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WALL MANIFOLDS WITH INVOLUTION

BY

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ABSTRACT. Consider smooth manifolds W with involution t and a Wall structure described by a map $f: W \to S^1$ such that $ft = f$. For such objects we define cobordism theories $\mathbf{W}_*^I$ (in case W is closed, t unrestricted), $\mathbf{W}_*^F$ (for W closed, t fixed-point free), and $\mathbf{W}_*^{\mathrm{rel}}$ (W with boundary, t free on W). We prove that there is an exact sequence

$$0 \to \mathbf{W}_*^I \to \mathbf{W}_*^{\mathrm{rel}} \to \mathbf{W}_*^F \to 0.$$

As a corollary, $\mathbf{W}_*^I$ imbeds in the cobordism of unoriented manifolds with involution. We also describe how $\mathbf{W}_*^I$ determines the 2-torsion in the cobordism of oriented manifolds with involution.

1. **Introduction.** C. T. C. Wall defined the class of manifolds now named for him in order to compute the 2-torsion of oriented cobordism [8]. In this paper we will show how a similar construction determines, in principle, the 2-torsion of the cobordism of oriented manifolds with involution.

Let W be a smooth, unoriented manifold. A *Wall structure* on W is a homotopy class of maps $f: W \to S^1$ such that $f^*(\iota)$ is the Stiefel-Whitney class $w_1(W)$, where $\iota \in H^1(S^1; Z_2) \cong Z_2$ is the generator. Equivalently, a Wall structure is given by a class $x \in H^1(W; Z)$ whose reduction mod 2 is $w_1(W)$. If $t: W \to W$ is a smooth involution, we say that (W, t, f) is a *Wall manifold with involution* provided $f(w) = f(t(w))$ for all $w \in W$. We define a cobordism theory $\mathbf{W}_*^I$ for such objects, a similar theory $\mathbf{W}_*^F$ for Wall manifolds with free involution, and a theory $\mathbf{W}_*^{\mathrm{rel}}$ for Wall manifolds with boundary and an involution free on the boundary. The major result is that there is an exact sequence

$$0 \to \mathbf{W}_*^I \to \mathbf{W}_*^{\mathrm{rel}} \to \mathbf{W}_*^F \to 0.$$

By comparison with a theorem of Conner and Floyd [3, p. 73] this proves that $\mathbf{W}_*^I$ imbeds in the cobordism $I_*(Z_2)$ of manifolds with involution.

In the last section we will define a cobordism theory $\mathbf{O}_*(Z_2)$ of oriented manifolds with involution, which contains as a summand the theory $\mathcal{O}_*(Z_2)$ studied by

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Rosenzweig [5] and Conner [2]. There is a Rohlin sequence

$$\begin{array}{ccc} \mathbf{W}_*^I & \longrightarrow & \mathbf{O}_*(Z_2) \\ & \nwarrow \quad \swarrow_{2} & \\ & \mathbf{O}_*(Z_2) & \end{array}$$

which is exact at the upper corners (but not at the lower one). This outlines a possible calculation of the 2-torsion of $\mathbf{O}_*(Z_2)$, although the practical difficulties remain considerable.

2. **Involutions and Wall structures.** Let $(W, f: W \longrightarrow S^1)$ be a Wall manifold, and let t be an involution on W, with $\pi: W \longrightarrow W/t$ as quotient map. If $x \in H^1(W; Z)$ is the class whose mod 2 reduction is $w_1(W)$, we have this equivalent to our definition of a structure-preserving map:

Proposition 2.1. *(W, t, f) is a Wall manifold with involution if and only if there is a class $y \in H^1(W/t; Z)$ such that $\pi^*(y) = x$.*

The proof is trivial, since both conditions are equivalent to the existence of a map $h: W/t \rightarrow S^1$ such that $f \cong h\pi$.

