

HARMONIC ANALYSIS ON F -SPACES WITH A BASIS

BY

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ABSTRACT. We establish Bochner's theorem and the Levy continuity theorem in the case that the underlying space is a real F -space with a basis, and then examine the infinitely divisible probability measures on a class of such spaces.

1. Introduction. We generalize Bochner's theorem and the Levy continuity theorem of Fourier analysis to the case that the underlying space E is a real F -space with basis and then study the infinitely divisible measures on such spaces. This extends the results of [2] which handles the situation where E is a Hilbert space and of [3] where E was an l_p space, $0 < p < \infty$. It is also related to the work of V. Sazonov ([6], [7]), Yu. V. Prokhorov ([5], [6]), and to the study of the infinitely divisible measures on a separable Hilbert space which is due to S.R.S. Varadhan in [11].

We first establish some terminology in §2. In §3 we give Bochner's theorem (Theorem 3.1) and in Corollary 3.1 we link Theorem 3.1 to the results of [2] and [3]. §4 is devoted to the continuity theorem. These results are of some interest in themselves but it is their application to Gaussian measures and other infinitely divisible laws which is of most interest to us. In fact, the proofs of Theorem 3.1 and Theorem 4.1 are much the same as those given in [3] when $E = l_p$ ($2 \leq p < \infty$) so we do not provide the details here.

In §5 we give some applications to Gaussian measures. In particular, Theorem 5.1 characterizes the Fourier transform of a Gaussian measure, Theorem 5.2 gives necessary and sufficient conditions in terms of the Fourier transforms for a family of Gaussian measures to be conditionally compact, and Theorem 5.3 deals with the convergence of Gaussian measures on E provided, in each case, E has what we call Property B. Corollaries 5.2 and 5.3 then relate these results to the case $E = l_p$, $2 \leq p < \infty$. In fact, Corollary 5.3 was previously known to N. N. Vakhania ([8], [9]). Finally, in §6 we characterize a large class of spaces whose quasi-norm is accessible in both directions as Orlicz spaces. This result is found in

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Theorem 6.3 and Corollary 6.2 again relates to the l_p spaces, $2 \leq p < \infty$. In Corollary 6.3 we also have a similar characterization of those spaces with Property B.

In the last sections of the paper we study the infinitely divisible laws on these Orlicz spaces. In particular, §7 is devoted to the representation theorem of the Fourier transform of an infinitely divisible law, §8 centers on the limiting behavior of sums of small independent random variables with values in the Orlicz spaces, and in §9 we apply the results of §8 to obtain a central limit theorem.

2. Terminology and preliminaries. Throughout the paper, E will denote a real F -space with basis $\{b_n\}$. Here, as is usual, an F -space is a topological vector space whose topology is given by an invariant metric, i.e. $d(x, y) = d(x - y, 0)$, which is also complete in this metric. We will let $\|x\| = d(x, 0)$. Then $\|\cdot\|$ has all the properties of a norm, except possibly homogeneity, and since E is a topological vector space it does satisfy the property that $\lim_n \|a_n x_n - ax\| = 0$ whenever $\{a_n\}$ is a sequence of reals converging to a and $\{x_n\}$ is a sequence in E converging to x . In this case, we will refer to $\|\cdot\|$ as a *quasi-norm* so as to distinguish it from a norm.

Since E has a basis, for each $x \in E$ there is a unique sequence of real numbers $\{\beta_n\}$ such that $\lim_k \|x - \sum_{n=1}^k \beta_n b_n\| = 0$, and it is easy to see that E is separable. We will write the expansion of x as $\sum_{n=1}^{\infty} \beta_n(x) b_n$ and this emphasizes that the coefficients generate coordinate functionals on E . It is clear that these coordinate functionals are linear and it is well known that they are continuous as well. Further, it is possible to assume without loss of generality (and we do) that $\|x\| = \sup_k \|\sum_{n=1}^k \beta_n(x) b_n\|$. We also make the additional assumption that the basis elements $\{b_n\}$ are adjusted so that $\|b_n\| \leq 1$ (this is always possible). In case $\|\cdot\|$ is actually a norm and E is then a Banach space we can and will assume $\|b_n\| = 1$ for $n = 1, 2, \dots$.

As usual E^* will denote the space of continuous linear functionals on E . Then E^* is a linear space and the E topology on E^* , i.e., the weak-star topology, is the topology obtained by taking as a base all sets of the form $N(f, A, \epsilon) = \{g: |f(x) - g(x)| < \epsilon, x \in A\}$ where $f \in E^*$, A is a finite subset of E , and $\epsilon > 0$.

If $x \in E$ we will define, for $N = 1, 2, \dots$,

$$P_N x = \sum_{k=1}^N \beta_k(x) b_k, \quad Q_N x = \sum_{k=N+1}^{\infty} \beta_k(x) b_k,$$

and for $y \in E^*$, $N = 1, 2, \dots$, we define

$$P_N y = \sum_{k=1}^N \beta_k(\cdot) y(b_k).$$

Lemma 2.1. *The set $\bigcup_{N=1}^{\infty} P_N(E^*)$ is dense in E^* with respect to the E topology on E^* . In fact, if $y \in E^*$, there exists a sequence $\{y_N\}$ in E^* such that $\{y_N\}$ converges to y in the E topology on E^* .*

Proof. Suppose $y \in E^*$. Then for $x \in E$ we have by the continuity of y that $y(x) = \sum_{k=1}^{\infty} \beta_k(x)y(b_k)$. Put $y_N = \sum_{k=1}^N \beta_k(\cdot)y(b_k)$. Then $y_N \in P_N(E^*)$ for $N = 1, 2, \dots$ and $\{y_N\}$ converges to y in the E topology on E^* , so the lemma follows.

The vector space of all sequences of real numbers with the topology of coordinatewise convergence is denoted by l . The subset of l consisting of all bounded sequences will be denoted by l_{∞} and l_{∞}^+ will symbolize the positive cone of l_{∞} , i.e. $x = \{x_i\} \in l_{\infty}^+$ iff $x_i > 0$ for each i and $x \in l_{\infty}$. For $0 < p < \infty$ we let l_p denote the subset of l consisting of all sequences which have $\|x\| = \{\sum_{i=1}^{\infty} |x_i|^p\}^{1/p}$ finite where $x = \{x_i\}$. Thus, for $1 \leq p < \infty$, we have $\|\cdot\|$ as the usual norm for l_p , and for $0 < p < 1$, $\|\cdot\|^p$ generates an invariant metric under which l_p is an F -space. The positive cone of l_p will be denoted by l_p^+ .

3. Bochner's theorem on E . Let μ be a probability measure on the Borel subsets of E where by Borel sets we mean the minimal sigma-algebra containing the open sets. Then the Fourier transform of μ is a complex-valued function ϕ defined on E^* (the topological dual of E) satisfying $\phi(x) = \int_E \exp\{i\langle x, y \rangle\} d\mu(y)$ where $\langle x, y \rangle = x(y)$.

If we let $\mathcal{F}$ denote the algebra of cylinder sets, i.e. sets of the form $\{y \in E : [(x_1, y), \dots, (x_n, y)] \in \Lambda\}$ where $x_1, \dots, x_n \in E^*$ and Λ is a Borel subset of R_n , and $\mathcal{B}(\mathcal{F})$ the minimal sigma-algebra containing $\mathcal{F}$, then it is easy to see that ϕ uniquely determines μ on $\mathcal{B}(\mathcal{F})$. Now $\mathcal{B}(\mathcal{F})$ is contained in the Borel subsets of E and since E is a complete separable metric space it is known that they actually coincide. Hence there is a one-to-one correspondence between probability measures on the Borel subsets of E and their Fourier transforms. To simplify many subsequent statements in the remainder of the paper we assume that all measures are defined on the Borel subsets of E .

If $\lambda \in l_{\infty}^+$ and $\{\mu_{\alpha} : \alpha \in A\}$ is a family of probability measures on E such that

$$(3.1) \quad \mu_{\alpha} \left\{ x \in E : \sum_{k=1}^{\infty} \lambda_k [\beta_k(x)]^2 < \infty \right\} = 1$$

for each $\alpha \in A$, we say λ is sufficient for the family $\{\mu_{\alpha} : \alpha \in A\}$. In other words, the map $x \rightarrow (\sqrt{\lambda_1} \beta_1(x), \sqrt{\lambda_2} \beta_2(x), \dots)$ is an l_2 -valued random variable for each μ_{α} .

If for each $y \in E$ we have $\sup_n |\beta_n(y)| < \infty$, it then follows that any $\lambda \in l_1^+$ is sufficient for any family of probability measures on E .

A family of probability measures $\{\mu_\alpha: \alpha \in A\}$ on E is a λ -family for some $\lambda \in l_\infty^+$ if λ is sufficient for $\{\mu_\alpha: \alpha \in A\}$ and for every $\epsilon, \delta > 0$ there is a sequence $\{\epsilon_N\}$ such that

$$\mu_\alpha \left\{ x \in E: \sum_{k=N+1}^{\infty} \lambda_k [\beta_k(x)]^2 < \delta \right\} > 1 - \epsilon$$

implies

$$\mu_\alpha \left\{ x \in E: \left\| \sum_{k=N+1}^{\infty} \beta_k(x) b_k \right\| < b(\delta) \right\} > 1 - (\epsilon + \epsilon_N)$$

where $\lim_N \epsilon_N = 0$ and b is a strictly increasing continuous function on $[0, \infty)$ with $b(0) = 0$.

It is quite clear that any family of probability measures on a real separable Hilbert space is a λ -family with $\lambda = (1, 1, \dots)$ and $b(\delta) = \delta^{1/2}$.

Lemma 3.1. *If $\{\mu_\alpha: \alpha \in A\}$ is a conditionally compact set of probability measures on E then $\{\mu_\alpha: \alpha \in A\}$ is a λ -family for every $\lambda \in l_\infty^+$ which is sufficient for $\{\mu_\alpha: \alpha \in A\}$.*

Proof. Let $\epsilon, \delta > 0$ be given and choose $\lambda \in l_\infty^+$ such that λ is sufficient for $\{\mu_\alpha: \alpha \in A\}$. Since $\{\mu_\alpha: \alpha \in A\}$ is conditionally compact there exists a compact set K in E such that $\mu_\alpha(K) > 1 - \epsilon$ for each $\alpha \in A$. Hence there exist points $p_1, \dots, p_T \in K$ such that if $S(p, \gamma) = \{x \in E: \|x - p\| < \gamma\}$ then $K \subseteq \bigcup_{i=1}^T S(p_i, \delta/3)$.

Now select N such that $\left\| \sum_{k=N+1}^{\infty} \beta_k(p_i) b_k \right\| < \delta/3$ for $i = 1, \dots, T$. Then for $x \in K$ we have $\|x - p_i\| < \delta/3$ for some p_i and hence

$$\begin{aligned} \left\| \sum_{k=N+1}^{\infty} \beta_k(x) b_k \right\| &\leq \left\| \sum_{k=N+1}^{\infty} \beta_k(x) b_k - \sum_{k=N+1}^{\infty} \beta_k(p_i) b_k \right\| + \left\| \sum_{k=N+1}^{\infty} \beta_k(p_i) b_k \right\| \\ &\leq \|x - p_i\| + \left\| \sum_{k=1}^N [\beta_k(x) - \beta_k(p_i)] b_k \right\| + \delta/3 \leq \delta \end{aligned}$$

since $\|x - p_i\| = \sup_N \left\| \sum_{k=1}^N [\beta_k(x) - \beta_k(p_i)] b_k \right\|$. Thus there exists an N such that for each $\alpha \in A$

$$\mu_\alpha \left\{ x \in E: \left\| \sum_{k=N+1}^{\infty} \beta_k(x) b_k \right\| < \delta \right\} > 1 - \epsilon$$

so clearly $\{\mu_\alpha: \alpha \in A\}$ is a λ -family.

Now let $\alpha(\cdot)$ be a convex function on $[0, \infty)$ such that $\alpha(0) = 0$ and $\alpha(s) > 0$ if $s > 0$. Further, assume for every compact set K of E there exists an $r > 0$ such that $y \in K$ implies $A(y) = \sum_{i=1}^{\infty} \alpha[\beta_i^2(y)] < r$, and for every $r > 0$ there exists $M > 0$ such that $A(y) < r$ implies $\sum_{i=1}^{\infty} \alpha[\beta_i^2(y)] \leq M\gamma(\|y\|)$ where $\gamma(\cdot)$ is another continuous function on $[0, \infty)$ such that $\gamma(0) = 0$. If the quasi-norm $\|\cdot\|$ on E admits the existence of functions $\alpha(\cdot)$ and $\gamma(\cdot)$ having the above properties we will say that it is *accessible*. We also note that if $\alpha(\cdot)$ and $\gamma(\cdot)$ satisfy the conditions indicated then $\alpha(\cdot)$ is continuous and strictly increasing on $[0, \infty)$, $\gamma(s) > 0$ for $s > 0$, and that $\gamma(\cdot)$ can be taken to be increasing on $[0, \infty)$. We note that the accessibility of the quasi-norm on E implies that the map $y \rightarrow (\beta_1(y), \beta_2(y), \dots)$ is continuous from the compact subsets of E into the (topological) subspace l_α of l consisting of those sequences $x = \{x_i\}$ for which $\sum_{i=1}^{\infty} \alpha(x_i^2)$ is finite. Under additional conditions on α this subspace becomes an Orlicz space (cf. §6).

We now provide some examples of spaces which have an accessible norm. First of all notice that the l_p spaces, $2 \leq p < \infty$, have an accessible norm with $\alpha(s) = s^{p/2}$ and $\gamma(s) = s^p$. Similarly, if $E = l_p$, $1 \leq p < 2$, then E has an accessible norm with $\alpha(s) = s$ and $\gamma(s) = s^2$; if $E = l_p$, $0 < p < 1$, then E has an accessible quasi-norm with $\alpha(s) = s$ and $\gamma(s) = s^{2/p}$. For an example of a space with accessible norm which is not an l_p space, consider the Banach space E of real sequences $\{x_i\}$ with norm

$$\|\{x_i\}\|^2 = \sum_{j=1}^{\infty} \left[\sum_{i=2^{j-1}}^{2^j-1} |x_i|^{(j+1)/j} \right]^{2j/(1+j)}$$

Here $\beta_i(x) = x_i$, thus if $\alpha(s) = s$ we have

$$\begin{aligned} \sum_{j=1}^{\infty} \alpha[\beta_j^2(x)] &= \sum_{j=1}^{\infty} x_j^2 = \sum_{j=1}^{\infty} \sum_{i=2^{j-1}}^{2^j-1} x_i^2 \\ &\leq \sum_{j=1}^{\infty} \left[\sum_{i=2^{j-1}}^{2^j-1} |x_i|^{(j+1)/j} \right]^{2j/(1+j)} = \|\{x_i\}\|^2. \end{aligned}$$

Hence E has an accessible norm if $\alpha(s) = s$ and $\gamma(s) = s^2$. Other similar examples follow immediately. For examples of a different type consider those in §6.

If the quasi-norm on E is accessible then by the τ_α -topology we will mean the topology on E^* generated by taking as a subbase all translates of sets of the

form $\{x \in E^*: T(x, x) < 1\}$ as $T(\cdot, \cdot)$ varies over the symmetric, positive definite, bilinear forms on E^* which are jointly weak-star sequentially continuous on E^* and satisfying $\sum_{k=1}^{\infty} \alpha(t_{kk}) < \infty$ where $t_{kk} = T(\beta_k, \beta_k)$. Bilinear forms having the above properties will be called α -operators. In view of Lemma 2.1 and the bilinearity it is easy to see that T is uniquely determined by $t_{ij} = T(\beta_i, \beta_j)$, $i, j = 1, 2, \dots$.

Lemma 3.2. *The sets $\{x \in E^*: T(x, x) < 1\}$ and all translates form a basis for the τ_α -topology as $T(\cdot, \cdot)$ ranges over the α -operators.*

The proof of Lemma 3.2 follows in a standard manner and hence is omitted.

