is nearly simple if and only if $\dim P$ is odd.*

Proof. If P is nearly simple then there is a ϕ with $P(\phi)$ simple. Relative to some basis, ϕ is represented by the matrix X of Theorem 4.2. Thus, $\dim N$ is even so $\dim P$ is odd. Conversely, the ϕ represented by X in Theorem 4.2 yields a simple algebra $P(\phi)$.

Theorem 4.4. *Let P be an associative commutative algebra over a splitting field F of char. $\neq 2, 3$ with $N \cdot N = 0$. Then P is nearly simple if and only if*

(23) *there is an identity element in P ,*

(24) *for every primitive idempotent e , $\dim P_{11}(e) \geq 3$,*

(25) *either 1 is not primitive or $\dim P$ is odd.*

Proof. If P is nearly simple then (23) is satisfied. If 1 is a primitive idempotent, P is nearly nodal and Theorem 4.3 tells us that $\dim P$ is odd. Thus (25) is satisfied. We will now prove (24).

If e is primitive with $\dim P_{11}(e) = 1$ then $P_{11}(e) = \{\alpha e: \alpha \text{ in } F\}$ is an ideal in any algebra $P(\phi)$. If $e = 1$, Theorem 4.3 implies $\dim P_{11}(e) \neq 2$. Suppose $\dim P_{11}(e) = 2$ and $e \neq 1$ and let $A = P(\phi)$. Then $A = A_{11}(e) + A_{00}(e)$ and $A_{11}(e) \cap N = \{\alpha x: \alpha \text{ in } F\}$ for some x with $x^2 = 0$. If y is in $A_{00}(e)$ then $xy = yx = 0$. Hence $A_{11}(e) \cap N$ is a nil ideal in A and A is not simple.

Suppose P satisfies (23), (24), and (25) with 1 not primitive and write $P = P_{11}(e_1) \oplus \cdots \oplus P_{11}(e_q)$ with each e_i primitive. We will define two antiflexible maps ϕ and ψ on each $P_{11}(e_m)$ and then extend them bilinearly to all of P . Let $\{x_i\}_{i=1}^n$ be a basis for $P_{11}(e_m) \cap N$. If n is even let $n = 2k$ while if n is odd let $n = 2k + 1$. Define

$$\phi(x_i, x_j) = \begin{cases} x_{ij} \cdot 1, & \text{if } i \leq 2k \text{ and } j \leq 2k, \\ 0, & \text{otherwise,} \end{cases}$$

where $X = ((x_{ij}))$ is the matrix of Theorem 4.2. If n is even define $\psi(x_i, x_j) = 0$, while if n is odd define

$$\psi(x_i, x_j) = \begin{cases} e_m, & i = 1, j = n, \\ -e_m, & i = n, j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Extend ϕ and ψ to all of $P_{11}(e_m)$ bilinearly with $\phi(e, N) = \psi(e, N) = \phi(N, e) = \psi(N, e) = 0$.

After extending ϕ, ψ to all of P define $A = P(\phi + \psi)$. We claim that A is

simple and totally antiflexible. It is clear that ϕ and ψ are antiflexible so we need only show that A is simple. Let J be an ideal of A , $J \neq 0$. Since $e_i J \subseteq J$ then, for some m , $J \cap P_{11}(e_m) \neq 0$. It is easy to show that $J \cap (P_{11}(e_m) \cap N) \neq 0$ so choose $x \neq 0$ in $J \cap P_{11}(e_m) \cap N$. Also, let $\{x_i\}_{i=1}^n$ be a basis for $P_{11}(e_m) \cap N$ with k as previously defined. Now, $x = \sum_{i=1}^n \alpha_i x_i$. If $n = 2k$ it is clear that there is a y with $\phi(x, y) = 1$ so $J = A$ so we will assume that $n = 2k + 1$. If $\phi(x, x_j) = 0$ for $j \leq 2k$ then, since X is nonsingular, $\alpha_i = 0$ for $i \leq 2k$. Hence $x = \alpha_n x_n$ with $\alpha_n \neq 0$ and $e_m = \psi(x_1/\alpha_n, x)$ is in J . However, $x_1 = e_m x_1$ is then in J so $1 = \phi(x_2, x_1)$ is in J . Consequently $J = A$ and we have proved A simple.