Next, suppose that t is fixed point free. Then W/t is a smooth manifold with boundary $(\partial W)/t$, and π is a principal Z_2-bundle. If $y \in H(W/t; Z)$ meets the conditions of (2.1), then

$$\pi^*(w_1(W/t)) = w_1(W),$$
$$\pi^*(\rho(y)) = \rho(\pi^*(y)) = w_1(W), \qquad \rho = \text{mod } 2 \text{ reduction}.$$

It does not follow that $w_1(W/t) = \rho(y)$, but recall that in the Gysin sequence of π there is an exact sequence

$$H^0(W/t; Z_2) \longrightarrow H^1(W/t; Z_2) \xrightarrow{\pi^*} H^1(W; Z_2),$$

where the left map is multiplication by the characteristic class $w_1(\pi)$. This proves

Proposition 2.2. *If W/t is connected, then either $\rho(y) = w_1(W/t)$ or else $\rho(y) = w_1(W/t) + w_1(\pi)$.*

This provides a tool for studying W/t, which need not be a Wall manifold.

Our cobordism theories are established by standard arguments. Let $i: \partial W \subset W$, and recall that $w_1(\partial W) = i^* w_1(W)$. The *boundary* of a Wall manifold with involution (W, t, g) is the Wall manifold with involution (W, ti, gi). Two closed, n-dimensional Wall manifolds with involution are *cobordant* if their disjoint union bounds a Wall manifold with involution. It is easy to check that this defines an equivalence relation, and that the set $\mathbf{W}_n^I$ of equivalence classes is a Z_2-vector space with operation induced by disjoint union. Further, the direct sum $\mathbf{W}_*^I$ of the $\mathbf{W}_n^I$ is a module over the Wall cobordism ring $\mathbf{W}_*$, with multiplication induced by Cartesian product, and by use of the multiplication $S^1 \times S^1 \longrightarrow S^1$ of complex numbers of norm 1. A similar argument, with all involutions required to be free, produces the cobordism $\mathbf{W}_*^F$ of Wall manifolds with free involution.

Finally, if (W_1, t_1, g_1) and (W_2, t_2, g_2) are n-dimensional Wall manifolds with involutions t_i free on ∂W_i, then they are cobordant if there is a Wall manifold with involution (V, u, b), having $W_1 \cup_\emptyset W_2$ as a submanifold of ∂V, u free on $\partial V - \operatorname{int}(W_1 \cup W_2)$, $u|W_i = t_i$ and $b|W_i = g_i$. This is also an equivalence relation, and the equivalence classes form a $\mathbf{W}_*$-module $\mathbf{W}_*^{\mathrm{rel}}$.

The assignment $\partial(W, t, g) = (\partial W, ti, gi)$ defines a homomorphism $\mathbf{W}_n^{\mathrm{rel}} \longrightarrow \mathbf{W}_{n-1}^F$, also called ∂. Let $r\colon \mathbf{W}_n^F \longrightarrow \mathbf{W}_n^I$ and $s\colon \mathbf{W}_n^I \longrightarrow \mathbf{W}_n^{\mathrm{rel}}$ be the obvious forgetful homomorphisms. It is straightforward, if tedious, to prove the cobordism exact sequence.

Proposition 2.3. *The sequence*

$$\mathbf{W}_*^I \xrightarrow{s} \mathbf{W}_*^{\mathrm{rel}} \xrightarrow{\partial} \mathbf{W}_*^F \quad (\text{with } r\colon \mathbf{W}_*^F \to \mathbf{W}_*^I)$$

is exact.

In the next section we will need to know something about the characteristic classes of a projective space bundle. Suppose $x^k \longrightarrow B$ is a vector bundle of dimension k, and let $Sx \longrightarrow B$ be its $(k-1)$-sphere bundle. The following calculation follows from well-known results of Borel and Hirzebruch, and is proved in [3, Theorem 23.3]:

Proposition 2.4. *Let $Px = Sx/t$, where $t\colon Sx \longrightarrow Sx$ is the involution antipodal on each fiber, and let π and p be the obvious projections*:

$$\begin{array}{ccc} Sx & \xrightarrow{\pi} & Px \\ & \searrow \quad \swarrow_{p} & \\ & B & \end{array}$$

Then $w_1(Px) = p^*(w_1(B) + w_1(x)) + kw_1(\pi)$.