The classical version of Bochner's theorem asserts that a function $\phi(x)$, $x \in R_n$, is the Fourier transform of some probability measure on R_n iff ϕ is positive definite, $\phi(0) = 1$, and ϕ is continuous at zero. In a real separable Hilbert space H it is possible to introduce a topology τ (which is determined by certain compact operators) such that a function ϕ on $H^* = H$ is the Fourier transform of some probability measure on H iff ϕ is positive definite, $\phi(0) = 1$, and ϕ is continuous at zero in the τ -topology. The Hilbert space result is due independently to L. Gross [2] and to V. Sazonov's earlier work [6]. In [3] the work of [2] and [6] is generalized to the l_p spaces. Here we will give a Bochner-type theorem in case E has an accessible quasi-norm. Its proof is much like that of Theorem 2.3 of [3] and involves analogues of Lemmas 2.3 and 2.4 of [3]. Due to these similarities the proof of Theorem 3.1 will not be included. However, the formulation of this theorem provided us with some interesting questions and corollaries and it is these that we begin to examine in the following sections.

Theorem 3.1. *If E has an accessible quasi-norm then a function ϕ on E^* is the Fourier transform of a probability measure iff*

- (i) $\phi(0) = 1$, ϕ is positive definite,
- (ii) ϕ is continuous in the τ_α -topology,
- (iii) the family of measures $\{\mu_N\}$ corresponding to $\phi(P_N(\cdot))$ has a subsequence which is a λ -family for some $\lambda \in l_\infty^+$ satisfying $\lim_k \sum_{i=k}^{\infty} \lambda_i t_{ii} = 0$ whenever $\sum_{i=1}^{\infty} \alpha(t_{ii}) < \infty$. Here $\alpha(\cdot)$, of course, is the function associated with the accessibility of the quasi-norm.

If $E = l_p$, $2 \leq p < \infty$, then a linear operator T from l_p^* into l_p is an S_p -operator if T can be represented as an infinite symmetric positive-definite matrix (t_{ij}) such that $\sum_{j=1}^{\infty} t_{jj}^{p/2} < \infty$. Here, by positive definite, we mean that $\sum_{i,j=1}^n t_{ij} x_i x_j \geq 0$ for all $x \in R_n$ and all integers n . The τ_p -topology on l_p^* is that topology generated by taking as a subbase all translates of all sets of the form $\{x \in l_p^*: \langle Tx, x \rangle < 1\}$ as T varies over the S_p -operators. In view of Lemma 3.2, it is easy to see that such sets actually form a basis. Furthermore, our next

lemma identifies the τ_p -topology in terms of a τ_α -topology. Its proof is direct and hence is omitted.

Lemma 3.3. *If $E = l_p$, $2 \leq p < \infty$, then the norm of E is accessible with $\alpha(s) = s^{p/2}$, $\gamma(s) = s^p$ and the τ_α -topology on E^* is equivalent to the τ_p -topology.*

Corollary 3.1. *If $E = l_p$, $2 \leq p < \infty$, and ϕ is defined on l_p^* then ϕ is the Fourier transform of a probability measure on l_p iff*

- (1) $\phi(0) = 1$, ϕ is positive definite,
- (2) ϕ is τ_p -continuous,
- (3) the family of measures $\{\mu_N\}$ corresponding to $\{\phi(P_N(\cdot))\}$ is a λ -family for some $\lambda \in l_{p/2}^{*+}$.

Proof. The sufficiency of (1), (2), (3) follows immediately from Theorem 3.1 and the previous lemma since $\lambda \in l_{p/2}^{*+}$ implies $\sum_{i=1}^{\infty} \lambda_i t_{ii} < \infty$ for all $\{t_{ii}\}$ satisfying $\sum_{i=1}^{\infty} t_{ii}^{p/2} < \infty$. Conversely, if ϕ is the Fourier transform on l_p^* of a probability measure on l_p then Theorem 3.1 and the previous lemma imply (1), (2) easily and (3) will also hold for every $\lambda \in l_{p/2}^{*+}$ since $\{\mu_N\}$ is conditionally compact and Lemma 3.1 applies.

4. The continuity theorem. Suppose l is the space of all real sequences with the topology of coordinatewise convergence and $P_\lambda(\cdot)$ is the product probability on l such that the i th coordinate is Gaussian with mean zero and variance $\lambda_i > 0$. If μ is a measure on E we choose $\lambda \in l_\infty^+$ so that λ is sufficient for μ (such a λ always exists if $\sup_n |\beta_n(y)| < \infty$ for all $y \in E$), and for $x \in l$ we define a "stochastic linear functional" on E in the following manner:

$$\langle x, y \rangle = \lim_N \sum_{k=1}^N x_k \beta_k(y).$$

Throughout this section we assume that $\sup_n |\beta_n(y)| < \infty$ for all $y \in E$. In case E is a Banach space this is not an additional assumption.

Lemma 4.1. *The stochastic linear functional $\langle x, y \rangle = \lim_N \sum_{k=1}^N x_k \beta_k(y)$ is Borel measurable on $l \times E$ and if $F = \{(x, y): \langle x, y \rangle \text{ exists and is finite}\}$ then $(P_\lambda \times \mu)(F) = 1$.*

The proof of Lemma 4.1 is similar to that of Lemma 3.1 in [3] and, for that reason, is omitted.

If μ is a measure on the Borel subsets of E we define the extended Fourier transform $\tilde{\phi}(\cdot)$ on l as follows:

$$\tilde{\phi}(x) = \int_E \exp\{i \langle x, y \rangle\} d\mu(y) \quad (x \in l).$$

Then $\tilde{\phi}(\cdot)$ is a Borel measurable function on l which is defined almost everywhere with respect to the measure P_{λ} . Furthermore, since each $x \in E^*$ generates the unique sequence of real numbers $\tilde{x} = \{x(b_1), x(b_2), \dots\}$, we may consider E^* as a linear subset of l , and hence the terminology extended Fourier transform, since for $x \in E^*$, $\phi(x) = \tilde{\phi}(\tilde{x})$.

Theorem 4.1. *Let $\{\mu_k\}$ be a sequence of probability measures on E with Fourier transforms $\{\phi_k\}$. Then $\{\mu_k\}$ converges weakly to a measure μ with Fourier transform ϕ iff $\{\mu_k\}$ is a λ -family for some $\lambda \in l_{\infty}^+$ which is also sufficient for μ , $\{\phi_k\}$ converges to ϕ on a subset of E^* which is dense in E^* with respect to weak-star sequential convergence, and $\{\tilde{\phi}_k\}$ converges in P_{λ} measure to $\tilde{\phi}$.*

Here again our proof parallels that of Theorem 5.1 of [3] and can be carried out in an analogous manner with appropriate changes in this context. Hence it is omitted. It should be mentioned, however, that the above result generalizes Theorem 5.1 of [3] where $E = l_p$, $2 \leq p < \infty$. However, it is not quite equivalent to the result of L. Gross (the case $E = l_2$) but, as the results in §4 of [3] indicate, it is the great symmetry of Hilbert space which enables the improved version in [2].

The next corollary is well known and in view of Lemma 3.1 is an immediate corollary to Theorem 4.1.

Corollary 4.1. *Let $\{\mu_k\}$ be a sequence of probability measures on E with Fourier transforms $\{\phi_k\}$. Then $\{\mu_k\}$ converges weakly to a measure μ iff $\{\mu_k\}$ is conditionally compact and $\{\phi_k\}$ converges on a subset of E^* which is dense in E^* with respect to weak-star sequential convergence. Further, the Fourier transform of μ is $\phi = \lim \phi_k$ on E^* .*

5. Application to Gaussian measures. A probability measure μ on E is a Gaussian measure with mean vector $a \in E$ if the distribution of the function $\langle y, x \rangle$ is Gaussian with mean $\langle y, a \rangle$ for each $y \in E^*$.

For example, if (t_{ij}) is a positive definite, symmetric matrix satisfying $\sum_{i=1}^{\infty} t_{ii}^{1/2} < \infty$ and E is a Banach space with basis $\{b_n\}$, then there exists a unique Gaussian measure μ with mean zero on E such that

$$t_{ij} = \int_E \beta_i(x) \beta_j(x) d\mu(x) \quad (i, j = 1, 2, \dots).$$

Further, the Fourier transform of μ is $\phi(x) = \exp\{-\frac{1}{2} \langle Tx, x \rangle\}$ where $T = (t_{ij})$ maps E^* onto E in such a manner that for $x \in E^*$

$$(5.1) \quad Tx = \sum_{i=1}^{\infty} \left[\sum_{j=1}^{\infty} t_{ij} x(b_j) \right] b_i.$$

To see that $Tx \in E$ simply observe that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |t_{ij}| \leq \left[\sum_{i=1}^{\infty} t_{ii}^{1/2} \right]^2 < \infty$$

and use the fact that E is a Banach space with basis.

Suppose E has an accessible quasi-norm. Then, if E^* is such that the sequence $\{x_n\}$ converging to x in the E topology on E^* implies $\sup_{n,i} |x_n(b_i)| < \infty$, we will say E has Property A.

It is easy to see from the uniform boundedness principle that if E is a Banach space with accessible norm then E has property A. Hence the examples of §3 all have Property A.

Lemma 5.1. *If E has Property A and $\{u_j\} \in l_1^+$ is such that $\sum_{j=1}^{\infty} \alpha(u_j) < \infty$, then $U(x, y) = \sum_{j=1}^{\infty} u_j x(b_j) y(b_j)$ is an α -operator on E^* and $U(x, x) = 0$ iff $x = 0$.*

The proof of Lemma 5.1 is immediate and hence is omitted.

Lemma 5.2. *If E has Property A and μ is a Gaussian measure on E with mean a then the Fourier transform of μ is $\exp\{i\langle x, a \rangle - \frac{1}{2}T(x, x)\}$ where $T(\cdot, \cdot)$ is an α -operator on E^* such that $T(\beta_i, \beta_j) = \int_E \beta_i(x-a) \beta_j(x-a) d\mu(x)$ for $i, j = 1, 2, \dots$.*

Proof. Let $\phi(x)$ denote the Fourier transform of μ . By Theorem 3.1, $\phi(\cdot)$ is τ_α -continuous and also weak-star sequentially continuous (it is a Fourier transform). Hence

$$\phi(x) = \lim_N \phi(P_N x) = \lim_N \exp \left\{ i \langle P_N x, a \rangle - \frac{1}{2} \sum_{j,k=1}^N s_{jk} x(b_j) x(b_k) \right\}$$

where $P_N(x) = \sum_{j=1}^N \beta_j(\cdot) x(b_j)$ and $s_{jk} = \int_E \beta_j(y-a) \beta_k(y-a) d\mu(y)$. Thus

$$S(x, x) \equiv \lim_N \sum_{i,j=1}^N s_{ij} x(b_i) x(b_j)$$

exists for all $x \in E^*$. We define for $x, y \in E^*$

$$S(x, y) = \lim_N \sum_{i,j=1}^N s_{ij} x(b_i) y(b_j).$$

Since it equals $\frac{1}{2}[S(x+y, x+y) - S(x, x) - S(y, y)]$, $S(x, y)$ now exists for all $x, y \in E^*$. Further, as is easily seen from the definition, $S(\cdot, \cdot)$ is bilinear on E^* . We now will show $S(\cdot, \cdot)$ is an α -operator and the proof is then completed by setting $T = S$. Since $\phi(x) = \exp\{-\frac{1}{2}S(x, x)\}$ is τ_α -continuous there exists an α -operator V such that $V(x, x) \leq 1$ implies $S(x, x) < \epsilon < 1$. We can (and do) assume $V(x, x) > 0$ for all $x \neq 0$ (possible by previous lemma). If $V(y_0, y_0) =$

$c^2 > 0$ then $V(y_0/c, y_0/c) = 1$ so $S(y_0/c, y_0/c) < \epsilon$ and $S(y_0, y_0) < \epsilon c^2 = \epsilon V(y_0, y_0)$. Thus $s_{jj} = S(\beta_j, \beta_j) < \epsilon V(\beta_j, \beta_j) = \epsilon v_{jj} < v_{jj}$ so $\alpha(s_{jj}) < \alpha(v_{jj})$ since $\alpha(\cdot)$ is increasing. Thus $\sum_{j=1}^{\infty} \alpha(s_{jj}) < \sum_{j=1}^{\infty} \alpha(v_{jj}) < \infty$ as V is an α -operator. The remaining properties of an α -operator are easily seen to hold for $S(\cdot, \cdot)$. For example,

$$\phi(x) = \exp\{i\langle x, a \rangle - \frac{1}{2} S(x, x)\}$$

is τ_α -continuous on E^* and since $\exp\{i\langle x, a \rangle\}$ is τ_α -continuous it follows that $S(x, x)$ is τ_α -continuous. Hence the lemma holds.

Lemma 5.3. *If E has Property A and $\{\mu_N\}$ is a sequence of Gaussian measures with mean vectors $\{a_N\}$ such that $\{\mu_N\}$ converges weakly to μ , then μ is Gaussian with mean $a = \lim_N a_N$ and*

$$(5.2) \quad \lim_N \sum_{j=1}^{\infty} \alpha(|t_{jj}^N - t_{jj}|) = 0$$

where $t_{jj}^N = \int_E \beta_j^2(x - a_N) d\mu_N(x)$, $t_{jj} = \int_E \beta_j^2(x - a) d\mu(x)$ for $j = 1, 2, \dots$.

Proof. For each measure μ_N we define the measure $\nu_N(A) = \mu_N(-A)$ and corresponding to μ we define the measure $\nu(A) = \mu(-A)$. Here, of course, A varies over the Borel subsets of E . Then $\{\nu_N\}$ converges weakly to ν . Now μ_N has Fourier transform $\exp\{i\langle x, a_N \rangle - \frac{1}{2} T_N(x, x)\}$ where T_N is an α -operator on E^* such that $T_N(\beta_i, \beta_j) = \int_E \beta_i(x - a) \beta_j(x - a) d\mu_N(x)$ for $i, j = 1, 2, \dots$, and hence ν_N has Fourier transform $\exp\{i\langle x, -a_N \rangle - \frac{1}{2} T_N(x, x)\}$. Thus the sequence of convolution measures $\{\mu_N * \nu_N\}$ has Fourier transforms $\exp\{-T_N(x, x)\}$ and, furthermore, they converge weakly to $\mu * \nu$ since $\{\mu_N\}$ and $\{\nu_N\}$ converge weakly to μ and ν . If $\Gamma_N(A) = (\mu_N * \nu_N)(\sqrt{2}A)$ for $N = 1, 2, \dots$ and $\Gamma(A) = (\mu * \nu)(\sqrt{2}A)$ for Borel sets A in E then $\{\Gamma_N\}$ converges weakly to Γ and the Fourier transform of Γ_N is $\exp\{-\frac{1}{2} T_N(x, x)\}$. If δ_N is the measure which places mass one at a_N then $\mu_N = \delta_N * \Gamma_N$ and since both $\{\Gamma_N\}$ and $\{\mu_N\}$ converge weakly to Γ and μ this implies $\{\delta_N\}$ converges weakly to the measure δ with unit mass at $a = \lim_N a_N$. To see this last assertion observe that for every $\epsilon > 0$ there exists a compact set K in E such that $\mu_N(K) > 1 - \epsilon$ and $\Gamma_N(K) > 1 - \epsilon$ for $N = 1, 2, \dots$. Thus

$$\begin{aligned} 1 - \epsilon &< \mu_N(K) = \int_E \delta_N(K - x) d\nu_N(x) \\ &\leq \int_K \delta_N(K - x) d\nu_N(x) + \epsilon \end{aligned}$$

and hence there exists an $x_N \in K$ such that $\delta_N(K - x_N) > 1 - 2\epsilon$. If $\epsilon < \frac{1}{2}$ this implies $a_N \in K - x_N$ and hence we find the sequence $\{a_N\}$ lies in the set of

differences $C = \{x - y: x, y \in K\}$. Now C is compact, thus $\{a_N\}$ is conditionally compact and since $\{\mu_N\}$ converges to μ we must have only one limit point for $\{a_N\}$. Thus $\lim_N a_N = a$ exists and $\{\delta_N\}$ converges weakly to δ , as asserted.