Theorem 4.5. *Let P be an associative commutative algebra over a splitting field F of char. $\neq 2, 3$ with $N \cdot N = 0$. Then P is nearly semisimple if and only if*

(26) P is not nil,

(27) for every primitive idempotent e , $\dim P_{11}(e) \neq 2$,

(28) e principal implies $\dim P_{00}(e) \neq 1$,

(29) e principal and primitive implies $\dim P_{11}(e)$ is odd and $\dim P_{00}(e)$ is even.

Proof. If e is primitive then $\dim(P_{11}(e) \cap N) = \dim P_{11}(e) - 1$. Let P be nearly semisimple so that some $P(\phi)$ is semisimple. Clearly, (26) is satisfied. If $\dim P_{11}(e) = 2$ then, as above, $P_{11}(e) \cap N$ is a nil ideal of $P(\phi)$. Thus (27) holds. If e is principal and not the identity element then adjoin an identity element 1 to $P(\phi)$. It is routine to show $1 - e$ primitive and the algebra formed is semisimple. Hence, (28) is true. Now, if e is primitive and principal with $P_{00}(e) = 0$ then $P(\phi)$ is nodal and simple so $\dim P_{11}(e)$ is odd. Suppose e is primitive and principal with $P_{00}(e) \neq 0$ and let $A = P(\phi)$ be semisimple. We know $A_{11}(e)A_{00}(e) = A_{00}(e)A_{11}(e) = 0$ so $\phi(A_{11}(e), A_{00}(e)) = \phi(A_{00}(e), A_{11}(e)) = 0$. By Lemma 4.1, $\{\phi(x, y)\} \subseteq \{\alpha e\}$. Thus for x, y in $A_{11}(e)$, $\phi(x, y) = \alpha_{xy} e$ and the restriction of ϕ to $A_{11}(e) \cap N$ yields a mapping S from $(A_{11}(e) \cap N) \times (A_{11}(e) \cap N)$ to F . By Theorem 3.3 and the fact that $\phi(A_{11}(e), A_{00}(e)) = \phi(A_{00}(e), A_{11}(e)) = 0$, S is nonsingular and $\dim(P_{11}(e) \cap N)$ is even. Similarly, $\dim P_{00}(e) = \dim(P_{00}(e) \cap N)$ is even. This establishes (29).

Conversely, let P satisfy (26), (27), (28) and (29). Since P is associative and commutative,

$$P = P_{11}(e_1) \oplus \cdots \oplus P_{11}(e_q) \oplus P_{00}(e)$$

where each e_i is primitive and $e = e_1 + \cdots + e_q$ is principal. Of course, $P_{00}(e)$ may be zero. As before, we will define two antiflexible maps ϕ and ψ on each $P_{11}(e_m)$ and on $P_{00}(e)$ and then extend them bilinearly to all of P . Let $\{x_i\}_{i=1}^n$

be a basis for $P_{11}(e_m) \cap N$ or $P_{00}(e)$. If n is even let $n = 2k$ and if n is odd let $n = 2k + 1$. Define

$$\phi(x_i, x_j) = \begin{cases} x_{ij}e & \text{if } i \leq 2k \text{ and } j \leq 2k, \\ 0, & \text{otherwise,} \end{cases}$$

where $X = ((x_{ij}))$ is the matrix of Theorem 4.2. If n is even, define $\psi(x_i, x_j) = 0$; otherwise

$$\psi(x_i, x_j) = \begin{cases} e, & i = 1, j = n, \\ -e, & i = n, j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Extend ϕ and ψ to all of $P_{11}(e_m)$ or $P_{00}(e)$ bilinearly.

After extending ϕ, ψ to all of P , define $A = P(\phi + \psi)$. We claim that A is semisimple and totally antiflexible. Clearly, A is totally antiflexible. If x is nonzero in N , $x = x_1 + \cdots + x_q + x_0$; x_i in $P_{11}(e_i) \cap N$ for $i > 0$, x_0 in $P_{00}(e)$. If $x_j \neq 0$ then there is a y in $P_{11}(e_j)$ if $j > 0$ or $P_{00}(e)$ if $j = 0$ such that $\phi(x_j, y) \neq 0$. Since $\phi(x_i, y) = 0$, $i \neq j$, we have $\phi(x, y) \neq 0$. Since $\{\phi(x, y)\}$ contains no nil elements, Corollary 3.1 implies A is semisimple.