Corollary 2.5. *If the disk bundle Dx is a Wall manifold and k is even, then Px is also a Wall manifold.*

Proof. If $r\colon Dx \longrightarrow B$ is the projection, note that $w_1(Dx) = r^*(w_1(B) + w_1(x))$ is an integral class, by hypothesis. Since B is homotopy equivalent to Dx it follows that $w_1(Px) = p^*(w_1(B) + w_1(x))$ is also integral. ***

3. Imbedding $\mathfrak{W}_*^I$ in $I_*(Z_2)$. We begin by calculating $\mathfrak{W}_*^F$ from Proposition 2.2. Suppose (W, f) is a Wall manifold and $g\colon W \to RP^\infty$ is any smooth map. The pullback of the universal Z_2-bundle $S^\infty \to RP^\infty$ is a cover $q\colon W' \to W$, and $w_1(W') = q^*w_1(W) = (fq)^*(\iota)$. Therefore, if $t_q\colon W' \to W$ reverses the sheets of the cover, (W', t_q, fq) is a Wall manifold with free involution. This construction defines a homomorphism

$$k\colon \mathfrak{W}_*(RP^\infty) \to \mathfrak{W}_*^F.$$

Now let V be any smooth manifold, and $f\colon V \to S^1$ any smooth map. Corresponding to the cohomology class $f^*(\iota) + w_1(V) \in H^1(V; Z_2)$ is a map $V \to RP^\infty$ and hence a Z_2-bundle $q\colon W' \to V$. By construction,

$$(fq)^*(\iota) + w_1(W') = q^*(f^*(\iota) + w_1(V)) = q^*(w_1(q)) = 0.$$

Therefore, (W', t_q, fq) is again a Wall manifold with free involution. This defines a homomorphism

$$j\colon \mathfrak{N}_*(S^1) \to \mathfrak{W}_*^F.$$

There are monomorphisms $a\colon \mathfrak{W}_* \to \mathfrak{N}_*(S^1)$ and $b\colon \mathfrak{W}_* \to \mathfrak{W}_*(RP^\infty)$ given by

$$a(W, f) = (W \to * \in S^1) \quad \text{and} \quad b(W, f) = (W, f; W \to * \in RP^\infty).$$

(See [6] for a discussion of these and other results from the theory of Wall cobordism.) Since $ja(W, f) = kb(W, f) = (W \times Z_2, -1, *)$, we have the commutative diagram:

$$\begin{array}{ccc} \mathfrak{W}_*(RP^\infty) & \xrightarrow{k} & \mathfrak{W}_*^F \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle j} \\ \mathfrak{W}_* & \xrightarrow{a} & \mathfrak{N}_*(S^1) \end{array}$$

The computation of $\mathfrak{W}_*^F$ is now easy:

Theorem 3.1. *k and j are monic, and*

$$k + j\colon \mathfrak{W}_*(RP^\infty) \oplus \mathfrak{N}_*(S^1) \to \mathfrak{W}_*^F$$

is epic, with kernel $a\mathfrak{W}_* \oplus b\mathfrak{W}_*$.

Proof. k must be a monomorphism, for there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{W}_*(RP^\infty) & \xrightarrow{\phi} & \mathfrak{N}_*(RP^\infty) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathfrak{W}_*^F & \longrightarrow & \mathfrak{N}_*(Z_2) \end{array}$$

in which the horizontal maps forget Wall structure, and ϕ is known to be monic. Define an inverse for j, assigning to the class of (W, t, g) the class of the induced map $g_\#: W/t \to S^1$.