Since $\lim_N a_N = a$ exists it now follows that μ is a Gaussian measure (possibly degenerate) with mean a and hence it has Fourier transform $\exp\{i\langle x, a \rangle - \frac{1}{2}T(x, x)\}$ where T is an α -operator on E^* such that

$$T(\beta_i, \beta_j) = t_{ij} = \int_E \beta_i(x - a)\beta_j(x - a)d\mu(x) \quad (i, j = 1, 2, \dots).$$

To complete the proof we need only verify (5.2). Now $\{\Gamma_N\}$ converges weakly to Γ , thus for every $\epsilon > 0$ there is a compact set K such that $\Gamma_N(K) > 1 - \epsilon/2$ for $N = 1, 2, \dots$ and hence

$$\begin{aligned} 1 - \exp\{-\frac{1}{2}T_N(x, x)\} &= \int_E [1 - \cos\langle x, y \rangle] d\Gamma_N(y) \\ (5.3) \qquad \qquad \qquad &\leq \frac{1}{2} \int_K \langle x, y \rangle^2 d\Gamma_N(y) + \epsilon. \end{aligned}$$

Let $S_N(x, x) = \frac{1}{2} \int_K \langle x, y \rangle^2 d\Gamma_N(y)$ for $N = 1, 2, \dots$. Since $t/2 \leq 1 - e^{-t}$, $0 \leq t \leq 1$, (5.3) implies $\frac{1}{4}T_N(x, x) \leq S_N(x, x) + \epsilon$ provided $T_N(x, x) \leq 2$. However, if $T_N(x, x) = c^2 > 0$ then $\frac{1}{4}T_N(x/c, x/c) = \frac{1}{4} \leq S_N(x/c, x/c) + \epsilon \leq 2S_N(x/c, x/c)$ provided $\epsilon < 1/8$. Thus $T_N(x, x) \leq 8S_N(x, x)$ for all $x \in E^*$. Hence

$$t_{jj}^N = T_N(\beta_j, \beta_j) \leq 8S_N(\beta_j, \beta_j) = 4 \int_K \beta_j^2(y) d\Gamma_N(y) = \int_K \beta_j^2(2y) d\Gamma_N(y)$$

for $N, j = 1, 2, \dots$ and we therefore find, since α is convex and the quasi-norm is accessible, that

$$\begin{aligned} \sum_{j=1}^{\infty} \alpha(t_{jj}^N) &\leq \sum_{j=1}^{\infty} \int_K \alpha[\beta_j^2(2y)] d\Gamma_N(y) \\ &= \int_K \sum_{j=1}^{\infty} \alpha[\beta_j^2(2y)] d\Gamma_N(y) \\ &\leq \int_K M\gamma(\|2y\|) d\Gamma_N(y) \end{aligned}$$

where M depends only on K and not N . That is, since K is compact there exists an $r > 0$ such that $\sum_{j=1}^{\infty} \alpha[\beta_j^2(2y)] < r$ for all $y \in K$, and hence there exists an $M > 0$ such that $y \in K$ implies

$$\sum_{j=L+1}^{\infty} \alpha[\beta_j^2(2y)] \leq M\gamma(\|Q_L(2y)\|).$$

Here M depends only on K through our choice of r and not on L . We then have

$$\begin{aligned} \sup_N \sum_{j=L+1}^{\infty} \alpha(t_{jj}^N) &\leq \sup_N \int_K \sum_{j=L+1}^{\infty} \alpha[\beta_j^2(2y)] d\Gamma_N(y) \\ &\leq \sup_N M \int_K \gamma(\|Q_L(2y)\|) d\Gamma_N(y) \end{aligned}$$

and as L approaches infinity $\gamma(\|Q_L(2y)\|)$ tends to zero on K boundedly, i.e. $\gamma(\|Q_L(2y)\|) \leq \gamma(\sup_{y \in K} 2\|2y\|) < \infty$, thus

$$\limsup_L \sup_N \sum_{j=L+1}^{\infty} \alpha(t_{jj}^N) = 0.$$

We then have the conclusion (5.3) because $\lim_N t_{jj}^N = t_{jj}$ for $j = 1, 2, \dots$ and hence for every $\epsilon > 0$ and integer L there exists N sufficiently large such that

$$\begin{aligned} \sum_{j=1}^{\infty} \alpha(|t_{jj}^N - t_{jj}|) &\leq \epsilon + \sum_{j=L}^{\infty} \alpha(|t_{jj}^N - t_{jj}|) \\ (5.4) \quad &\leq \epsilon + \sum_{j=L}^{\infty} \alpha(t_{jj}^N) + \sum_{j=L}^{\infty} \alpha(t_{jj}). \end{aligned}$$

Now $\lim_L \sum_{j=L}^{\infty} \alpha(t_{jj}) = 0$ since T is an α -operator and $\lim_L \sup_N \sum_{j=L}^{\infty} \alpha(t_{jj}^N) = 0$ by previous remarks, so (5.4) implies (5.2).

An F -space E has an *accessible quasi-norm in both directions* if there exist functions $\alpha, \gamma_1, \gamma_2$ such that E has an accessible quasi-norm with respect to α and γ_2 and, for every $y \in E$, $\gamma_1(\|y\|) \leq \sum_{j=1}^{\infty} \alpha(\beta_j^2(y))$. Here $\gamma_1(\cdot)$ is an increasing continuous function on $[0, \infty)$ such that $\gamma_1(0) = 0$, $\gamma_1(s) > 0$ for $s > 0$, and γ_2, α satisfy the conditions required in the definition of an accessible quasi-norm. Geometrically, the above inequality implies that the inverse of the map $y \rightarrow (\beta_1(y), \beta_2(y), \dots)$ is continuous from the Orlicz space related to l_α into E under additional conditions (cf. (6.3)) on α .

Let E be a real separable F -space having an accessible quasi-norm in both directions. Then E has Property B provided E has Property A and there is a constant $M > 0$ satisfying

$$\int_{-\infty}^{\infty} \alpha(u^2) d\nu(u) \leq M \alpha \left[\int_{-\infty}^{\infty} u^2 d\nu(u) \right]$$

for all Gaussian measures ν on $(-\infty, \infty)$ with mean zero.

Lemma 5.4. *Let E have Property B and suppose T is an α -operator on E^* . Then $\phi(x) = \exp\{-\frac{1}{2}T(x, x)\}$ is the Fourier transform of a Gaussian measure μ on E . Further, $T(\beta_i, \beta_j) = \int_E \beta_i(y)\beta_j(y) d\mu(y)$ for $i, j = 1, 2, \dots$.*

Proof. Let μ_N denote the Gaussian measure on E corresponding to the

Fourier transform $\exp\{-\frac{1}{2}T(P_N x, P_N x)\}$ for $N = 1, 2, \dots$. We show $\{\mu_N\}$ is conditionally compact and hence, by Corollary 4.1, $\{\mu_N\}$ converges to the measure μ with Fourier transform $\phi(x)$. Let $\epsilon, \delta > 0$. Then

$$\begin{aligned}\mu_N\{y: \|Q_L(y)\| > \delta\} &= \mu_N\{y: \gamma_1(\|Q_L(y)\|) > \gamma_1(\delta)\} \\ &\leq \mu_N\left\{y: \sum_{j=L+1}^{\infty} \alpha(\beta_j^2(y)) > \gamma_1(\delta)\right\} \\ &\leq \frac{M}{\gamma_1(\delta)} \sum_{j=L+1}^{\infty} \alpha\left[\int_E \beta_j^2(y) d\mu_N(y)\right] = \frac{M}{\gamma_1(\delta)} \sum_{j=L+1}^{\infty} \alpha(t_{jj}^N).\end{aligned}$$

However, $t_{jj}^N = \int_E \beta_j^2(y) d\mu_N(y) = t_{jj}$ for $j = 1, \dots, N$ and $t_{jj}^N = 0$ for $j > N$, thus $\mu_N\{y: \|Q_L(y)\| > \delta\} < \epsilon$ for $N = 1, 2, \dots$ provided L is sufficiently large. Proceeding with standard techniques (see, for example, Lemma 2.4 of [5]) we see $\{\mu_N\}$ is conditionally compact. As remarked previously, Corollary 4.1 now implies $\{\mu_N\}$ converges weakly to a measure μ with Fourier transform $\phi(x)$, and this completes the proof.

In view of Lemma 3.1 it is easy to see that Theorem 3.1 also easily applies to the previous lemma.

Theorem 5.1. *If E has Property B then μ is a Gaussian measure with mean a iff the Fourier transform of μ is*

$$\phi(x) = \exp\{i\langle x, a \rangle - \frac{1}{2}T(x, x)\}$$

where T is the α -operator on E^* satisfying $T(\beta_i, \beta_j) = \int_E \beta_i(y - a)\beta_j(y - a) d\mu(y)$ for $i, j = 1, 2, \dots$.

Proof. Combine Lemmas 5.2 and 5.4.

Theorem 5.2. *If E has Property B and $\{\mu_\lambda: \lambda \in \Lambda\}$ is a family of Gaussian measures with means a_λ and Fourier transforms*

$$\phi_\lambda(x) = \exp\{i\langle x, a_\lambda \rangle - \frac{1}{2}T_\lambda(x, x)\}$$

then $\{\mu_\lambda: \lambda \in \Lambda\}$ is conditionally compact iff

- (1) $\{a_\lambda: \lambda \in \Lambda\}$ is conditionally compact in E ;
- (2) the family $\{\phi_\lambda(P_N x): \lambda \in \Lambda\}$ is equicontinuous on $P_N E^*$ for $N = 1, 2, \dots$;
- (3) $\lim_L \sup_\lambda \sum_{j=L+1}^{\infty} \alpha(t_{jj}^\lambda) = 0$.

Proof. If $\{\mu_\lambda: \lambda \in \Lambda\}$ is conditionally compact then every sequence in $\{\mu_\lambda: \lambda \in \Lambda\}$ has a weakly convergent subsequence. Consequently, by Lemma 5.3, (1) holds and (2) follows since $\{\mu_\lambda^{P_N}: \lambda \in \Lambda\}$ is conditionally compact for $N =$

1, 2, Now if $\lim_L \sup_{\lambda} \sum_{j=L+1}^{\infty} \alpha(t_{jj}^{\lambda}) = r > 0$ then there exists $\lambda_1, \lambda_2, \dots$ such that $\sum_{j=L+1}^{\infty} \alpha(t_{jj}^{\lambda_k}) > r/2$ for all $L \leq k$. However, the sequence of measures $\{\mu_{\lambda_k}\}$ has a convergent subsequence, so assuming $\{\mu_{\lambda_k}\}$ itself converges weakly to a measure μ we have by Lemma 5.3 that

$$(5.5) \quad \lim_k \sum_{j=1}^{\infty} \alpha(|t_{jj}^{\lambda_k} - t_{jj}|) = 0$$

where $t_{jj} = \int_E \beta_j^2(y - a) d\mu(y)$ for $j = 1, 2, \dots$. Thus

$$(5.6) \quad \sum_{j=L+1}^{\infty} \alpha(t_{jj}^{\lambda_k}) \leq \sum_{j=L+1}^{\infty} \alpha[t_{jj} + |t_{jj}^{\lambda_k} - t_{jj}|]$$

$$\leq \frac{1}{2} \sum_{j=L+1}^{\infty} \alpha(2t_{jj}) + \frac{1}{2} \sum_{j=L+1}^{\infty} \alpha(2|t_{jj}^{\lambda_k} - t_{jj}|)$$

where the last inequality follows since $\alpha(\cdot)$ is convex on $[0, \infty)$. Now since E has an accessible quasi-norm in both directions we have, given $y \in E$, that there exists $M > 0$ such that $\gamma_1(\|y\|) \leq \sum_{j=1}^{\infty} \alpha[\beta_j^2(y)] \leq M\gamma_2(\|y\|)$ and also that $\gamma_1(\|\sqrt{2}y\|) \leq \sum_{j=1}^{\infty} \alpha[2\beta_j^2(y)] \leq M\gamma_2(\|\sqrt{2}y\|)$. We now show that $\sum_{j=1}^{\infty} \alpha(t_{jj}) < \infty$ implies $\sum_{j=1}^{\infty} \alpha(2t_{jj}) < \infty$. Let $s_n = \sum_{j=1}^n t_{jj}^{1/2} b_j$ for $n = 1, 2, \dots$. Then $\{s_n\}$ is a Cauchy sequence in E . That is, if $n > k$ then $\|s_n - s_k\| = \|\sum_{j=k+1}^n t_{jj}^{1/2} b_j\|$.

Now $\gamma_1(\|s_n - s_k\|) \leq \sum_{j=1}^{\infty} \alpha(\beta_j^2(s_n - s_k))$ and since $\beta_j^2(s_n - s_k) = t_{jj}$ for $k+1 \leq j \leq n$ we have $\gamma_1(\|s_n - s_k\|) \leq \sum_{j=k+1}^n \alpha(t_{jj})$, and $\sum_{j=1}^{\infty} \alpha(t_{jj}) < \infty$ along with the properties of γ_1 implies $\lim_{n,k} \|s_n - s_k\| = 0$. Thus $\{s_n\}$ converges to a vector s and $\{\sqrt{2}s_n\}$ converges to $\sqrt{2}s$ so

$$\sum_{j=1}^{\infty} \alpha(2t_{jj}) = \sum_{j=1}^{\infty} \alpha(2\beta_j^2(s)) \leq M\gamma_2(\|\sqrt{2}s\|) < \infty$$

for some constant $M > 0$. We now verify that (5.5) implies

$$(5.7) \quad \lim_k \sum_{j=1}^{\infty} \alpha(2|t_{jj}^{\lambda_k} - t_{jj}|) = 0.$$

By the above argument the vectors $v_k = \sum_{j=1}^{\infty} |t_{jj}^{\lambda_k} - t_{jj}|^{1/2} b_j$ converge to zero since $\gamma_1(\|v_k\|) \leq \sum_{j=1}^{\infty} \alpha(|t_{jj}^{\lambda_k} - t_{jj}|)$ and (5.5) implies this last term goes to zero as k goes to infinity. Thus the sequence $\{\sqrt{2}v_k\}$ also converges to zero, and hence

$$0 \leq \lim_k \sum_{j=1}^{\infty} \alpha(2|t_{jj}^{\lambda_k} - t_{jj}|) \leq M \lim_k \gamma_2(\|\sqrt{2}v_k\|) = 0$$

for some constant M , so (5.7) holds. Applying (5.7) and $\sum_{j=1}^{\infty} \alpha(2t_{jj}) < \infty$ to (5.6) we see

$$\limsup_L \sup_k \sum_{j=L+1}^{\infty} \alpha(t_{jj}^{\lambda_k}) = 0.$$

This contradicts $\sum_{j=L+1}^{\infty} \alpha(t_{jj}^{\lambda_k}) > \tau/2$ for all $L \leq k$ and hence (3) holds.

For the converse assume (1), (2), (3) hold. Let $\epsilon, \delta > 0$. Then

$$(5.8) \quad \begin{aligned} \mu_{\lambda} \left\{ y: \left\| \sum_{j=L+1}^{\infty} \beta_j(y) b_j \right\| > \delta \right\} &= \mu_{\lambda} \{ y: \gamma_1(\|Q_L y\|) > \gamma_1(\delta) \} \\ &\leq \mu_{\lambda} \left\{ y: \sum_{j=L+1}^{\infty} \alpha(\beta_j^2(y)) > \gamma_1(\delta) \right\} \leq \frac{M}{\gamma_1(\delta)} \sum_{j=L+1}^{\infty} \alpha(t_{jj}^{\lambda}). \end{aligned}$$

Thus by (3) we have for L sufficiently large that $\mu_{\lambda} \{ y: \|Q_L y\| > \delta \} < \epsilon$ for all $\lambda \in \Lambda$. Now arguing as indicated in Lemma 5.4 we see (5.8) and condition (2) implies $\{\mu_{\lambda}: \lambda \in \Lambda\}$ is conditionally compact, so the theorem is proved.

Theorem 5.3. *If E has Property B and $\{\mu_k\}$ is a sequence of Gaussian measures with mean vectors $\{a_k\}$ and Fourier transforms*

$$\phi_k(x) = \exp\{i\langle x, a_k \rangle - \frac{1}{2} T_k(x, x)\},$$

then $\{\mu_k\}$ converges weakly to a measure μ iff

$$(i) \lim_k \phi_k(x) \text{ exists on } \bigcup_{N=1}^{\infty} P_N(E^*),$$

$$(ii) \lim_{k,n} \sum_{j=1}^{\infty} \alpha(|t_{jj}^k - t_{jj}^n|) = 0.$$

Further, the measure μ is Gaussian with mean $a = \lim a_k$ and Fourier transform $\phi(x) = \exp\{i\langle x, a \rangle - \frac{1}{2} T(x, x)\}$ where $t_{ij} = T(\beta_i, \beta_j) = \int_E \beta_i(y - a) \beta_j(y - a) d\mu(y)$ for $i, j = 1, 2, \dots$.