These two theorems characterize those associative commutative algebras that are either nearly simple or nearly semisimple when $N \cdot N = 0$. Also, Theorems 4.1 and 4.3 characterize the nodal simple antiflexible algebras in which $N \cdot N = 0$.

5. Nodal algebras of type (n, n) and $(n - 1, n)$. We now focus attention on nodal algebras. If A is such an algebra then $\dim A = 1 + \dim N$. The following is immediate from Theorem 3.1.

Lemma 5.1. *If ϕ is an antiflexible map on an associative commutative algebra P of char. $\neq 2, 3$ then for x in P and integers n, α with $n \geq \alpha \geq 1$, $\phi(x^{n-\alpha}, x^\alpha) = \alpha\phi(x^{n-1}, x)$ and $n\phi(x^{n-1}, x) = 0$.*

Theorem 5.1. *Suppose N is an associative commutative nilpotent algebra over a field F . If $\dim N'_i = 1$ then there is an x in N_{i-1} (if $i = 1$, set $x = 1$ in F) and an a in N such that xa is not in N_{i+1} . If x in N_{i-1} (if $i = 1$, x in F) and a in N are such that xa is not in N_{i+1} and if c is in N_j for $j \geq i$ then $c = \alpha xa^{j-i+1} + n$ for α in F and n in N_{j+1} . Furthermore, if $j \geq i$ then $\dim N'_j = 1$ or $N_j = 0$.*

Proof. Since $\dim N'_i = 1$ then there is a y in N_i such that if c is in N_i then $c = \alpha y + n$ for α in F and n in N_{i+1} . By definition, $N_i = N_{i-1}N$ so there is an x in N_{i-1} and a in N with xa not in N_{i+1} . That is, $xa = \beta y + n_1$ with $\beta \neq 0$, β in F and n_1 in N_{i+1} . Clearly, $c = (\alpha/\beta)xa + n - (\alpha/\beta)n_1$ and $n - (\alpha/\beta)n_1$ is in N_{i+1} . Such a formula holds for any x in N_{i-1} , a in N with xa not in N_{i+1} .

We fix x and a and note that the general result holds for $j = i$. Suppose it holds for $j = k$. If d is in N_{k+1} then $d = \sum_{m=1}^s \beta_m c_m b_m$ for c_m in N_k , b_m in N and β_m in F . However, $c_m = \gamma_m x a^{k-i+1} + n_m$ for γ_m in F and n_m in N_{k+1} . We now observe that $x b_m$ is in N_i so $x b_m = \delta_m x a + n'_m$ with δ_m in F and n'_m in N_{i+1} . Thus

$$\begin{aligned} d &= \sum_{m=1}^s \beta_m c_m b_m \\ &= \sum_{m=1}^s \beta_m \gamma_m x b_m a^{k-i+1} + \sum_{m=1}^s \beta_m n_m b_m \\ &= \sum_{m=1}^s \beta_m \gamma_m \delta_m x a^{k-i+2} + \sum_{m=1}^s (\beta_m \gamma_m n'_m a^{k-i+1} + \beta_m n_m b_m). \end{aligned}$$

However, $n'_m a^{k-i+1}$ and $n_m b_m$ are each in N_{k+2} so $d = \alpha' x a^{k-i+2} + n'$ for α' in F and n' in N_{k+2} . Finally, if $j \geq i$, either $N_j = 0$ or $\dim N'_j = 1$.

The following follows from a footnote in [8, p. 10].

Lemma 5.2. *If N is an associative commutative nilpotent algebra of class k over a field F of char. 0 or char. $\geq k$ then there is an x in N such that $x^{k-1} \neq 0$.*

Lemma 5.3. *If N is an associative commutative nilpotent algebra of class k with $\dim N'_m = 1$ over a field F of char. $> m$ then there is an x in N with $x^{k-1} \neq 0$.*

Proof. Write $Q = N - N_{m+1}$ and note that Q is of class $m+1$. Let $[x]$ be the image of x in N in the natural map from $N \rightarrow N - N_{m+1}$. Since $\text{char } F \geq m+1$, there is an element $[y]$ in Q with $[y]^m \neq 0$. Thus, y^m is not in N_{m+1} . If $m = 1$, set $x = 1$ while, if $m > 1$, set $x = y^{m-1}$. In either case define $a = y$. We have x in N_{m-1} and a in N with xa not in N_{m+1} . Since $N_{k-1} \neq 0$ there is a nonzero element c in N_{k-1} . From Theorem 5.1 and the fact that $N_k = 0$, $c = \alpha x a^{k-m} = \alpha y^{k-1}$. Therefore, $y^{k-1} \neq 0$.