Proposition 2.2 shows that $k + j$ is epic. Now suppose $\pi: W' \to W$ is a double cover, and $h: W \to S^1$ is such that $(W', t_\pi, h\pi)$ is a cobordism between $(M_1', t_\pi i_1, h\pi i_1)$, a representative of $m_1 \in \operatorname{Im} k$, and $(M_2', t_\pi i_2, h\pi i_2)$, a representative of $m_2 \in \operatorname{Im} j$, where $i_x: M_x' \subset W'$. Then W is a cobordism between $M_1 = \pi M_1'$ and $M_2 = \pi M_2'$, and by hypothesis, $h^*\iota|M_1 = w_1(M_1)$, while $h^*\iota|M_2 = w_1(M_2) + w_1(\pi|M_2)$. By (2.2), either $h^*\iota = w_1(W)$, in which case $w_1(\pi|M_2) = 0$ and $m_2 \in \operatorname{Im} ja$, or else $h^*\iota = w_1(W) + w_1(\pi)$, in which case $w_1(\pi|M_1) = 0$ and $m_1 \in \operatorname{Im} kb$. This completes the proof. ***

The next task is to describe the structure of $\mathbf{W}_*^{\mathrm{rel}}$.

Theorem 3.2. *There is an isomorphism*

$$\beta: \mathbf{W}_*^{\mathrm{rel}} \cong \widetilde{\mathbf{W}}_*\left(\bigvee_k TBO_k\right)$$

given by a standard "normal bundle to the fixed set" construction (as in [3, *p.* 73]).

Proof. To review the construction, let (M, t, g) represent an element of $\mathbf{W}_*^{\mathrm{rel}}$, let F^k be the k-dimensional component of the fixed set of t, and let $\nu^{n-k} \to F^k$ be its normal bundle in M. There is a tubular neighborhood of F^k equivariantly diffeomorphic to the disk bundle $D\nu^{n-k}$, where $D\nu^{n-k}$ is given an involution antipodal in each fiber. $D\nu^{n-k}$ thus acquires a Wall structure by imbedding in M, so classifying $D\nu^{n-k}$ produces an element of $\widetilde{\mathbf{W}}_n(TBO_{n-k})$.

Conversely, suppose $h: (M, \partial M) \to (TBO_j, \infty)$ represents some element of $\widetilde{\mathbf{W}}_n(TBO_j)$, with $f: M \to S^1$ defining the Wall structure. Set $BO_j \subset TBO_j$ as the zero section of the universal bundle γ^j, and make h transverse regular on BO_j. Then $F = h^{-1}BO_j$ has a tubular neighborhood D diffeomorphic to $Dh^*\gamma^j$, so that, if we give D the Wall structure induced by that of M, and the involution s antipodal in the fibers of $h^*\gamma^j$, we obtain a Wall manifold together with an involution whose fixed set is F. This construction produces an inverse for β, as in [3], but we must verify that $(D, s, f|D)$ is a Wall manifold with involution. Recall that there is a retraction $r: D \to F$ factoring through $\pi: D \to D/s$. Since r^* is an isomorphism, π^* is epic, and Proposition 2.1 applies. ***

Theorem 3.3. *The map* $\partial: \mathbf{W}_*^{\mathrm{rel}} \to \mathbf{W}_*^F$ *of* (2.3) *is an epimorphism. Thus there is an exact sequence*

$$0 \to \mathbf{W}_*^I \xrightarrow{s} \mathbf{W}_*^{\mathrm{rel}} \xrightarrow{\partial} \mathbf{W}_*^F \to 0.$$

Proof. There are two steps; we show that $\partial\widetilde{\mathbf{W}}_*(\bigvee_{k\text{ odd}}TBO_k)$ covers $\operatorname{Im} j$, and that $\partial\widetilde{\mathbf{W}}_*(\bigvee_{k\text{ even}}TBO_k)$ covers $\operatorname{Im} k$.

Lemma 3.4. *The composite*

$$\widetilde{\mathbf{W}}_*(TBO_1) \xrightarrow{\beta^{-1}} \mathbf{W}_*^{\mathrm{rel}} \xrightarrow{\partial} \mathbf{W}_*^F \xrightarrow{j^{-1}} \mathfrak{N}_*(S^1)$$

is well defined and an isomorphism.