The proof of Theorem 5.3 follows directly by using Lemma 5.3 and the techniques employed in the proof of Theorem 5.2, so it is omitted.

Corollary 5.1. *If E has Property B and μ is a measure on E such that every linear functional on E has a Gaussian distribution, then there exists a vector a in E such that a is the mean of μ , i.e. $\int_E x(y) d\mu(y) = x(a)$ for every $x \in E^*$.*

Proof. Since μ^{P_N} is a Gaussian measure on $P_N(E)$ with mean a_N in $P_N(E)$ for $N = 1, 2, \dots$, and $\{\mu^{P_N}\}$ converges to μ , it follows from Theorem 5.3 that $a = \lim_N a_N$ exists and the corollary follows.

As a result of Corollary 5.1 we find that in our definition of a Gaussian measure on E we need not assume that a mean vector a exists provided E has Property B.

If $E = l_p$ and $y = (y_1, y_2, \dots)$ then as is usual $\beta_j(y) = y_j$ for $j = 1, 2, \dots$.

Corollary 5.2. *If $E = l_p$, $2 \leq p < \infty$, and $\{\mu_k\}$ is a sequence of Gaussian probability measures on l_p with mean vectors $\{a_k\}$ and $t_{ij}^k = \int_E \beta_i(y - a_k) \beta_j(y - a_k) d\mu_k(y)$ for $i, j, k = 1, 2, \dots$, then $\{\mu_k\}$ converges weakly to a measure μ iff*

- (1) $\lim_k a_k$ exists;
- (2) $\lim_k t_{ij}^k$ exists for $i, j = 1, 2, \dots$;
- (3) $\lim_{k,n} \sum_{j=1}^{\infty} |t_{jj}^k - t_{jj}^n|^{p/2} = 0$.

Furthermore, μ is then the Gaussian measure with mean $a = \lim_k a_k$ and satisfying $t_{ij} = \int_E \beta_i(y - a) \beta_j(y - a) d\mu(y)$ for $i, j = 1, 2, \dots$.

Proof. This follows from Theorem 5.3 since if $E = l_p$, $2 \leq p < \infty$, then E has Property B. To see this let $\alpha(s) = s^{p/2}$ and $\gamma_1(s) = \gamma_2(s) = s^p$. Then l_p has an accessible norm in both directions with respect to $\alpha, \gamma_1, \gamma_2$. Further, for every Gaussian measure ν on $(-\infty, \infty)$ with mean zero we have

$$\int_{-\infty}^{\infty} \alpha(s^2) d\nu(s) = \int_{-\infty}^{\infty} |s|^p d\nu(s) = M\alpha\left[\int_{-\infty}^{\infty} s^2 d\nu(s)\right]$$

where $M = \Gamma((p+1)/2)[\Gamma(3/2)]^{-p/2} \pi^{(p-1)/2}$ and since l_p is a Banach space we see l_p has Property A. Combining these we find that l_p has Property B and that conditions (1), (2), and (3) are equivalent to conditions (i) and (ii) of Theorem 5.3. Thus the corollary is proved.

It should also be mentioned that (1), (2), (3) are sufficient for $\{\mu_k\}$ to converge weakly if $1 \leq p < 2$. The necessity in this case is unknown.

Corollary 5.3. *If $E = l_p$, $2 \leq p < \infty$, then*

$$\phi(x) = \exp\{i(x, a) - \frac{1}{2}(Tx, x)\} \quad (x \in E^*)$$

is the Fourier transform of a Gaussian measure on E iff $a \in E$ and T is an S_p operator.

Proof. Combine Corollary 5.2 and the ideas used in the proof of Corollary 5.3.

6. As pointed out in the proof of Corollary 5.2 the l_p spaces, $2 \leq p < \infty$, each have Property B. It is natural to ask if other spaces have it, or, for that matter, if there are other spaces which have a quasi-norm which is accessible in both directions. In this section we characterize all possible spaces with quasi-norm accessible in both directions provided the function $\alpha(\cdot)$ used in the definition of an accessible quasi-norm satisfies the additional condition

$$(6.1) \quad \alpha(2s) \leq M\alpha(s)$$

for all $s \geq 0$ and some $M < \infty$. This result is found in Theorem 6.3. Furthermore, if, in addition to (6.1), there is a constant C such that

$$(6.2) \quad \int_{-\infty}^{\infty} \alpha(u^2) d\nu(u) \leq C \alpha \left[\int_{-\infty}^{\infty} u^2 d\nu(u) \right]$$

for all Gaussian measures ν on $(-\infty, \infty)$ with mean zero, then our characterization applies to spaces with Property B. Throughout this section $\alpha(\cdot)$ will always have the usual properties required in the definition of accessibility as well as (6.1). Condition (6.2) will be used only in connection with Property B.

Since $\alpha(\cdot)$ is convex we have for $a, b \geq 0$ that $\alpha(a+b) \leq \frac{1}{2}[\alpha(2a) + \alpha(2b)]$ and by (6.1) we find

$$(6.3) \quad \alpha(a+b) \leq (M/2)[\alpha(a) + \alpha(b)].$$

Now we define E_α as the space of real sequences satisfying $\sum_{i=1}^{\infty} \alpha(y_i^2) < \infty$. Using (6.3) it follows that E_α is a vector space over the reals. If $x, y \in E_\alpha$ we define

$$(6.4) \quad J(x, y) = \sum_{i=1}^{\infty} \alpha((x_i - y_i)^2).$$

Then $J(x, y)$ is invariant under translations and we denote by $\mathcal{G}_\alpha$ all subsets G of $E_\alpha \times E_\alpha$ such that there exists an $r > 0$ with the property that $J(x, y) < r$ implies $(x, y) \in G$. Then $\mathcal{G}_\alpha$ is a uniformity for E_α [4, p. 176] and the topology τ of the uniformity $\mathcal{G}_\alpha$ consists of the family of subsets A of E_α such that for each $x \in A$ there is a $G \in \mathcal{G}_\alpha$ such that $\{y: (x, y) \in G\} \subseteq A$. Now the subfamily $\mathcal{B}_\alpha = \{G_1, G_2, \dots\}$ of $\mathcal{G}_\alpha$ where $G_n = \{(x, y): J(x, y) < 1/n\}$ for $n = 1, 2, \dots$ is a countable symmetric base for $\mathcal{G}_\alpha$ and since $J(x, y) > 0$ for $x \neq y$ it follows that the topology τ is Hausdorff. Hence the metrization theorem [4, p. 185] tell us that (E_α, τ) is metrizable.

In fact, using the proof and the notation employed in the metrization theorem we will show that there exists an invariant metric $d(\cdot, \cdot)$ on E_α , i.e. $d(x, y) = d(x+z, y+z)$ for all $z \in E_\alpha$, such that $d(\cdot, \cdot)$ generates the topology τ , the quasi-norm $\|x\| = d(0, x)$ is accessible in both directions, and that E_α is an F -space with respect to τ .

Linking our situation to the metrization theorem we let $U_0 = E_\alpha \times E_\alpha$, $U_1 = \{(x, y): J(x, y) < 1\}$, and $U_n = \{(x, y): J(x, y) < (9M^4)^{-n+1}\}$ for $n = 1, 2, \dots$ where M is as in (6.1). Then defining $d(\cdot, \cdot)$ as in [4, p. 185] we have that $d(\cdot, \cdot)$ is a metric satisfying

$$(6.5) \quad U_n \subseteq \{(x, y): d(x, y) < 2^{-n}\} \subseteq U_{n-1}$$

for $n = 1, 2, \dots$ and that $d(x, y) \leq \frac{1}{2}$ for all $x, y \in E_\alpha$. Moreover, examining the definition of $d(\cdot, \cdot)$ we find $d(\cdot, \cdot)$ is invariant since our sets $\{U_n\}$ are invariant in

the sense that $(x, y) \in U_n$ iff $(x+z, y+z) \in U_n$ for every $z \in E_\alpha$. In view of (6.5) if $x \in E_\alpha$ then $(x, 0) \in U_{n-1} - U_n$ ($n = 2, 3, \dots$) iff $\Gamma^{n-1} \leq J(x, 0) < \Gamma^{n-2}$ and $2^{-n-1} \leq d(x, 0) < 2^{-n+1}$ where $\Gamma = (9M^4)^{-1}$. Hence if $\gamma_1(s), \gamma_2(s)$ are continuous strictly increasing functions on $[0, \infty)$ such that $\gamma_i(0) = 0$ for $i = 1, 2$, and satisfying

$$(6.6) \quad \gamma_2(2^{-n-1}) \geq \Gamma^{n-2}, \quad \gamma_1(2^{-n+1}) \leq \Gamma^{n-1}$$

for $n = 1, 2, 3, \dots$ we have

$$(6.7) \quad \gamma_1(d(x, 0)) \leq J(x, 0) \leq \gamma_2(d(x, 0))$$

provided $(x, 0) \in U_1$, i.e. $J(x, 0) < 1$. If $(x, 0) \in U_0 - U_1$ then $J(x, 0) \geq 1$, $\frac{1}{2} \geq d(x, 0) \geq \frac{1}{4}$, and since $\gamma_2(\frac{1}{4}) \geq \Gamma^{-1} = 9M^4$, $\gamma_1(1) \leq 1$ we have

$$(6.8) \quad \gamma_1(d(x, 0)) \leq J(x, 0) \leq M_1 \gamma_2(d(x, 0))$$

provided $J(x, 0) \leq M_1 \Gamma^{-1}$.

Let $c > 0$ be such that $\gamma_1^{-1}(\alpha(c^2)) \leq 1$.

Theorem 6.1. *The metric space (E_α, τ) with quasi-norm $\|x\| = d(x, 0)$ as above is an F -space with basis $\{b_n\}$. Here b_n is the vector with c as the n th coordinate and all other coordinates zero. Further, $\|b_n\| \leq 1$ for $n = 1, 2, \dots$ and the quasi-norm is accessible in both directions with respect to the functions $\alpha, \gamma_1, \gamma_2$ where γ_1, γ_2 are as in (6.6).*

To show (E_α, τ) is an F -space we first must show that it is complete in the metric $d(\cdot, \cdot)$ and that it is also a linear topological space in the topology τ . Then the assertions about $\{b_n\}$ are immediate in view of (6.8), the definition of E_α , and our choice of $c > 0$. Finally, the accessibility of the quasi-norm will follow from (6.7) and (6.8) provided we can show that for any compact set K of (E_α, τ) there is an $r < \infty$ such that $K \subseteq \{x: J(x, 0) < r\}$. These facts will be obtained in the next several lemmas and hence Theorem 6.1 holds.

Lemma 6.1. *The metric space (E_α, τ) with the invariant metric $d(\cdot, \cdot)$ as defined above is a complete metric space.*

The proof of Lemma 6.1 follows in exactly the same manner as in the l_p spaces and hence is omitted.

Lemma 6.2. *(E_α, τ) is a linear topological space.*

Proof. Since (E_α, τ) is a complete metric space in the invariant metric $d(\cdot, \cdot)$ it suffices to show by Theorem 12 of [1, p. 53] that the mapping $\psi(b, x) = bx$ from $R \times E_\alpha$ into E_α is continuous in b for each x and continuous in x for each b . First fix x and choose $\{b_n\}$ in R converging to b . Then

$$\gamma_1(d((b - b_n)x, 0)) \leq J((b - b_n)x, 0) = \sum_{j=1}^{\infty} \alpha[(b - b_n)^2 x_j^2] < \infty,$$

and as n approaches infinity the last term converges to zero by the dominated convergence theorem since $(b - b_n)^2 x_i^2 \leq x_i^2$ for $(b - b_n)^2 < 1$ and $\alpha(\cdot)$ is increasing. Now fix b and choose $\{p_n\}$ converging to x in E_α . Then, if k is an integer greater than b^2 we find

$$\begin{aligned} \gamma_1(d(bp_n, bx)) &\leq J(bp_n, bx) = \sum_{j=1}^{\infty} \alpha(b^2(x_{n,j} - x_j)^2) \\ (6.9) \quad &\leq \sum_{j=1}^{\infty} \alpha(k(x_{n,j} - x_j)^2) \leq M^k \sum_{j=1}^{\infty} \alpha((x_{n,j} - x_j)^2) \leq M^k \gamma_2(d(p_n - x, 0)) \end{aligned}$$

since $\{p_n\}$ converging to x implies that $d(p_n - x, 0) < 1/4$ for n sufficiently large. Now $\lim_n \gamma_2(d(p_n - x, 0)) = 0$ so $\lim_n d(bp_n, bx) = 0$ and hence the lemma is proved.

Lemma 6.3. *If $J(y, x)$ is defined as in (6.4) and K is a compact subset of (E_α, τ) then there exists an $r > 0$ such that $K \subseteq \{y: J(y, 0) < r\}$.*

Proof. If no such r exists there is a sequence $\{y_n\}$ in K such that $\{y_n\}$ converges to y and $J(y_n, 0) > n$. This contradicts the fact that $\sup_n J(y_n, 0)$ is bounded for Cauchy sequences and hence such an r exists so the lemma is proved.

Lemma 6.4. *If $\{b_n\}$ is defined as in Theorem 6.1 then $\{b_n\}$ is a basis for (E_α, τ) and $\|b_n\| \leq 1$ for $n = 1, 2, \dots$.*

Proof. Combine the definition of E_α and $c > 0$ along with (6.8).

At this point we have now proved Theorem 6.1. The impact of Theorem 6.1 is that any function $\alpha(\cdot)$ satisfying (6.1) and the usual conditions we have imposed can be used to produce an F -space which has an accessible quasi-norm in both directions. We now proceed to characterize these spaces as well as those with Property B.

If $\alpha(t)$ is a convex function on $[0, \infty)$ such that $\alpha(0) = 0$, $\alpha(t) > 0$ for $t > 0$, then, as mentioned previously, $\alpha(t)$ is also continuous and strictly increasing on $[0, \infty)$. Further, $\Gamma(t) = \alpha(t^2)$ then has the same properties and it follows that $\Gamma(t) = \int_0^t \rho(s) ds$ where $\rho(0) = 0$ and $\rho(s)$ is nondecreasing on $[0, \infty)$. We assume, without loss of generality, that $\rho(s)$ is left continuous and define $\Lambda(t) = \int_0^t \psi(s) ds$ where $\psi(s)$ is the inverse of $\rho(s)$ on $[0, \infty)$ defined with the understanding that $\psi(s)$ is left continuous where it is finite and that if $\rho(t)$ makes a jump at $t = a$ then $\psi(s) = a$ for $\rho(a-) < s \leq \rho(a+)$, while, if $\rho(t) = c$ for $a < u \leq b$ but $\rho(t) < c$ for $t < a$ then $\psi(c) = a$. Furthermore, $\psi(0) = 0$ and if $\lim_{t \rightarrow \infty} \rho(t) = \gamma < \infty$ then

$\psi(s) = +\infty$ for $s > \gamma$. Then Γ and Λ are complementary in the sense of Young [10, p. 77] and by $\mathcal{S}_\Gamma^*$ we mean all real sequences $\{x_i\}$ such that

$$\sum_{i=1}^{\infty} \Gamma(|x_i|) = \sum_{i=1}^{\infty} \alpha(x_i^2) < \infty.$$

Similarly, $\mathcal{S}_\Lambda^*$ is all sequences such that $\sum_{i=1}^{\infty} \Lambda(|x_i|) < \infty$.