Theorem 5.2. *Suppose N is an associative commutative nilpotent algebra of dimension $n-1$ over a field F . If N is of class k and if $\text{char } F = 0$ or $\text{char } F \geq k$ or $\text{char } F \geq n-k+2$ then there is an x in N with $x^{k-1} \neq 0$.*

Proof. By Lemma 5.2, we need only consider the case $k > \text{char } F \geq n-k+2$. By Lemma 5.3, it is sufficient to have $\dim N'_m = 1$ where $m = n-k+1$. Assume $\dim N'_i > 1$ for $i \leq m$ so $\dim N'_i \geq 2$ for $i \leq m$. Now, $n-1 = \dim N'_1 + \cdots + \dim N'_{k-1} \geq 2m + \dim N'_{m+1} + \cdots + \dim N'_{k-1} \geq 2m + (k-m-1) = m+k-1 = n$ which is impossible. Thus, $\dim N'_m = 1$.

Theorem 5.3. *If N is of type (n, n) then there is an element a in N such that N is spanned by $a, a^2, \dots, a^{n-1}$.*

Proof. When $k = n$, the hypotheses of Theorem 5.2 are always satisfied. Theorem 5.3 becomes a corollary to Theorem 5.2.

As an immediate corollary to Lemma 5.1, we have

Lemma 5.4. *If ϕ is an antiflexible map on an associative commutative algebra P of char. $\neq 2, 3$ then for x in P and integers n, α with $n \geq \alpha \geq 1$, $\phi(x^{n-\alpha}, x^\alpha) = 0$ if $n \not\equiv 0 \pmod{p}$.*

Theorem 5.4. *Let $P = F \cdot 1 \oplus N$ where N is an associative commutative nilalgebra of type (n, n) over a field F of char. $\neq 2, 3$. The algebra P is nearly simple if and only if char F divides n .*

Proof. We first assume P is nearly simple. By Theorem 5.3, there is an element a in N with N spanned by $a, a^2, \dots, a^{n-1}$. It is easy to verify each M_i is spanned by a^{n-i} and each N_i is spanned by $a^i, \dots, a^{n-1}$. Now, Lemma 3.1 implies $\phi(a^i, a^j) = 0$ whenever $i + j > n$. Theorem 3.3 (or 3.4) then states $\phi(a^{n-1}, y) \neq 0$ for some $y = \alpha_1 a + \dots + \alpha_{n-1} a^{n-1}$. We conclude $\phi(a^{n-1}, a) \neq 0$. However $n\phi(a^{n-1}, a) = 0$ so $n \equiv 0 \pmod{p}$ for $p = \text{char } F$.

Now, assume $n = kp$ for $p = \text{char } F$. If we define

$$\phi(a^i, a^j) = \begin{cases} 0, & i + j \neq n, \\ j, & i + j = n, \end{cases}$$

then the proof in [6] for the case $k = 1$ will generalize and $P(\phi)$ is a simple nodal algebra.

We have determined all nearly simple associative commutative algebras of class 2. In classifying nearly simple associative commutative algebras of type (m, n) , we can assume $m \geq 3$.

Having determined nearly simple associative commutative algebras of type (n, n) , our next interest is those of type $(n-1, n)$. If $\dim N'_1 = 1$ then $\dim N'_i = 1$ for all $i \leq n-2$ so that $\dim N = n-2$. Since $\dim N = n-1$ we conclude $\dim N'_1 = 2$ and $\dim N'_i = 1$ for $2 \leq i \leq n-2$. We have proved the following lemma.