Proof. Let $(W, t, f\pi)$, where $\pi: W \to W/t$, represent an element of $\operatorname{Im} j \in \mathbf{W}_*^F$. Let $d: D\xi \to W/t$ be the disk bundle of the line bundle associated with π. By definition of j,

$$w_1(\xi) = w_1(\pi) = f^*(\iota) + w_1(W/t),$$
$$w_1(D\xi) = d^*(w_1(\xi) + w_1(W/t)) = (fd)^*(\iota),$$

so $D\xi$ is a Wall manifold with boundary, and $(W, t, f\pi)$ represents the image of $[D\xi] \in \widetilde{\mathbf{W}}_*(TBO_1)$. Thus $j^{-1}\partial\beta^{-1}$ is defined and epic. A simple calculation shows that $\mathfrak{N}_n(S^1)$ has the same rank as $\widetilde{\mathbf{W}}_{n+1}(TBO_1) \cong \mathbf{W}_n(RP^\infty)$, so $j^{-1}\partial\beta^{-1}$ is an isomorphism.

Lemma 3.5. *The composite*

$$\widetilde{\mathbf{W}}_*(TBO_2) \xrightarrow{\beta^{-1}} \mathbf{W}_*^{\mathrm{rel}} \xrightarrow{\partial} \mathbf{W}_*^F \xrightarrow{k^{-1}} \widetilde{\mathbf{W}}_*(RP^\infty)$$

is well defined and an epimorphism.

Proof. If $a \in \widetilde{\mathbf{W}}_*(TBO_2)$, $\partial\beta^{-1}(a)$ is represented as the sphere bundle Sx of some 2-dimensional bundle x, together with the antipodal involution. By Corollary 2.5, the quotient space Px is also a Wall manifold, so $\partial\beta^{-1}(a) \in \operatorname{Im} k$.

We will exhibit a $\mathbf{W}_*$-basis of $\widetilde{\mathbf{W}}_*(RP^\infty)$ which lies in the image of $k^{-1}\partial\beta^{-1}$. To be specific, there are manifolds W^j, in each dimension $j \geq 1$, with these properties:

(1) There is an $\alpha \in H^1(W^j; Z_2)$ so that α^j is the generator of $H^j(W^j; Z_2) \cong Z_2$.

(2) Each W^j is a Wall manifold.

(3) Each $W^j = Px_j$ for some two-dimensional bundle x_j.

(4) $w_1(W^j)$ is the image of some integral class in the cohomology of the base of x_j.

If $h_j: W^j \to RP^\infty$ classifies α, then the first and second properties assure that (W^j, h_j) represents a set of generators of $\widetilde{\mathbf{W}}_*(RP^\infty)$ over $\mathbf{W}_*$. The third and fourth properties, together with Proposition 2.4, assure that the Dx_j are Wall manifolds, so that the W^j lie in the image of $k^{-1}\partial\beta^{-1}$. Since all three maps are clearly $\mathbf{W}_*$-homomorphisms, the lemma will follow.

The necessary examples were constructed by P. G. Anderson in [1]. If $\lambda \to RP^n$ is the canonical line bundle, define $(m+n)$-manifolds

$$M(m, n) = P(\lambda \oplus m \longrightarrow RP^n).$$

Then

$$H^*(M(m, n); Z_2) = Z_2[\alpha, \beta]/\langle \alpha^{m+1} + \alpha^m\beta, \beta^{n+1}\rangle, \qquad \dim \alpha = \dim \beta = 1.$$

If m is odd and n is even, $M(m, n)$ is orientable. If m is odd and $n = 1$, $M(m, 1)$ is a Wall manifold, and $w_1(M(m, 1)) = \beta + (m+1)\alpha = \beta$ is induced from an integral class in the cohomology of the base S^1. Next, let $\lambda' \longrightarrow M(m, 1)$ be the canonical line bundle. Define an $(m+2)$-manifold

$$M(1, m, 1) = P(\lambda' \oplus 1 \longrightarrow M(m, 1)).$$

Then

$$H^*(M(1, m, 1); Z_2) = Z_2[\alpha, \beta, \gamma]/\langle \alpha^2 + \alpha\beta, \beta^{m+1} + \beta^m\gamma, \gamma^2\rangle,$$
$$\dim \alpha = \dim \beta = \dim \gamma = 1.$$

For m even, $M(1, m, 1)$ is a Wall manifold, and $w_1(M(1, m, 1)) = \gamma + (m+2)\beta = \gamma$ is induced from an integral class in the cohomology of the base $M(m, 1)$. All these calculations are expanded in [1].