If $x = \{x_i\}$ is a sequence we define

$$\|x\|_\Gamma = \sup_y \left\{ \sum_{i=1}^{\infty} |x_i y_i| : \sum_{i=1}^{\infty} \Lambda(|y_i|) \leq 1 \right\},$$

$$\|x\|_\Lambda = \sup_y \left\{ \sum_{i=1}^{\infty} |x_i y_i| : \sum_{i=1}^{\infty} \Gamma(|y_i|) \leq 1 \right\}.$$

The Orlicz space $\mathcal{S}_\Gamma$ ($\mathcal{S}_\Lambda$) is the collection of all real sequences such that $\|x\|_\Gamma$ ($\|x\|_\Lambda$) is finite. Since $\alpha(t)$ satisfies (6.1) it follows that $\Gamma(2t) = \alpha(4t^2) \leq M\alpha(2t^2) \leq M^2\alpha(t^2) = M^2\Gamma(t)$ and hence we know [10, p. 81] that $\mathcal{S}_\Gamma = \mathcal{S}_\Gamma^*$. Further, it is known that $\mathcal{S}_\Gamma = \mathcal{S}_\Gamma^*$ is a real separable Banach space in the norm $\|x\|_\Gamma$ and since $\Gamma(2t) \leq M^2\Gamma(t)$ for $t \geq 0$ we also have that $\{p_n\}$ converges to p in norm provided

$$(6.10) \quad \lim_n \sum_{i=1}^{\infty} \Gamma(|x_{i,n} - x_i|) = \lim_n \sum_{i=1}^{\infty} \alpha((x_{i,n} - x_i)^2) = 0.$$

These results follow as in [10, pp. 82–85]. Now it is clear that $E_\alpha = \mathcal{S}_\Gamma^*$ and since $\mathcal{S}_\Gamma^* = \mathcal{S}_\Gamma$ we have $E_\alpha = \mathcal{S}_\Gamma$. Consequently, $E_\alpha = \mathcal{S}_\Gamma$ is an F -space in the topology τ and in the topology τ_Γ generated by the norm $\|x\|_\Gamma$. Further, $\tau_\Gamma \subseteq \tau$ since $\{p_n\}$ converging to p with respect to the τ topology implies $\lim_n d(p_n, p) = 0$, and hence (6.10) holds so $\{p_n\}$ converges to p in the norm $\|\cdot\|_\Gamma$, i.e. in the τ_Γ topology. Thus any closed set with respect to τ_Γ is also closed with respect to τ and hence $\tau_\Gamma \subseteq \tau$ as indicated.

Theorem 6.2. *The metric space (E_α, τ) is a Banach space in the norm $\|\cdot\|_\Gamma$.*

Proof. Since the topologies τ and τ_p both make $E_\alpha = \mathcal{S}_\Gamma$ an F -space and they are comparable it follows that they are identical [1, p. 58]. Thus the theorem follows since (E_α, τ_Γ) is a Banach space.

Corollary 6.1. *The metric space (E_α, τ) with quasi-norm $\|\cdot\|$ and basis $\{b_n\}$ as defined in Theorem 6.1 is such that*

(1) *If K is any compact subset of E_α and $\{\beta_n\}$ represents the coefficient functionals relative to the basis $\{b_n\}$, then $\sup_{n,y \in K} |\beta_n(y)| < \infty$.*

(2) *The sigma-algebra generated by the weakly open subsets of E_α is equal to the sigma-algebra generated by the τ -open subsets of E_α .*

(3) (E_α, τ) has Property A.

(4) If $\alpha(\cdot)$ also satisfies (6.2), then (E_α, τ) has Property B.

Proof. If K is compact then there exists an $r > 0$ such that $y \in K$ implies $J(y, 0) < r$. Now, $\beta_n(y) = y_n/c$ where y_n is the n th coordinate of y and $c > 0$ is the constant chosen as in Theorem 6.1. Further, $\alpha(y_n^2/c^2) < r$ and hence $|y_n/c| \leq \{\alpha^{-1}(r)\}^{1/2}$ so $\sup_{y \in K, n} |\beta_n(y)| \leq \{\alpha^{-1}(r)\}^{1/2} < \infty$ and (1) holds.

To see that (2) holds combine Theorem 6.2 and the remarks made in the second paragraph of §3.

To prove (E_α, τ) has Property A we note that Theorem 6.1 implies $d(\cdot, \cdot)$ defines an accessible quasi-norm for (E_α, τ) , and by Theorem 6.2 that (E_α, τ) is also a Banach space in the norm $\|\cdot\|_\Gamma$. Hence $\{x_n\}$ in E_α^* converging to x in the E_α topology on E_α^* implies $\sup_{n,i} |x_n(b_i)| < \infty$ provided $\sup_i \|b_i\|_\Gamma < \infty$. Now $\|b_i\|_\Gamma = \sup_y \{|cy| : \Lambda(|y|) \leq 1\}$ for $i = 1, 2, \dots$ and this last quantity is finite, so $\sup_i \|b_i\|_\Gamma < \infty$ as desired.

Condition (4) now follows from (3) and the definition of Property B.

Theorem 6.3. If E has a quasi-norm which is accessible in both directions with respect to the functions α, ρ_1, ρ_2 and $\alpha(\cdot)$ satisfies condition (6.1), then E and the Orlicz space E_α are homeomorphic and isomorphic.

Proof. We denote the quasi-norm on E by $\|\cdot\|$. If $p \in E$ we let $\Phi(p) = (\beta_1(p), \beta_2(p), \dots)$. Since $\|\cdot\|$ is accessible, $p \in E$ implies $\sum_{j=1}^\infty \alpha[\beta_j^2(p)] < \infty$, thus $\Phi(p) \in E_\alpha$. On the other hand, if $(c_1, c_2, \dots) \in E_\alpha$, then $\sum_{i=1}^\infty \alpha(c_i^2) < \infty$ and if $p_n = \sum_{i=1}^n c_i b_i$ for $n = 1, 2, \dots$ then $\{p_n\}$ is a Cauchy sequence in E . Let $p = \lim_n p_n$. Then $\Phi(p) = (\beta_1(p), \beta_2(p), \dots) = (c_1, c_2, \dots)$ so Φ is a one-to-one map of E onto E_α which clearly preserves the algebraic relations. To see that Φ is also a homeomorphism we first note that since $\|\cdot\|$ is accessible in both directions it follows that a sequence $\{p_n\}$ in E converges to zero iff $\lim_n \sum_{j=1}^\infty \alpha[\beta_j^2(p_n)] = 0$. Then, if $\Phi(p_n) = (\beta_1(p_n), \beta_2(p_n), \dots)$ we see from previous considerations that $\{\Phi(p_n)\}$ converges to zero in the norm on E_α iff $\lim_n \sum_{j=1}^\infty \alpha[\beta_j^2(p_n)] = 0$. Consequently, Φ is a homeomorphism and the theorem is proved.

Corollary 6.2. If E has a quasi-norm which is accessible in both directions with respect to the functions α, ρ_1, ρ_2 and $\alpha(\cdot)$ satisfies condition (6.1), then E is isomorphic and homeomorphic to l_p for some $p, 2 \leq p < \infty$, iff

$$(6.11) \quad \lim_{t \downarrow 0} \frac{\alpha(t^2)}{t^p} = k < \infty,$$

$$(6.12) \quad \lim_{t \downarrow 0} \frac{\alpha(t^2)}{t^p} = l > 0.$$

Proof. If (6.11) and (6.12) hold then E is isomorphic to l_p with respect to the map $\Phi(x) = (\beta_1(x), \beta_2(x), \dots)$. In view of (6.11) a sequence $\{x_n\}$ converges to zero in E if

$$\lim_n \sum_{j=1}^{\infty} (k + \epsilon) |\beta_j(x_n)|^p = 0.$$

Thus $\{x_n\}$ converges to zero in E if $\{x_n\}$ converges to zero in l_p for some p such that (6.11) holds. On the other hand, if $\{x_n\}$ converges to zero in E then by (6.12) it follows that $\{x_n\}$ converges to zero in l_p . Thus if (6.11) and (6.12) hold we also find E and l_p are homeomorphic as well as isomorphic.

Conversely, if E is homeomorphic and isomorphic to l_p then

$$(6.13) \quad \lim_n \sum_{k=1}^{\infty} \alpha(\beta_k^2(x_n)) = 0 \quad \text{iff} \quad \lim_n \sum_{k=1}^{\infty} |\beta_k(x_n)|^p = 0$$

for a sequence of points $\{x_n\}$ in $E = l_p$. If $\lim_{t \downarrow 0} \alpha(t^2)/t^p = \infty$ we choose $0 < \epsilon_n < 1$ such that $\alpha([\epsilon_n/n^{1/p}]^2) > \epsilon_n^p$ and let k_n be such that $1 < k_n(\epsilon_n)^p < 2$. Then for $n = 1, 2, \dots$ we define the point x_n in E such that $\beta_k(x_n) = 0$ if $k > k_n$ and $\beta_k(x_n) = \epsilon_n/n^{1/p}$ for $1 \leq k \leq k_n$. Thus

$$\sum_{k=1}^{\infty} \alpha(\beta_k^2(x_n)) \geq \sum_{k=1}^{k_n} |\epsilon_n|^p > 1, \quad \sum_{k=1}^{\infty} |\beta_k(x_n)|^p = \frac{k_n |\epsilon_n|^p}{n} < \frac{2}{n},$$

so (6.13) is violated. If $\lim_{t \downarrow 0} \alpha(t^2)/t^p = 0$ there exists $0 < \epsilon_n < 1$ such that $\alpha(\epsilon_n^2) < \epsilon_n^p/n$ for $n = 1, 2, \dots$. Let k_n be an integer such that $1 < k_n(\epsilon_n)^p < 2$ and define the point x_n in E such that $\beta_k(x_n) = 0$ if $k > k_n$ and $\beta_k(x_n) = \epsilon_n$ for $1 \leq k \leq k_n$. Then $1 < \sum_{k=1}^{\infty} |\beta_k(x_n)|^p = k_n \epsilon_n^p < 2$ but $\sum_{k=1}^{\infty} \alpha(\beta_k^2(x_n)) \leq k_n \epsilon_n^p/n < 2/n$, so (6.13) is again violated. Combining the above we see that the theorem is proved.

Now Theorem 6.3 points out that E is an Orlicz space provided E has a quasi-norm which is accessible in both directions with respect to the functions α , ρ_1 , ρ_2 and $\alpha(\cdot)$ satisfies condition (6.1). Furthermore, Corollary 6.2 gives necessary and sufficient conditions for E to be l_p for some p , $2 \leq p < \infty$. For an example of a space (E_α, τ) constructed as in Theorem 6.1 which is not an l_p space, consider E_α where $\alpha(t) = 0$ for $t = 0$, $\alpha(t) = t^2[-\log t]$ for $0 < t \leq 1/e^2$, and $\alpha(t) = 2t^2$ for $t > 1/e^2$. In fact, by the previous corollary, $E_\alpha \neq l_p$ for any p , $2 \leq p < \infty$, since $\lim_{t \downarrow 0} \alpha(t^2)/t^4 = +\infty$ and $\lim_{t \downarrow 0} \alpha(t^2)/t^{4-\delta} = 0$ for any

$\delta > 0$. Now further calculation also shows that E_α , in this case, also has Property B. Hence we have an example of a space which satisfies our most rigid requirements and is something other than an l_p space, $2 \leq p < \infty$.

Corollary 6.3. *If E has Property B and the function $\alpha(\cdot)$ satisfies (6.1), then $\alpha(\cdot)$ also satisfies (6.2) and E is homeomorphic and isomorphic to the Orlicz space E_α .*

The proof of Corollary 6.3 follows immediately by combining the results of Corollary 6.1 and Theorem 6.3.

7. Infinitely divisible measures on spaces with Property B. The study of infinitely divisible measures on a real separable Hilbert space is due to S.R.S. Varadhan [11], and here we consider similar questions for F -spaces with Property B where the function $\alpha(\cdot)$ satisfies (6.1) and (6.2). In view of Theorem 6.3 and Corollary 6.3 it follows that we are actually studying infinitely divisible measures on the Orlicz space E_α . Furthermore, as pointed out in the proof of Corollary 5.2, the l_p spaces, $2 \leq p < \infty$, each have Property B, so our results will extend those of Varadhan. It should be mentioned, however, that Varadhan's work is used at several points throughout this section.

Throughout the section E_α will denote the Hilbert space l_2 or an E_α space where $\alpha(\cdot)$ satisfies (6.1), (6.2), and if $\alpha_c(\cdot)$ is the complementary function of $\alpha(\cdot)$ in the sense of Young [10, p. 77], then $\alpha_c(\cdot)$ should also satisfy (6.1). Notice that if $E_\alpha = l_2$ then a natural choice for the function α is $\alpha(t) = t$. Hence $\alpha_c(s) = 0$ on $[0, 1]$ but $\alpha_c(s) = \infty$ for $s > 1$. Thus $\alpha_c(\cdot)$ does not satisfy (6.1) when $E_\alpha = l_2$ and this is a special case outside our general setup. Nevertheless, it is easily handled.

In terms of the results of §6 we have that E_α is equivalent to the Orlicz space S_Γ where $\Gamma(t) = \alpha(t^2)$. We will let S_α, S_{α_c} denote the Orlicz spaces given by $\alpha(\cdot)$ and $\alpha_c(\cdot)$, respectively. Then the dual space of S_α can be identified as S_{α_c} and since $\alpha_c(\cdot)$ also satisfies (6.1), except when $E_\alpha = l_2$, it follows that the dual of S_{α_c} is S_α [10, p. 150].

To simplify our notation we will use the sequence space representation for all of the spaces of the remainder of the paper.

The following terminology is used. If $F(\cdot)$ is a finite Borel measure on E_α then $e(F)$ is the probability measure defined by $\exp\{-F(E_\alpha)\} \cdot \sum_{k=0}^{\infty} F^k/k!$ where F^k is F convolved k times and $F^0 = \delta_0$ where δ_x is the unit mass at x . A family $\mathcal{F}$ of probability measures on E_α is *shift-compact* if for every sequence $\{\nu_n\}$ in $\mathcal{F}$ there exists a sequence $\{x_n\}$ in E_α such that $\{\nu_n * \delta_{x_n}\}$ is compact.

For each vector $\lambda = (\lambda_1, \lambda_2, \dots)$ in the positive cone of S_{α_c} which is of norm at most one-half, we define the Hilbert space H_λ as all sequences $x =$

$(x_1, x_2, \dots)$ such that $\|x\|_\lambda^2 = \sum_{i=1}^\infty \lambda_i x_i^2 < \infty$. In the special case $E_\alpha = l_2$ we have $S_{\alpha_c} = l_\infty$ and for simplicity we take $\lambda = (1, 1, \dots)$. Then $H_\lambda = l_2$ and we shall assume without loss of generality that $\alpha(t) \equiv t$.

Lemma 7.1. E_α is a Borel subset of H_λ for each λ in the positive cone of S_{α_c} . Furthermore, every Borel subset of E_α is a Borel subset of H_λ .

Proof. If $E_\alpha = l_2$ and $\lambda = (1, 1, \dots)$ the result is obvious. In general to see $E_\alpha \subseteq H_\lambda$ we need only notice that by Young's inequality [10, p. 77] $\lambda_i x_i^2 \leq \alpha_c(\lambda_i) + \alpha(x_i^2)$. Hence for $x = (x_1, x_2, \dots)$ in E_α and λ as indicated, we have $x \in H_\lambda$. To show that the Borel subsets of E_α also are Borel subsets of H_λ , we need only show that the norm $\|\cdot\|_\Gamma$ is a Borel function on H_λ . Now

$$\|x\|_\Gamma = \sup_y \left\{ \sum_{i=1}^\infty |x_i y_i| : \sum_{i=1}^\infty \Lambda(|y_i|) \leq 1 \right\}$$

where Λ is the function complementary to Γ in the sense of Young. Since $\Gamma(t) = \alpha(t^2)$ it follows that $\Gamma'(t) = 2t\alpha'(t^2)$ almost everywhere on $[0, \infty)$ and hence $\Gamma'(0)$ can be defined as zero. Further, since $\alpha'(t^2)$ is bounded on $[0, \epsilon]$ for any $\epsilon > 0$ it follows that $\Gamma'(t)$ is continuous at $t = 0$. Now $\alpha(\cdot)$ is assumed strictly increasing and convex on $[0, \infty)$, so we see $\alpha'(t^2)$ is positive almost everywhere there, and furthermore, $\Gamma'(t) = 2t\alpha'(t^2)$ is strictly increasing on $[0, \infty)$. Let $\psi(s)$ denote a left continuous inverse of $\Gamma'(t) = 2t\alpha'(t^2)$. Then $\psi(0) = 0$ and since $\Gamma'(t)$ is continuous at zero and strictly positive for all $t > 0$ it follows that $\psi(s)$ is strictly positive for $s > 0$. Thus $\Lambda(t) = \int_0^t \psi(s) ds > 0$ for any $t > 0$, and hence if $y = (y_1, y_2, \dots)$ is such that $\sum_{i=1}^\infty \Lambda(|y_i|) < \infty$ it follows that $\lim y_i = 0$. If W denotes all sequences with rational coordinates and with only finitely many nonzero coordinates satisfying the condition $\sum_{i=1}^\infty \Lambda(|y_i|) \leq 1$, it follows that W is countable and $\|x\|_\Gamma = \sup_W \{ \sum_{i=1}^\infty |x_i y_i| \}$. Hence $\|x\|_\Gamma$ is measurable on H_λ since for each $y \in W$ the function $\sum_{i=1}^\infty |x_i y_i|$ is measurable on H_λ .