Lemma 5.5. *If N is of type $(n-1, n)$ then N is of type $(n-1, n, 2)$.*

Theorem 5.5. *Let N be an associative commutative nilalgebra of type $(n-k, n, k+1)$ over a field F with char. $\neq 2$. If $n \geq k+3$ then there are elements $a, b_i, i = 1, \dots, k$, with N spanned by $a, \dots, a^{n-k-1}, b_1, \dots, b_k$; $ab_i = 0$; $b_i^2 = \alpha_i a^{n-k-1}$; $b_i b_j = \lambda_{ij} a^{n-k-1}$.*

Proof. By Lemma 5.3, since char $F \neq 2$ and $\dim N'_2 = 1$, there is an element

a in N with $a^{n-k-1} \neq 0$. Let $c_1, \dots, c_k$ be chosen so they are not in N_2 and $a, \dots, a^{n-k-1}, c_1, \dots, c_k$ are a basis for N . (This is possible since $\dim N'_1 = k+1$.) We know $a^2, \dots, a^{n-k-1}$ form a basis for N_2 ,

$$ac_j = \sum_{i=2}^{n-k-1} \beta_{ij} a^i = a \left(\sum_{i=2}^{n-k-1} \beta_{ij} a^{i-1} \right), \quad j = 1, \dots, k.$$

Define $b_j = c_j - \sum_{i=2}^{n-k-1} \beta_{ij} a^{i-1}$, $j = 1, \dots, k$. Clearly $ab_j = 0$, $j = 1, \dots, k$. Now, b_j^2 is in N_2 so $b_j^2 = \sum_{i=2}^{n-k-1} \gamma_{ij} a^i$ and

$$0 = (ab_j)b_j = ab_j^2 = \sum_{i=2}^{n-k-2} \gamma_{ij} a^{i+1}.$$

Hence, $\gamma_{ij} = 0$ for $j = 1, \dots, k$ and $i = 2, \dots, n-k-2$. Defining $\alpha_j = \gamma_{n-k-1,j}$ we have $b_j^2 = \alpha_j a^{n-k-1}$.

Lemma 5.6. *If ϕ is an antiflexible map on an associative commutative algebra P of char. $\neq 2, 3$ in which $ab = 0$ then $\phi(a^r, b^s) = 0$ if $r > 1$ or $s > 1$.*

Proof. If $r > 1$ then

$$\phi(a^r, b^s) + \phi(b^s a, a^{r-1}) + \phi(b^s a^{r-1}, a) = 0.$$

Since $b^s a = b^s a^{r-1} = 0$, $\phi(a^r, b^s) = 0$. The proof when $s > 1$ is similar.

Theorem 5.6. *Let $P = F \cdot 1 \oplus N$ where N is an associative commutative nil-algebra of type $(n-k, n, k+1)$ with $n-k > 2$ over a field F of char. $\neq 2, 3$.*

The algebra P is nearly simple if and only if the following hold:

(a) *N is spanned by $a, \dots, a^{n-k-1}, b_1, \dots, b_k$ where $ab_i = b_i^2 = b_i b_j = 0$, $i, j = 1, \dots, k$.*

(b) *Either $n-k = \text{char } F$ with k even or $n-k = m \text{ char } F$ for $m > 1$.*

Proof. By Theorem 5.5, there are elements $a, b_1, \dots, b_k$ with N spanned by $a, \dots, a^{n-k-1}, b_1, \dots, b_k$. Furthermore, $ab_i = 0$, $b_i^2 = \alpha_i a^{n-k-1}$, $b_i b_j = \lambda_{ij} a^{n-k-1}$ for all i, j where each α_i, λ_{ij} is in F . From this, it is clear that M is a subspace of the space spanned by $a^{n-k-1}, b_1, \dots, b_k$.

Assume P is nearly simple. Then there is a ϕ with $P(\phi)$ simple. We first show that each b_i is in M . To do this, it is necessary and sufficient to prove that each $\alpha_i = 0$ and each $\lambda_{ij} = 0$. If $x \neq 0$ is in M , Theorem 3.4 assures the existence of a y in N with $\phi(x, y) \neq 0$. Thus, if x in M has the property that $\phi(x, y) = 0$ for all y in N then $x = 0$. Since a^{n-k-1} is in M , each b_i^2 and each $b_i b_j$ are in M .