The required manifolds are

$$W^j = \begin{cases} M(1, 2k), & \text{if } j = 2k+1 \geq 3, \\ M(1, 2k-2, 1), & \text{if } j = 2k \geq 4. \end{cases}$$

In the low dimensions, $W^1 = S^1 = P(1 \oplus 1 \longrightarrow pt)$ and $W^2 = M(1, 1) =$ the Klein bottle. Properties (2), (3), and (4) are clear from Anderson's construction; property (1) for the class $\alpha \in H^1(W^j; Z_2)$ is immediate from the indicated structure of $H^*(W^j; Z_2)$.

This completes the proof of Theorem 3.3. ***

Corollary 3.6. *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{W}_*^I & \xrightarrow{s} & \mathfrak{W}_*^{\mathrm{rel}} & \xrightarrow{\partial} & \mathfrak{W}_*^F & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & I_*(Z_2) & \longrightarrow & \mathfrak{N}_*\left(\bigvee_k TBO_k\right) & \longrightarrow & \mathfrak{N}_*(Z_2) & \longrightarrow & 0 \end{array}$$

the lower sequence being that of [3, *Theorem* 28.1], *and* a *and* b *are monomorphisms.*

This follows at once, since the map

$$\mathfrak{W}_*\left(\bigvee_k TBO_k\right) \longrightarrow \mathfrak{N}_*\left(\bigvee_k TBO_k\right),$$

forgetting Wall structure, is known to be monic.

4. **Oriented manifolds with involution.** Rosenzweig [5] and Conner [2] have studied the cobordism group $\mathcal{O}_*(Z_2)$ of orientation-preserving involutions on closed oriented manifolds. In this section we consider this subject from a different point of view.

Let $t: M \to M$ be an involution on an oriented manifold with boundary. Clearly, t will be orientation-preserving on ∂M if and only if it is orientation-preserving on M. If ∂M is the disjoint union of N_1 and N_2 then we say that (M, t) is a cobordism between $(N_1, t|N_1)$ and $(N_2, t|N_2)$, and thus construct the cobordism group $\mathbf{O}_*(Z_2)$ of oriented manifolds with involution (no restriction on t's action on the orientation). $\mathbf{O}_*(Z_2)$ is obviously a module over the oriented cobordism ring Ω_*, with multiplication induced by the Cartesian product. $\mathcal{O}_*(Z_2) \subset \mathbf{O}_*(Z_2)$, and indeed

$$\mathbf{O}_*(Z_2) = \mathcal{O}_*(Z_2) \oplus \mathcal{R}_*(Z_2),$$

where $\mathcal{R}_*(Z_2)$ can be interpreted as the cobordism of oriented manifolds with orientation-reversing involution.

In order to relate $\mathbf{O}_*(Z_2)$ to $\mathbf{W}_*(Z_2)$, one would like to define a map $\partial: \mathbf{W}_*(Z_2) \to \mathbf{O}_*(Z_2)$, of degree -1, which associates to the class of (W, t, f) the class of $(N, t|N)$, where N is a submanifold of W invariant under t and dual to $w_1(W)$. This makes sense if, for some point $* \in S^1$, f is equivariantly homotopic to a map $g: W \to S^1$ which is transverse regular at $*$, for then $N = g^{-1}(*)$ will have the needed properties (see Wall's original construction [8, §1]).