We will use the symbol $\|\cdot\|_\Gamma$, $\|\cdot\|_{\Gamma^c}$, $\|\cdot\|_\lambda$, $\|\cdot\|_\alpha$, $\|\cdot\|_{\alpha_c}$ for the norm on E_α , E_α^* , H_λ , S_α , and S_{α_c} , respectively.

Lemma 7.2. Let F_n be a sequence of finite Borel measures on E_α such that $e(F_n)$ is shift-compact. Then

$$(7.1) \quad \sup_n \int_{\mathcal{U}} \|x\|_\lambda^2 dF_n(x) < \infty$$

where $\mathcal{U} = \{x : \sum_{i=1}^\infty \alpha(x_i^2) \leq 1\}$.

Proof. If $E_\alpha = l_2$, $\alpha(t) = t$, $\lambda = (1, 1, \dots)$ this follows directly from [11, p. 223]. Otherwise, if $x = \{x_i\} \in \mathcal{U}$ then by [10, p. 82]

$$(7.2) \quad \|x\|_{\lambda}^2 = \sum_{i=1}^{\infty} \lambda_i x_i^2 \leq \|\lambda\|_{\alpha_c} \|x_i^2\|_{\alpha}$$

$$\leq \|\lambda\|_{\alpha_c} \left\{ 1 + \sum_{i=1}^{\infty} \alpha(x_i^2) \right\} \leq 1$$

since $\sum_{i=1}^{\infty} \alpha(x_i^2) \leq 1$ and $\|\lambda\|_{\alpha_c} \leq 1/2$. Thus $\mathcal{U}$ is a measurable subset of the unit ball of H_{λ} where the measurability of $\mathcal{U}$ follows since $\sum_{i=1}^{\infty} \alpha(x_i^2)$ is a measurable function in H_{λ} . Now using the arguments employed in the proof of Theorems 6.1 and 6.2 we can show that a sequence of points $\{p_N\}$ in $E_{\alpha} = S_{\Gamma}$ converges to a point p in the norm $\|\cdot\|_{\Gamma}$ iff

$$(7.3) \quad \lim_N \sum_{i=1}^{\infty} \Gamma(|x_{N,i} - x_i|) = \lim_N \sum_{i=1}^{\infty} \alpha(|x_{N,i} - x_i|^2) = 0$$

where $p_N = \{x_{N,i}\}$, $p = \{x_i\}$. Further, the right-hand limit in (7.3) equals zero iff the sequence $\{q_N\}$ converges to zero in S_{α} where q_N has i th coordinate $(x_{N,i} - x_i)^2$. Hence by (7.2), $\lim \|p_N - p\|_{\Gamma} = 0$ implies $\lim_N \|p_N - p\|_{\lambda} = 0$, so f continuous on H_{λ} implies f is continuous with respect to the norm $\|\cdot\|_{\Gamma}$ when restricted to E_{α} . Thus $e(F_n)$ shift-compact on E_{α} implies it is shift-compact on H_{λ} and since the unit ball of H_{λ} contains $\mathcal{U}$ it follows from [11, p. 223] that (7.1) holds.

Lemma 7.3. Suppose F_n is an increasing sequence of finite measures on E_{α} such that $e(F_n)$ is shift-compact. Then F_n increases to a measure F which is finite on the complement of every neighborhood of zero and such that

$$(7.4) \quad J = \sum_{i=1}^{\infty} \alpha \left[\int_{\mathcal{U}} x_i^2 dF(x) \right] < \infty,$$

and

$$(7.5) \quad J = \lim_n \sum_{i=1}^{\infty} \alpha \left[\int_{\mathcal{U}} x_i^2 dF_n(x) \right]$$

where $\mathcal{U} = \{x: \sum_{i=1}^{\infty} \alpha(x_i^2) \leq 1\}$.

Proof. Since F_n is increasing we let $F = \lim_n F_n$. Now $e(F_n)$ shift-compact implies by [11, p. 214] and Remark 2.3 of [11] that F is finite on the complement of every neighborhood of zero in E_{α} and by Lemma 7.2 that

$$\sum_{i=1}^{\infty} \lambda_i \int_{\mathcal{U}} x_i^2 dF_n(x) < \infty.$$

If $E_{\alpha} = l_2$, $\lambda = (1, 1, \dots)$, $\alpha(t) = t$ the lemma now follows. In general, since $\lambda = \{\lambda_i\}$ is an arbitrary vector in the positive cone of S_{α_c} of norm less than or equal

to one-hair and S_α is the dual of S_{α_C} we find that the sequence $\{\int_{\mathcal{U}} x_i^2 dF(x)\}$ is in S_α . This implies (7.4), and (7.5) then follows from the dominated convergence theorem since F_n increases to F , $\alpha(\cdot)$ is continuous, and

$$\lim_n \int_{\mathcal{U}} x_i^2 dF_n(x) = \int_{\mathcal{U}} x_i^2 dF(x)$$

for $i = 1, 2, \dots$.

Lemma 7.4. *Let $\mu_n = e(F_n)$ where F_n is an increasing sequence of measures on E_α and suppose the sequence $\mu_n * \delta_{x_n}$ converges weakly to μ for some sequence of points $\{x_n\}$ in E_α . Then F_n increases to a measure F as in Lemma 7.3 such that (7.4) and (7.5) hold. Furthermore, the Fourier transform of μ is*

$$(7.6) \quad \phi(y) = \exp \left\{ i(x_0, y) + \int_{E_\alpha - \mathcal{U}} \left[e^{i(x, y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\Gamma^2} \right] dF(x) \right. \\ \left. + \int_{\mathcal{U}} \left[e^{i(x, y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\lambda^2} \right] dF(x) \right\}$$

where $\mathcal{U} = \{x \in E_\alpha : \sum_{i=1}^\infty \alpha(x_i^2) \leq 1\}$.

Proof. Let ρ_n be the shift of μ_n by

$$(7.7) \quad z_n = - \int_{E_\alpha - \mathcal{U}} \frac{x}{1 + \|x\|_\Gamma^2} dF_n(x) - \int_{\mathcal{U}} \frac{x}{1 + \|x\|_\lambda^2} dF_n(x)$$

where the integrals are Bochner integrals. Now the function in the first integral in (7.6) is uniformly bounded for any y in a bounded sphere in the dual of E_α , and by (6.5) $\mathcal{U}$ contains an open neighborhood of zero. Thus $\{F_n\}$ converges weakly to F on $E_\alpha - \mathcal{U}$ [11, p. 214], and if S is a bounded sphere in the dual of E_α then

$$(7.8) \quad \lim_n \sup_{y \in S} \left| \int_{E_\alpha - \mathcal{U}} \left[e^{i(x, y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\Gamma^2} \right] d(F - F_n)(x) \right| = 0.$$

We now consider the second integral in (7.6). Now $|e^{i(x, y)} - 1 - i(x, y)/(1 + \|x\|_\lambda^2)| \leq \frac{1}{2}(x, y)^2 + |(x, y)|\|x\|_\lambda^2$ and $|(x, y)| \leq \|x\|_\Gamma \|y\|_{\Gamma^C} \leq 2\|y\|_{\Gamma^C}$ since $x \in \mathcal{U}$ which implies $\|x\|_\Gamma \leq 2$. Let $K_n = F - F_n$. Then, for S any bounded sphere in the dual of E_α , we have

$$(7.9) \quad \lim_n \sup_{y \in S} \int_{\mathcal{U}} |(x, y)| \|x\|_\lambda^2 dK_n(x) = 0$$

since F_n increases to F and (7.1) holds in the present setting. Thus to show the second integral in (7.6) is the limit of the corresponding integrals when F_n replaces F uniformly for y in a bounded sphere S of the dual of E_α we need

only verify that

$$(7.10) \quad \lim_n \sup_{y \in S} \int_{\mathfrak{U}} (x, y)^2 dK_n(x) = 0.$$

Let $s_{ij}^{(n)} = \int_{\mathfrak{U}} x_i x_j dK_n(x)$ for $i, j, n = 1, 2, \dots$. Then $|s_{ij}^{(n)}| \leq \int_{\mathfrak{U}} |x_i x_j| dK_n(x) \leq [s_{ii}^{(n)}]^{1/2} [s_{jj}^{(n)}]^{1/2}$ and since $s_{jj}^{(n)} \leq \int_{\mathfrak{U}} x_j^2 dF(x)$ it follows from (7.4) that

$$(7.11) \quad \sum_{j=k}^{\infty} \alpha(s_{jj}^{(n)}) \leq \sum_{j=k}^{\infty} \alpha\left(\int_{\mathfrak{U}} x_j^2 dF(x)\right) < \infty$$

for $k = 1, 2, \dots$. Now for each y in the dual of E_{α} we have $(x, y)^2 = \sum_{i,j=1}^{\infty} x_i x_j y_i y_j$ and since

$$\begin{aligned} \int_{\mathfrak{U}} \sum_{i,j}^{\infty} |x_i x_j y_i y_j| dK_n(x) &= \sum_{i,j} \int_{\mathfrak{U}} |x_i x_j y_i y_j| dK_n(x) \\ &\leq \left\{ \sum_{i=1}^{\infty} [s_{ii}^{(n)}]^{1/2} |y_i| \right\}^2 < \infty \end{aligned}$$

it follows that

$$(7.12) \quad \int_{\mathfrak{U}} (x, y)^2 dK_n(x) = \sum_{i,j=1}^{\infty} s_{ij}^{(n)} y_i y_j,$$

$$(7.13) \quad \int_{\mathfrak{U}} (x, y)^2 dK_n(x) \leq \left(\sum_{j=1}^{\infty} [s_{jj}^{(n)}]^{1/2} |y_j| \right)^2 \leq \|[s_{jj}^{(n)}]^{1/2}\|_{\Gamma}^2 \|y\|_{\Gamma^c}^2.$$

Thus (7.10) holds if

$$(7.14) \quad \lim_n \|[s_{jj}^{(n)}]^{1/2}\|_{\Gamma} = 0.$$

Using (7.3), we see (7.14) holds iff

$$(7.15) \quad \lim_n \sum_{j=1}^{\infty} \alpha(s_{jj}^{(n)}) = 0,$$

and by (7.11) we have (7.15) if

$$(7.16) \quad \lim_n s_{jj}^{(n)} = 0.$$

Now (7.16) holds since $\lim_n \int_{\mathfrak{U}} x_j^2 dF_n(x) = \int_{\mathfrak{U}} x_j^2 dF(x)$, for $j = 1, 2, \dots$.

If $F_{n,1}$ and $F_{n,2}$ denote F_n restricted to $\mathfrak{U}$ and $E_{\alpha} - \mathfrak{U}$, respectively, we have that

$$(7.17) \quad e(F_n) * \delta_{x_n} = e(F_{n,1}) * e(F_{n,2}) * \delta_{z_n} * \delta_{x_n - z_n}$$

where z_n is defined as in (7.7). Now the sequence $\{e(F_n) * \delta_{x_n}\}$ converges weakly to μ thus (7.17) implies

$$(7.18) \quad \nu_n = e(F_{n,1}) * e(F_{n,2}) * \delta_{z_n}$$

is shift-compact. Using (7.8), (7.9), and (7.10) it follows that the Fourier transforms of the sequence of measures $\{\nu_n\}$ converge uniformly on bounded spheres of the dual of E_α to the right-hand side of (7.6) without the term $\exp\{i(x_0, y)\}$. Thus $\{\nu_n\}$ actually converges to a measure ν , and hence by (7.17) and the fact that $e(F_n) * \delta_n$ converges to μ the sequence of points $\{x_n - z_n\}$ converges to a point $x_0 \in E_\alpha$. Furthermore, we have the Fourier transform of μ is $\exp\{i(x_0, y)\}$ multiplied by the Fourier transform of ν so the lemma is proved.

In §5 we said a measure on E_α was Gaussian with mean $a \in E_\alpha$ if the distribution of every linear functional x in the dual of E_α is Gaussian with mean $x(a)$. Varadhan [11, p.215] defines a Gaussian measure as any measure μ which is infinitely divisible and such that if $\mu = e(F) * \gamma$ where γ is infinitely divisible then F is degenerate at zero. The definition of infinite divisibility is the usual one. The purpose of our next result is to show that these definitions coincide on E_α .

Theorem 7.1. *If μ is a measure on E_α the following are equivalent:*

(i) *μ has Fourier transform of the form*

$$\phi(x) = \exp\{i\langle x, a \rangle - \frac{1}{2}T(x, x)\}$$

where T is an α -operator on the dual of E_α satisfying

$$T(\beta_i, \beta_j) = \int_{E_\alpha} \beta_i(y - a)\beta_j(y - a) d\mu(y)$$

for $i, j = 1, 2, \dots$ and $a \in E_\alpha$.

(ii) *For each vector f in the dual of E_α the linear functional $f(\cdot)$ on E_α has a Gaussian distribution with mean $f(a)$ for some a in E_α .*

(iii) *μ is infinitely divisible and if $\mu = e(F) * \gamma$ where γ is infinitely divisible then F is degenerate at zero.*

Proof. By Theorem 5.1, Corollary 5.1, Theorem 6.3 and Corollary 6.1 it follows that (i) and (ii) are equivalent. Now assume (ii) holds and let f be any linear functional on E_α . Now μ is infinitely divisible since (ii) implies (i) and the class of α -operators is closed under multiplication by constants. Thus if $\mu = e(F) * \gamma$ where γ is infinitely divisible we must show F is degenerate at zero. Now an easy computation shows that the measure $\mu^f(A) = \mu(f^{-1}(A))$ defined on the real line satisfies

$$\mu^f = e(F^f) * \gamma^f$$

where F^f and γ^f are defined in the same manner as μ^f . Similarly, γ being

infinitely divisible implies that γ^f is infinitely divisible, and since μ^f is Gaussian on the real line we then have F^f degenerate at zero. Thus for each continuous linear functional f on E_α the measure F^f is degenerate at zero and this implies F itself is concentrated at the origin in E_α which proves that (ii) implies (iii).

Now assume (iii) holds and let H_λ be defined as in Lemma 7.1. Then μ is infinitely divisible on H_λ and if $\mu = e(F) * \gamma$ where γ is infinitely divisible on H_λ and F is some finite measure on H_λ it follows that F is degenerate at zero. That is, if μ is a probability measure on H_λ with $\mu(E_\alpha) = 1$ and $\mu = e(F) * \gamma$ where F is some finite measure on H_λ and γ is a probability measure on H_λ then F and γ are concentrated on E_α . Hence (iii) holds when μ is viewed as a measure on H_λ and by [11, p. 226] every linear functional f on H_λ has a Gaussian distribution. Now f restricted to E_α is continuous and since E_α is dense in H_λ the restriction maps H_λ into the dual of E_α in one-to-one fashion. By Lemma 2.1 it then follows that each f in the dual of E_α is the weak-star limit of restrictions to E_α of elements in the dual of H_λ , and since limits of Gaussian distributions are Gaussian (possibly degenerate) it follows that each linear functional on E_α has a Gaussian distribution. In view of Corollary 5.1 it follows that (iii) implies (ii).

Theorem 7.2. *Let μ be a measure on E_α . Then μ is infinitely divisible on E_α iff the Fourier transform of μ is of the form*

$$\begin{aligned} \phi(y, \mu) = \exp \left\{ i(x_0, y) - \frac{1}{2} T(y, y) + \int_{E_\alpha - \mathcal{U}} \left[e^{i(x, y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\Gamma^2} \right] dF(x) \right. \\ \left. + \int_{\mathcal{U}} \left[e^{i(x, y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\lambda^2} \right] dF(x) \right\} \end{aligned} \quad (7.19)$$

where $x_0 \in E_\alpha$, T is an α -operator, $\mathcal{U} = \{x \in E_\alpha: \sum_{i=1}^\infty \alpha(x_i^2) \leq 1\}$, and F is a σ -finite measure on E_α with finite mass outside every neighborhood of zero and no mass at zero satisfying

$$\sum_{i=1}^\infty \alpha \left[\int_{\mathcal{U}} x_i^2 dF(x) \right] < \infty. \quad (7.20)$$

For each fixed λ the representation is unique.