Lemma 5.6 implies $\phi(b_i^2, a^j) = 0$ for all i, j . Since $n-k-1 > 1$, $\phi(b_i^2, b_j) = \alpha_i \phi(a^{n-k-1}, b_j) = 0$ for all i, j (also by Lemma 5.6). Thus, for each i , $\phi(b_i^2, y) = 0$ for each y in N so $b_i^2 = 0$. If $p > 1$, a^p is in N_2 so $\phi(b_i b_j, a^p) =$

$\lambda_{ij}\phi(a^{n-k-1}, a^p) = 0$ by Lemma 3.1. From Lemma 5.6, since $n - k - 1 > 1$, we derive $\phi(b_i b_j, b_p) = \lambda_{ij}\phi(a^{n-k-1}, b_p) = 0$. Finally by Theorem 3.1, $\phi(b_i b_j, a) = -\phi(b_i a, b_j) - \phi(b_j a, b_i) = 0$. We conclude $b_i b_j = 0$ and have shown that M is spanned by $a^{n-k-1}, b_1, \dots, b_k$.

Since a^{n-k-1} is in M there is a y in N with $\phi(a^{n-k-1}, y) \neq 0$. If $p > 1$, a^p is in N_2 so $\phi(M, a^p) = 0$. Also, as above, for each i , $\phi(a^{n-k-1}, b_i) = 0$. We conclude $\phi(a^{n-k-1}, a) \neq 0$. From Lemma 5.1, $(n-k)\phi(a^{n-k-1}, a) = 0$ so char F divides $n-k$. We further note that $\phi(a^{n-k-\alpha}, a^\alpha) = \alpha\phi(a^{n-k-1}, a)$ so, for α not divisible by char F , $\phi(a^{n-k-\alpha}, a^\alpha) \neq 0$.

Define $q = \text{char } F$ and assume $n-k = q$. Since P is spanned by $1, a, \dots, a^{n-k-1}, b_1, \dots, b_k$, if x and y are arbitrary,

$$\phi(x, y) = \delta_0 + \sum_{i=1}^{q-1} \delta_i a^i + \sum_{i=1}^k \gamma_i b_i.$$

From (6), we know that, for any z , $\phi(\phi(x, y), z) = 0$. Lemma 5.1 implies $\phi(a^i, a^j) = j\phi(a^{i+j-1}, 1)$ and $(i+j)\phi(a^{i+j-1}, a) = 0$. Hence, $\phi(a^i, a^j) = 0$ unless $i+j = q$. For $s > 1$, Lemma 5.6 implies $\phi(\sum_{i=1}^k \gamma_i b_i, a^s) = 0$. Thus, for $s > 1$,

$$\begin{aligned} 0 &= \phi(\phi(x, y), a^s) = \sum_{i=1}^{q-1} \delta_i \phi(a^i, a^s) \\ &= \delta_{q-s} \phi(a^{q-s}, a^s) = s\delta_{q-s} \phi(a^{q-1}, a). \end{aligned}$$

Since $1 < s < q$ and $\phi(a^{q-1}, a) \neq 0$, $\delta_{q-s} = 0$. Letting $y_1 = \delta_{q-1} a^{q-1} + \sum_{i=1}^k \gamma_i b_i$, we have $\phi(x, y) = y_1 + \delta_0$ with y_1 in M . Now, for any z , $0 = \phi(\phi(x, y), z) = \phi(y_1, z) + \phi(\delta_0, z) = \phi(y_1, z)$. If $y_1 \neq 0$ there must be a z in N with $\phi(y_1, z) \neq 0$ so we conclude $y_1 = 0$ and $\phi(x, y)$ is in $F \cdot 1$.

We know that for $b = \sum_{i=1}^k \eta_i b_i \neq 0$ there is a z in N with $\phi(b, z) \neq 0$. Since $ab = 0$, $\phi(b, a^s) = 0$ when $s > 1$. Thus, if b satisfies $\phi(b, b_j) = 0$ for all j then $\phi(b, a) \neq 0$. If we write $\phi(b, a) = \beta_1 \neq 0$ and $\phi(a^{q-1}, a) = \beta_2 \neq 0$, we have shown β_1 and β_2 to be in $F \cdot 1$. Define $x = \beta_2 b - \beta_1 a^{q-1}$ and verify $\phi(x, z) = 0$ for all z in P . However, the simplicity of $P(\phi)$ implies $\phi(x, z) \neq 0$ for some z . We have proved that for any b there is a j with $\phi(b, b_j) \neq 0$.