Lemma 4.1. *Let (M, t) be a manifold with involution, X any manifold, and $* \in X$. Any equivariant map $f: (M, t) \to (X, 1_X)$ can be equivariantly homotoped to a map g which is transverse regular at $*$. Further, if $A \subseteq M$ is an invariant closed set and f is transverse regular at $*$ for points of A, the homotopy can be kept fixed on a neighborhood of A.*

The technique of proof is standard; for a more extensive use see Stong [7, Lemma 4.2]. Briefly, we homotope f to be transverse regular on the fixed set F of t. Since F is an equivariant strong neighborhood deformation retract, we may assume f to be transverse regular on a neighborhood of F. We may extend the result to all of M if we can prove the theorem under the assumption that t is free. In that case, divide out the action of t, obtaining $h: M/t \to X$. h can be made transverse regular via some $H: M/t \times I \to X$. If $\pi: M \times I \to M/t \times I$ is the projection, $H\pi: M \times I \to X$ is the desired equivariant homotopy. ***

Using this lemma, it is easy to prove that ∂ is well defined. We now prove that there is a partial Rohlin sequence relating $\mathbf{W}_*(Z_2)$ and $\mathbf{O}_*(Z_2)$.

Theorem 4.2. *Let* $p: \mathbf{O}_*(Z_2) \to \mathbf{W}_*(Z_2)$ *be defined by assigning to the class of* (M, t) *the class of* $(M, t, *: M \to \text{point} \in S^1)$. *There is an exact sequence*

$$\mathbf{O}_*(Z_2) \xrightarrow{p} \mathbf{W}_*^I \xrightarrow{\partial} \mathbf{O}_*(Z_2) \xrightarrow{2} \mathbf{O}_*(Z_2),$$

where 2 *represents multiplication by* 2.

Note. There is clearly an inclusion Im 2 $\subset$ Ker p, but Conner has shown it to be proper [2, Theorem 5.8].

Proof. If W is orientable, $w_1(W) = 0$ and its dual submanifold is $\emptyset$, so $\partial p = 0$.

If $\partial(w) = 0$, and w is represented by (W, t, f) with f transverse regular at $* \in S^1$, then $N = f^{-1}(*)$ bounds some oriented manifold M with involution m. Now the normal bundle of N in W is trivial by construction, so one can form (S, s) from $(W \times I, t \times 1_I)$ and $(M \times I, m \times 1_I)$ by identifying an equivariant tubular neighborhood D of $N \times 1$ in $W \times 1$ with $(N \times I, t|N \times 1_I) \subset (M \times I, m \times 1_I)$. The result is a cobordism between $(W, t, f) = (W, t, f) \times 0$ and an object of the form $(\mathrm{cl}(W - D) \cup (M \times \partial I), t', f')$. We may assume, from the construction, that Im f' misses $*$, so this object lies in Im p.

To show that $2\partial = 0$, let $\partial(W, t, f) = (N, t|N)$ as before, and deform f so that, on a tubular neighborhood D of N, f factors through the projection $D \to N$. If $V \doteq W - \mathrm{Int}\, D$, then V is orientable, and $\partial(V, t|V) = 2(N, t|N)$.

Finally, suppose (M, t') is an oriented manifold with involution, and $(M, t') \times (S^0, 1) = \partial(U, u)$. Let $* \in S^1$ have a tubular neighborhood D with boundary S^0. Using a tubular neighborhood of ∂U, we can easily construct a map $f: U \to S^1 - \mathrm{Int}\, D \subset S^1$ which is the projection $M \times S^0 \to S^0$ when restricted to ∂U. Form (W, t) by sewing (U, u) and $(M \times D, t' \times 1_D)$ along the copies of $M \times S^0$, and extend f by the projection $M \times D \to D$ to define $g: W \to S^1$. It is clear that $g^*(\iota) = w_1(W)$, g is transverse regular at $*$, $g^{-1}(*) = M$, and g is equivariant, so (M, t') represents a class in Im ∂. ***

In particular, the 2-torsion of $\mathbf{O}_*(Z_2)$ is precisely Im ∂. Since $\mathbf{W}_*^I$ imbeds in $I_*(Z_2)$, which was computed by Conner and Floyd [4], it becomes possible, in principle, to compute the 2-torsion of $\mathbf{O}_*(Z_2)$ from the structure of $I_*(Z_2)$. There are, of course, two highly practical obstacles: the complexity of $I_*(Z_2)$ and the lack of a geometric splitting in the proof of Theorem 3.3.

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