Proof. First we assume μ has Fourier transform (7.19) and F satisfies (7.20). We now observe that $\mu = g * \nu$ where g and ν are measures on E_α with Fourier transforms

$$(7.21) \quad \hat{g}(y) = \exp \left\{ i(x_0, y) - \frac{1}{2} T(y, y) + \int_{E_\alpha - \mathfrak{U}} \left[e^{i(x, y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\Gamma^2} \right] dF(x) \right\},$$

$$(7.22) \quad \hat{\nu}(y) = \exp \left\{ \int_{\mathfrak{U}} \left[e^{i(x, y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\lambda^2} \right] dF(x) \right\}.$$

That the measure g exists follows immediately from Theorem 5.2 and that F restricted to $E_\alpha - \mathfrak{U}$ is finite. Furthermore, it is obvious that g is infinitely divisible. The existence of ν proceeds as follows. From (7.20) it follows that $\int_{\mathfrak{U}} \|x\|_\lambda^2 dF(x) < \infty$, and hence if F restricted to $\mathfrak{U}$ is viewed as a measure on H_λ we have the existence of the measure ν on H_λ [11, p. 225] with Fourier transform as in (7.22). Furthermore, ν is infinitely divisible on H_λ and since $\mu = g * \nu$ where μ and g assign mass one to E_α it follows that $\nu(E_\alpha) = 1$ and that ν is infinitely divisible on E_α . Thus both g and ν exist on E_α and they are infinitely divisible so μ is infinitely divisible.

Now we assume μ is infinitely divisible on E_α . To prove μ has a Fourier transform of the form (7.19) where F satisfies (7.20) we can proceed exactly as in [11]. That is, we choose a sequence of probability measures $e(F_n)$ where F_n increases to, say, F , $e(F_n) * \delta_{x_n}$ converges to a measure λ , and $\mu = \lambda * g$ where g is Gaussian. Then by Lemma 7.4 and Theorem 7.1 the Fourier transform of μ is of the form (7.19) and, by Lemma 7.3, F satisfies (7.20) along with the other conditions of the theorem. The uniqueness follows exactly as indicated in [11].

The following corollary relates to the results found in [11].

Corollary 7.1. *A function $\phi(y)$ is the Fourier transform of an infinitely divisible probability measure μ on l_2 iff*

$$(7.23) \quad \phi(y) = \exp \left\{ i(x_0, y) - \frac{1}{2} (Ty, y) + \int_{l_2} K(x, y) dF(x) \right\}$$

where $x_0 \in l_2$, $K(x, y) = e^{i(x, y)} - 1 - i(x, y)/(1 + \|x\|^2)$, T is an S -operator on l_2 , and F is a σ -finite measure on l_2 with finite mass outside every neighborhood of zero and satisfying $\int_B \|x\|^2 dF(x) < \infty$ where $B = \{x \in l_2: \|x\| \leq 1\}$.

Proof. In view of Theorems 7.1, 7.2, Corollary 5.3, and Corollary 3.1 it suffices to prove that $\gamma(y) = \exp \{ \int_{l_2} K(x, y) dF(x) \}$ is continuous in the τ_2 -topology since it follows easily that $\gamma(0) = 1$, γ is positive definite, and that any family of measures on l_2 is a λ -family with $\lambda = (1, 1, \dots)$. Further, since F is finite on $l_2 - B$ it follows that $\exp \{ \int_{l_2 - B} K(x, y) dF(x) \}$ is the Fourier transform of a probability measure on l_2 and hence it is τ_2 -continuous. Thus we need only show

$$(7.24) \quad \exp \left\{ \int_B K(x, y) dF(x) \right\}$$

is τ_2 -continuous. Now proceeding as in Lemma 7.4,

$$(7.25) \quad \left| \int_B K(x, y) dF(x) \right| \leq \frac{1}{2} \int_B (x, y)^2 dF(x) + \left[\int_B (x, y)^2 dF(x) \right]^{1/2} \cdot \left\{ \int_B \|x\|^4 dF(x) \right\}^{1/2}$$

and, since $\int_B (x, y)^2 dF(x)$ is an S -operator on l_2 , (7.25) implies (7.24) is τ_2 -continuous. This proves $\gamma(y)$ is τ_2 -continuous and completes the proof.

8. Infinitely divisible distributions as limiting distributions. We now consider the derivation of the limiting distribution of the sum of independent random variables taking values in the Orlicz space E_α as defined in §7. For each integer n , we are given k_n random variables which are mutually independent. We shall denote these by X_{nj} where $j = 1, \dots, k_n$ and letting $Y_n = \sum_{j=1}^{k_n} X_{nj}$ our purpose is to derive the limiting distribution of Y_n and Y_n suitably shifted. We need the following definition.

Let $\{X_{nj}\}$ be a sequence of random variables taking values in E_α and let the sequence of probability measures induced on E_α be denoted by $\{F_{nj}\}$. Then $\{X_{nj}\}$, or equivalently $\{F_{nj}\}$, is said to be *uniformly infinitesimal* if for every $\epsilon > 0$

$$\lim_n \sup_{1 \leq j \leq k_n} F_{nj}(\{x \in E_\alpha: \|x\|_\Gamma > \epsilon\}) = 0.$$

The following theorem on Poisson sums of random variables in a separable Banach space is due to Lucien LeCam [13].

Theorem 8.1. Let X_{nji} ($j = 1, \dots, k_n$; $n, i = 1, 2, \dots$) be independent random variables with values in a separable Banach space with the distribution of X_{nji} being F_{nj} for $i = 1, 2, \dots$. Let $\{N_{nj}\}$ be Poisson random variables independent among themselves as well as independent of all the X_{nji} and each with parameter one. Let G_n denote the distribution of

$$T_n = \sum_{j=1}^{k_n} \sum_{i=1}^{N_{nj}} X_{nji}.$$

If G_n is conditionally compact then $\prod_{j=1}^{k_n} F_{nj}$ is shift-compact where $\prod_{j=1}^{k_n} F_{nj}$ denotes the convolution of the indicated measures.

We now fix some notation to be used throughout the remainder of the paper. We let $B = \{x \in E_\alpha: \|x\|_\Gamma \leq \rho\}$ where $0 < \rho \leq 1$ and $B \subseteq \mathcal{U} = \{x: \sum_{i=1}^\infty \alpha(x_i^2) \leq 1\}$. The existence of such a ρ follows from the arguments of Theorems 6.1 and 6.3 and the definition of the quasi-norm of Theorem 6.1. Additional notation we will use is

$$(8.1) \quad \mu_n = \prod_{j=1}^{k_n} F_{n_j}, \quad x_{n_j} = \int_B x dF_{n_j}(x), \quad \beta_{n_j} = F_{n_j} * (\delta_{-x_{n_j}}),$$

$$x_n = \sum_{j=1}^{k_n} x_{n_j}, \quad F_n = \sum_{j=1}^{k_n} \beta_{n_j}, \quad \lambda_n = e(F_n) * \delta_{x_n}.$$

Lemma 8.1. *Let F_{n_j} ($j = 1, \dots, k_n$; $n = 1, 2, \dots$) be a uniformly infinitesimal sequence of probability measures on E_α and assume the notation in (8.1).*

Then

- (i) $\lim_n \max_{1 \leq j \leq k_n} \|x_{n_j}\|_\Gamma = 0$,
- (ii) *there exists an N such that for all $1 \leq j \leq k_n$ and $n \geq N$*

$$\left\| \int_B x d\beta_{n_j}(x) \right\|_\Gamma \leq 2\beta_{n_j}\{x: \|x\|_\Gamma > \rho/2\}.$$

Proof. Take $\epsilon > 0$ and let $V = \{x: \|x\|_\Gamma < \epsilon\}$. Then

$$\|x_{n_j}\|_\Gamma \leq \int_V \|x\|_\Gamma dF_{n_j}(x) + F_{n_j}\{x: \epsilon < \|x\|_\Gamma \leq \rho\} \leq \epsilon + F_{n_j}(E_\alpha - V).$$

and $\{F_{n_j}\}$ being uniformly infinitesimal implies

$$\lim_n \max_{1 \leq j \leq k_n} \|x_{n_j}\|_\Gamma < \epsilon$$

so (i) holds. To prove (ii) we first take N such that $\max_{1 \leq j \leq k_n} \|x_{n_j}\|_\Gamma < \rho/4$ for all $n \geq N$. Then for $n \geq N$ and $1 \leq j \leq k_n$,

$$\begin{aligned} \left\| \int_B x d\beta_{n_j}(x) \right\|_\Gamma &= \left\| \int_{\|x-x_{n_j}\|_\Gamma \leq \rho} (x - x_{n_j}) dF_{n_j}(x) \right\|_\Gamma \\ &= \left\| \int_{\|x-x_{n_j}\|_\Gamma \leq \rho} x dF_{n_j}(x) - x_{n_j} F_{n_j}\{x: \|x - x_{n_j}\|_\Gamma \leq \rho\} \right\|_\Gamma \\ &= \left\| \int_{\|x-x_{n_j}\|_\Gamma \leq \rho} x dF_{n_j}(x) - \int_B x dF_{n_j}(x) + x_{n_j} F_{n_j}\{x: \|x - x_{n_j}\|_\Gamma > \rho\} \right\|_\Gamma \\ &\leq \int_{3\rho/4 < \|x\|_\Gamma < 5\rho/4} \|x\|_\Gamma dF_{n_j}(x) + (\rho/4) \beta_{n_j}\{x: \|x\|_\Gamma > \rho\} \\ &\leq (\rho/4) [5F_{n_j}\{x: \|x\|_\Gamma > 3\rho/4\} + \beta_{n_j}\{x: \|x\|_\Gamma > \rho\}] \\ &\leq 2\beta_{n_j}\{x: \|x\|_\Gamma > \rho/2\} \end{aligned}$$

so (ii) holds since $\rho \leq 1$.

Theorem 8.2. Let $\{F_{nj}\}$ ($j = 1, \dots, k$; $n = 1, 2, \dots$) be a uniformly infinitesimal sequence of probability measures on E_α and assume the notation in (8.1). Then, if $\{\lambda_n\}$ is conditionally compact we have $\{\mu_n\}$ shift-compact, and for each uniformly bounded set S in the dual of E_α

$$(8.2) \quad \lim_{n \rightarrow \infty} \sup_{y \in S} |\phi(y, \lambda_n) - \phi(y, \mu_n)| = 0.$$

Proof. If $\{\lambda_n\}$ is conditionally compact then Theorem 8.1 implies $\prod_{j=1}^{k_n} \beta_{nj} = \prod_{j=1}^{k_n} F_{nj} * \delta_{x_n}$ is shift-compact and hence $\mu_n = \prod_{j=1}^{k_n} F_{nj}$ is shift-compact. To complete our proof we now need to establish (8.2). First we observe that since $\{F_{nj}\}$ is uniformly infinitesimal there exists an N such that $n \geq N$ implies

$$(8.3) \quad \sup_{y \in S} \max_{1 \leq j \leq k_n} |\phi(y, \beta_{nj}) - 1| < 1/2.$$

Then, since $|\log(1+z) - z| \leq 2|z|^2$ for $|z| \leq 1/2$ we have

$$\begin{aligned} & |\log \phi(y, \lambda_n) - \log \phi(y, \mu_n)| \\ &= \left| \sum_{j=1}^{k_n} [\phi(y, \beta_{nj}) - 1] + i(y, x_n) - \sum_{j=1}^{k_n} \log \phi(y, \beta_{nj}) - i(y, x_n) \right| \\ (8.4) \quad &= \left| \sum_{j=1}^{k_n} [\phi(y, \beta_{nj}) - 1 - \log \phi(y, \beta_{nj})] \right| \\ &\leq 2 \sum_{j=1}^{k_n} |\phi(y, \beta_{nj}) - 1|^2 \\ &\leq 2 \max_{1 \leq j \leq k_n} |\phi(y, \beta_{nj}) - 1| \sum_{j=1}^{k_n} |\phi(y, \beta_{nj}) - 1|. \end{aligned}$$

Thus, by (8.4),

$$(8.5) \quad \lim_{n \rightarrow \infty} \sup_{y \in S} |\log \phi(y, \lambda_n) - \log \phi(y, \mu_n)| = 0$$

since $\{F_{nj}\}$ uniformly infinitesimal implies

$$\lim_{n \rightarrow \infty} \sup_{y \in S} \max_{1 \leq j \leq k_n} |\phi(y, \beta_{nj}) - 1| = 0$$

and we also have $\overline{\lim}_n \sup_{y \in S} \sum_{j=1}^{k_n} |\phi(y, \beta_{nj}) - 1| < \infty$. To see this last assertion notice that

$$\begin{aligned}
\sum_{j=1}^{k_n} |\phi(y, \beta_{nj}) - 1| &= \sum_{j=1}^{k_n} \left| \int_{E_\alpha} [e^{i(x, y)} - 1] d\beta_{nj}(x) \right| \\
&= \sum_{j=1}^{k_n} \left| \int_B [e^{i(x, y)} - 1 - i(x, y)] d\beta_{nj}(x) + i \int_B (x, y) d\beta_{nj}(x) \right. \\
&\quad \left. + \int_{E_\alpha - B} [e^{i(x, y)} - 1] d\beta_{nj}(x) \right| \\
&\leq \sum_{j=1}^{k_n} \left[\int_B (x, y)^2 d\beta_{nj}(x) + \|y\|_{E_\alpha^*} \left\| \int_B x d\beta_{nj}(x) \right\|_\Gamma + 2\beta_{nj}(E_\alpha - B) \right] \\
&\leq \int_B (x, y)^2 dF_n(x) + 2[\|y\|_{E_\alpha^*} + 1] F_n\{x: \|x\|_\Gamma > \rho/2\}.
\end{aligned}
\tag{8.6}$$

Since $e(F_n)$ is conditionally compact this implies $e(\sum_1^{k_n} |\beta_{nj}|^2)$ is conditionally compact where $|\beta_{nj}|^2 = \beta_{nj} * \bar{\beta}_{nj}$ and $\bar{\beta}_{nj}(A) = \beta_{nj}(-A)$ for all Borel sets A . Therefore we obtain from the symmetry of $|\beta_{nj}|^2$ that

$$\overline{\lim}_n \sup_{y \in S} \sum_1^{k_n} \int_{\|x\|_\Gamma \leq 2\rho} (x, y)^2 |\beta_{nj}|^2(dx) < \infty.$$

Hence from conditional compactness of $e(F_n)$ we obtain by an argument similar to [11, p. 235] and Lemma 8.1(ii) that

$$\overline{\lim}_n \sup_{y \in S} \int_B (x, y)^2 F_n(dx) < \infty;$$

and further, by [11, p. 214] that $\sup_n F_n\{x: \|x\|_\Gamma > \rho/2\} < \infty$, so that (8.6) implies $\overline{\lim}_n \sup_{y \in S} \sum_1^{k_n} |\phi(y, \beta_{nj}) - 1|$ is finite. Thus (8.5) holds and this implies (8.2), so the theorem is proved.

Corollary 8.1. *Let $\{F_{nj}\}$ be uniformly infinitesimal. Then, if $\{\mu_n\}$ converges weakly to μ and $\{x_n\}$ is a compact set in E_α we have μ infinitely divisible.*

Proof. By Theorem 8.2 we have $\lim_n \phi(y, \lambda_n) = \lim_n \phi(y, \mu_n) = \phi(y, \mu)$ for each $y \in E_\alpha^*$, and by [11, Theorem 7.6] $\{\lambda_n\}$ converges weakly to a measure ν on H_λ . Since $\phi(y, \nu) = \phi(y, \mu)$ on H_λ it follows that $\mu = \nu$ on H_λ , and since $\mu(E_\alpha) = 1$ we have $\nu(E_\alpha) = 1$. Further, ν is infinitely divisible on H_λ by [11, Theorem 5.1] since each λ_n is infinitely divisible there, and hence $\mu = \nu$ is infinitely divisible on E_α .

9. Convergence of certain infinitely divisible laws and a central limit theorem. We first prove a theorem regarding the convergence of certain infinitely divisible laws on the Orlicz space E_α as defined in §7. The notation $\mu = [x_0, T, F]$ is used to denote the fact that μ is infinitely divisible with Fourier transform as in (7.19) having parameters x_0, T, F with respect to some fixed λ .