Define $\beta_{ij} = \phi(b_i, b_j)$, $i, j = 1, \dots, k$. For any set $\eta_1, \dots, \eta_k$ there is a j with $\sum_{i=1}^k \eta_i \beta_{ij} = \sum_{i=1}^k \eta_i \phi(b_i, b_j) \neq 0$. Defining B as the matrix $((\beta_{ij}))$ we conclude that B is a nonsingular matrix. If we let $Q = F \cdot 1 \oplus F \cdot b_1 \oplus \dots \oplus F \cdot b_k$ and let $\phi' = \phi$ restricted to Q , then $Q(\phi')$ is a nodal simple subalgebra of $P(\phi)$. Since $Q(\phi')$ is of class 2, Theorem 4.3 says that $\dim Q$ is odd so k is even.

For the converse, first assume that P satisfies (a) with k even. Write $r = n - k$. Define ϕ on the basis as follows:

$$(30) \quad \phi(a^i, a^j) = \begin{cases} 0, & i + j \neq r, \\ j, & i + j = r. \end{cases}$$

$$(31) \quad \phi(b_i, b_j) = x_{ij} \cdot 1 \quad \text{where } X = ((x_{ij})) \text{ is the matrix of Theorem 4.2.}$$

$$(32) \quad \phi(a^s, b_i) = \phi(b_i, a^s) = 0 \quad \text{for all } i, s \geq 1.$$

$$(33) \quad \phi(1, x) = \phi(x, 1) = 0 \quad \text{for all } x \text{ in } P.$$

Extend ϕ bilinearly to $P \times P$. Let $x = \delta_0 + \sum_{i=1}^{r-1} \delta_i a^i + \sum_{i=1}^k \gamma_i b_i$ be a nonzero element in an ideal J of $P(\phi)$. If some $\gamma_i \neq 0$ then there is a j with $\phi(x, b_j) = \pm \gamma_i$ in J . If each $\gamma_i = 0$ then, for some j , $\delta_j \neq 0$. If $\delta_j \neq 0$ and $\text{char } F$ divides j (or $j = 0$) then $\phi(xa, a^{r-j-1}) = (-j-1)\delta_j = -\delta_j$ is in J . If $\delta_j \neq 0$ with j and $\text{char } F$ relatively prime then $\phi(x, a^{r-j}) = -j\delta_j \neq 0$ in J . In any case, $F \cdot 1 \subseteq J$ so $J = P(\phi)$.

Now, suppose P satisfies (a) with $k = b + 1$ odd so b is even. Write $r = n - k$. We know $r = m \text{ char } F$ with $m > 1$. Let $q = \text{char } F$. Define ϕ on the basis as follows:

$$(34) \quad \phi(a^i, a^j) = \begin{cases} 0, & i + j \neq r, \\ j, & i + j = r. \end{cases}$$

$$(35) \quad \phi(b_i, b_j) = x_{ij} \cdot 1 \quad \text{for } i, j = 1, \dots, b \quad \text{where } X = ((x_{ij})) \text{ is the matrix of Theorem 4.2.}$$

$$(36) \quad \phi(b_k, a) = -\phi(a, b_k) = a^q.$$

$$(37) \quad \phi(a^s, b_i) = \phi(b_i, a^s) = 0 \quad \text{unless } s = 1 \text{ and } i = k.$$

$$(38) \quad \phi(1, x) = \phi(x, 1) = 0 \quad \text{for all } x \text{ in } P.$$

Extend ϕ bilinearly to $P \times P$. It is straightforward to verify $P(\phi)$ is simple.

As an immediate corollary, we have

Corollary 5. *Let $P = F \cdot 1 \oplus N$ where N is an associative commutative nil-algebra of type $(n-1, n)$ with $n-1 > 2$ over a field F of $\text{char.} \neq 2, 3$. The algebra P is nearly simple if and only if N is spanned by $a, \dots, a^{n-2}, b$ where $ab = b^2 = 0$ and $n-1 = m \text{ char } F$ with $m > 1$.*

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