Theorem 9.1. *Let $\mu_n = [x_n, T_n, F_n]$ with respect to some fixed λ and assume $\{x_n\}$ is conditionally compact with F_n an increasing sequence of finite measures on E_α . Then $\{\mu_n\}$ is conditionally compact iff*

- (i) F_n restricted to the complement of a neighborhood of zero is weakly conditionally compact,
- (ii) the sequence of bilinear forms

$$(9.1) \quad R_n(y, y) = \int_{\mathcal{U}} (x, y)^2 dF_n(x) + T_n(y, y)$$

satisfies

$$(9.2) \quad \sup_n \sum_{i=1}^{\infty} \alpha(r_{ii}^{(n)}) < \infty,$$

$$(9.3) \quad \lim_N \sup_n \sum_{i=N}^{\infty} \alpha(r_{ii}^{(n)}) = 0$$

where $r_{ii}^{(n)} = R_n(e_i, e_i)$, e_i is the sequence with one in the i th position and zero elsewhere, and $\mathcal{U} = \{x \in E_\alpha: \sum_{i=1}^{\infty} \alpha(x_i^2) \leq 1\}$,

- (iii) $\{\mu_n\}$ is a λ -family with respect to the fixed λ yielding the representations $\mu_n = [x_n, T_n, F_n]$.

Proof. First assume $\{\mu_n\}$ is conditionally compact. Then $\{x_n\}$ conditionally compact implies $\{e(F_n)\}$ shift-compact, so (i) follows from [11, p. 214] and Remark 2.3 of [11]. Since e_i represents the coefficient functional β_i scaled by a fixed positive constant for $i = 1, 2, \dots$, we have by Theorem 5.2(3), that (9.2) and (9.3) hold with $r_{ii}^{(n)}$ replaced by $t_{ii}^{(n)} = T_n(e_i, e_i)$. That is, we know the measures $[x_n, T_n, 0]$ are shift-compact, thus $[0, T_n, 0]$ is compact, so Theorem 5.2 applies, and the scale factor can be ignored since $\alpha(\cdot)$ satisfies (6.1). Furthermore, if

$$(9.4) \quad s_{ii}^{(n)} = \int_{\mathcal{U}} (x, e_i)^2 dF_n(x) \quad (n, i = 1, 2, \dots)$$

then

$$s_{ii}^{(n)} = \int_{\mathcal{U}} x_i^2 dF_n(x)$$

and letting $F = \lim F_n$ as in Lemma 7.3 we have by (7.5) that (9.2) and (9.3) hold

with $r_{ii}^{(n)}$ replaced by $s_{ii}^{(n)}$. Now $r_{ii}^{(n)} = s_{ii}^{(n)} + t_{ii}^{(n)}$ and since $\alpha(\cdot)$ satisfies (6.1) we have by (6.3) that (9.2) and (9.3) hold as stated. Finally, (iii) holds by Lemma 3.1. Now we assume (i), (ii), and (iii) hold. Now $\mathcal{U}$ contains an open neighborhood V of zero and we can assume F_n vanishes on the complement of V . The last remark follows from (i) since on $E_\alpha - V$ we have F_n converging to some finite measure, call it F , and hence if G_n is F_n restricted to $E_\alpha - V$ we have $e(G_n)$ converging to $e(F)$. Since the unit ball, call it W , of H_λ contains $\mathcal{U}$ and each F_n is concentrated on $V \subseteq \mathcal{U} \subseteq W$ it follows from (ii) that R_n is an α -operator. Hence Lemma 5.4 implies that $\exp\{-\frac{1}{2}R_n(y, y)\}$ is the Fourier transform of a Gaussian measure, say λ_n , on $E_\alpha \subseteq H_\lambda$. It follows from (ii) that $\{\lambda_n\}$ is conditionally compact on E_α and hence on H_λ , and hence that the operators A_n induced by R_n through λ_n on H_λ^* satisfy the conditions of Theorem 6.3 (see [11, p. 230]) so $\{\mu_n\}$ is conditionally compact on H_λ . Then using (iii) we have $\{\mu_n\}$ conditionally compact on E_α so the theorem is proved.

We now proceed to a central limit theorem.

Theorem 9.2. Let $\{F_{nj}\}$ ($j = 1, \dots, k_n; n = 1, 2, \dots$) be a uniformly infinitesimal sequence of probability measures on E_α and assume the notation in (8.1). Further, assume $\lim_n x_n = x_0$ and that

$$(9.5) \quad \sup_n \sum_{i=1}^{\infty} \alpha \left(\int_{\mathcal{U}} x_i^2 dF_n(x) \right) < \infty,$$

$$(9.6) \quad \limsup_N \sum_{i=N}^{\infty} \alpha \left(\int_{\mathcal{U}} x_i^2 dF_n(x) \right) = 0.$$

Then $\prod_{j=1}^{k_n} F_{nj} = \mu_n$ converges weakly to the Gaussian measure $\mu_0 = [x_0, T_0, 0]$ where T_0 is an α -operator if for every $\epsilon > 0$

- (a) $\lim_n \sum_{j=1}^{k_n} F_{nj} \{x: \|x - x_{nj}\|_\Gamma > \epsilon\} = 0$,
- (b) $\lim_n \sum_{j=1}^{k_n} \int_{\|x\|_\Gamma \leq \epsilon} (x - x_{nj}, y)^2 dF_{nj}(x) = T_0(y, y)$ for each $y \in E_\alpha^*$,
- (c) $e(F_n)$ is a λ -family.

Proof. Now (a) implies $\lim_n F_n(E_\alpha - V) = 0$ for every open neighborhood V of zero, and hence if G_n is F_n restricted to $E_\alpha - V$ we have $e(G_n)$ converging to the unit mass at zero. Now let H_n be F_n restricted to V where $V = \{x \in E_\alpha: \|x\|_\Gamma < \gamma\} \subseteq B$. Then $e(H_n) = [z_n, 0, H_n]$ for any λ where

$$(9.7) \quad z_n = \int_{\mathcal{U}} \frac{x}{1 + \|x\|_\lambda^2} dH_n(x) = \int_V \frac{x}{1 + \|x\|_\lambda^2} dF_n(x).$$

Now let $\epsilon > 0$ be given. Then

$$\begin{aligned} z_n &= \int_V x dF_n(x) - \int_V \frac{x \|x\|_\lambda^2}{1 + \|x\|_\lambda^2} dF_n(x) \\ &= x_n - \int_{B-V} x dF_n(x) - \int_V \frac{x \|x\|_\lambda^2}{1 + \|x\|_\lambda^2} dF_n(x) \end{aligned}$$

and since

$$\begin{aligned} (9.8) \quad & \left\| \int_{B-V} x dF_n(x) \right\|_\Gamma \leq \rho F_n\{B - V\}, \\ & \left\| \int_V \frac{x \|x\|_\lambda^2}{1 + \|x\|_\lambda^2} dF_n(x) \right\|_\Gamma \leq \epsilon \int_{\|x\|_\Gamma \leq \epsilon} \|x\|_\lambda^2 dF_n(x) + \gamma F_n\{x: \epsilon < \|x\|_\Gamma < \gamma\}, \end{aligned}$$

we have $\overline{\lim}_n \|z_n - x_n\|_\Gamma \leq \epsilon$ by applying (a) and (9.5). Thus $\lim_n z_n = \lim x_n = x_0$ and by (9.5), (9.6), and (a) we have by the reasoning used in Theorem 9.1 that $e(F_n)$ is conditionally compact on H_λ . Then by (c) it follows that $e(F_n)$ is conditionally compact on E_α . Further, in view of (a), (b) each linear functional has a limiting Gaussian distribution with mean zero and variance $T_0(y, y)$ with respect to all limits of $e(F_n)$. Hence $e(F_n)$ converges to $\gamma = [0, T_0, 0]$ and by Theorem 8.2 we then have $\prod_{j=1}^k F_{n_j}$ converging weakly to γ so the theorem is proved.

Corollary 9.1. *Let $\{F_{n_j}\}$ be a uniformly infinitesimal sequence of probability measures on E_α and assume the notation of (8.3). Further, assume $\lim x_n = x_0$ and that there exists a measure ρ on E_α such that*

$$(9.9) \quad \sum_{i=1}^{\infty} \alpha \left(\int_{E_\alpha} x_i^2 d\rho(x) \right) < \infty,$$

$$(9.10) \quad \int_V x_i^2 dF_n(x) \leq \int_{E_\alpha} x_i^2 d\rho(x) \quad (i = 1, 2, \dots).$$

Then conditions (a), (b), (c) of Theorem 9.2 imply that $\prod_{j=1}^k F_{n_j}$ converge weakly to $\mu_0 = [x_0, T_0, 0]$.

Proof. We need only show (9.5) and (9.6) hold and that is obvious by (9.9) and (9.10).

Corollary 9.2. *Let μ be a probability measure on E_α such that (9.9) holds for μ , $\int_{E_\alpha} \|x\|_\Gamma^2 d\mu(x)$ is finite, and $\int_{E_\alpha} x d\mu(x) = 0$. Let $Z_1, Z_2, \dots$ be independent*

random variables such that $\mathcal{L}(Z_k) = \mu$ for $k = 1, 2, \dots$ and define $S_n = Z_1 + \dots + Z_n$. Then $\mathcal{L}(S_n/\sqrt{n})$ converges weakly to the Gaussian measure $\mu_0 = [0, T_0, 0]$, where

$$T_0(y, y) = \int_{E_\alpha} (x, y)^2 d\mu(x)$$

provided the measures $\gamma_n = \mathcal{L}(S_n/\sqrt{n})$ or the measures $e(F_n)$ form a λ -family. Here F_n is as in (8.3) with $F_{nj} = \mathcal{L}(Z_j/\sqrt{n})$ for $j = 1, \dots, n$; $n = 1, 2, \dots$.

Proof. Since (9.9) holds and $\int_{E_\alpha} \|x\|_\Gamma^2 d\mu(x)$ is finite it follows that $T_0(y, y)$ is an α -operator and hence the Gaussian measure $\mu_0 = [0, T_0, 0]$ exists on $E_\alpha \subseteq H_\lambda$. Further, since the H_λ topology on E_α is weaker than the E_α topology there exists a constant $c > 0$ such that $c\|x\|_\lambda \leq \|x\|_\Gamma$ for each $x \in E_\alpha$ and hence

$$\int_{H_\lambda} \|x\|_\lambda^2 d\mu(x) = \int_{E_\alpha} \|x\|_\lambda^2 d\mu(x) \leq \left[\frac{1}{c}\right]^2 \int_{E_\alpha} \|x\|_\Gamma^2 d\mu(x) < \infty.$$

Thus γ_n converges weakly to μ_0 on H_λ [9, p. 1561] and in the case γ_n is a λ -family we then have γ_n converging to μ_0 on E_α . In the case $e(F_n)$ is a λ -family we apply Corollary 9.1. To verify the hypothesis of Corollary 9.1 we first observe that $\lim_n x_n = 0$. To see this notice that

$$x_n = n \int_B x dF_{nj}(x) = \sqrt{n} \int_{\sqrt{n}B} x d\mu(x)$$

and since $\int_B x d\mu(x) = 0$ we have

$$\|x_n\|_\Gamma = \left\| -\sqrt{n} \int_{E_\alpha - \sqrt{n}B} x d\mu(x) \right\|_\Gamma \leq \frac{1}{\sqrt{n}} \int_{E_\alpha} \|x\|_\Gamma^2 d\mu(x).$$

Further, for every $\epsilon > 0$,

$$\begin{aligned} \lim_n \sum_{j=1}^n F_{nj}\{x: \|x - x_{nj}\|_\Gamma > \epsilon\} &= \lim_n n\mu\{x: \|x - x_{nj}\|_\Gamma > \sqrt{n}\epsilon\} \\ &\leq \lim_n \frac{1}{\epsilon^2} \int_{\{x: \|x - x_{nj}\|_\Gamma > \sqrt{n}\epsilon\}} \|x - x_{nj}\|_\Gamma^2 d\mu(x) = 0 \end{aligned}$$

since $\int_{E_\alpha} \|x\|_\Gamma^2 d\mu(x)$ is finite and $\lim_n \|x_{nj}\|_\Gamma = 0$. Similarly,

$$\begin{aligned} \lim_n \sum_{j=1}^n \int_{\|x\|_\Gamma \leq \epsilon} (x - x_{nj}, y)^2 dF_{nj}(x) &= \lim_n \int_{\|x\|_\Gamma \leq \sqrt{n}\epsilon} (x - x_{nj}, y)^2 d\mu(x) \\ &= \int_{E_\alpha} (x, y)^2 d\mu(x) = T_0(y, y). \end{aligned}$$

Finally (9.10) holds since $\sup_n \int_{\mathbf{U}} x_i^2 dF_n(x) = \int_{E_\alpha} x_i^2 d\mu(x)$.

We now derive some recent results on the central limit theorem due to A.D. de Acosta [12]. A Banach space X with a Schauder basis $\{b_i\}$ and coordinate functionals $\{\beta_j\}$ is said to be of type p with respect to $\{b_i\}$ if

$$(9.11) \quad P_n \left[\sum_{j=1}^m \beta_j(x) b_j \right]^p \leq \sum_{j=n+1}^m |\beta_j(x)|^p$$

for all $x \in X$ and integers m, n ($1 \leq n \leq m$) where, for each n , $x \in X$,

$$P_n(x) = \inf \{ \|x - z\| : z \in S_n \}$$

with S_n being the subspace generated by $\{b_1, \dots, b_n\}$ for $n \geq 1$ and $S_0 = 0$.

Lemma 9.1. *Let X be of the type $p \geq 2$. Then there exists a linear isomorphism U on a subset X_0 of X onto l_2 such that $\|Ux\|_{l_2} \geq \|x\|_X$.*

Proof. Since X is of type $p \geq 2$ it is of type 2 [12]. Let us consider $X_0 = \{x: x \in X, \sum_{j=1}^{\infty} |\beta_j(x)|^2 < \infty\}$. Define U on X_0 to l_2 by $Ux = (\beta_1(x), \beta_2(x), \dots)$. Then U is an isomorphism of X_0 onto l_2 and taking $n = 0$ we have by (9.11) that $\|Ux\|_{l_2} \geq \|x\|_X$.

Now let T be a bilinear form defined on X^* . Then T is said to be of trace class p relative to $\{\beta_j\}$ if $\sum_{j=1}^{\infty} |T(\beta_j, \beta_j)|^p < \infty$.

We now state a lemma which strengthens the assertion of Theorem 4.1 (b) of [12].

Lemma 9.2. *Let X be a Banach space with Schauder basis of type $p \geq 2$. Let $\{\mu_\alpha: \alpha \in A\}$ be a set of probability measures on X . Suppose for every $\epsilon > 0$, there exists a family $\{T_{\alpha, \epsilon}\}$ of nonnegative bilinear forms of trace class one satisfying*

$$\sup_{\alpha \in A} \sum_{j=1}^{\infty} T_{\alpha, \epsilon}(\beta_j, \beta_j) < \infty,$$

$$\lim_N \sup_{\alpha \in A} \sum_{j=N}^{\infty} T_{\alpha, \epsilon}(\beta_j, \beta_j) = 0,$$

$$|1 - \operatorname{Re} \phi(y, \mu_\alpha)| \leq T_{\alpha, \epsilon}(y, y) + \epsilon \quad (y \in X^*).$$

Then $\{\mu_\alpha: \alpha \in A\}$ is supported on X_0 and it is conditionally compact on X_0 where the norm $\|x\|_{X_0} = \|Ux\|_{l_2}$.

Proof. Apply Theorem 2.2 of [3].

In view of Lemma 9.2 we can prove the central limit theorem of [12] with $p \geq 2$. That is, under the hypothesis of the theorem and by Lemma 9.2 the measures $\prod_{j=1}^k F_{n_j}$ are conditionally compact on X_0 . Further, the classical central limit theorem says the limiting measure must be Gaussian with Fourier transform as indicated. We also point out, but we will not provide the computations, that the hypotheses on the central limit theorem of [12] with $p \geq 2$ actually imply the hypotheses of Theorem 9.2 in case the measures sit on l_2 . This, however, in view of Lemma 9.2, is always the case under such conditions.

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