

TORSION DIFFERENTIALS AND DEFORMATION

BY

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ABSTRACT. Let S -scheme X be a Schlessinger deformation of a curve X_0 defined over a field k . In §§1 and 2, the dimension of the parameter space S , the relative differentials of X over S , and the fibres with singularity were studied, in case when X_0 is locally complete-intersection. In §3 we show that if k -scheme X_0 is a specialization of a smooth k -scheme, then the punctured spectrum $\text{Spex}(O_{X_0, x})$ has to be connected for every point $x \in X_0$ such that $\dim O_{X_0, x} \geq 2$. In turn we construct a rigid singularity on a surface. In the last section a few conjectures amplifying those of P. Deligne are made.

Let X_0 be an algebraic variety over a fixed perfect field k . A (formal) deformation⁽²⁾ of X_0 is meant by a pair (R, X) where R is a complete noetherian local k -algebra with the residue field k , and X a flat (formal) R -scheme together with an isomorphism $X_0 \xrightarrow{\sim} k \otimes_R X$. A (formal) deformation (R, X) of X_0 is called a (formal) versal deformation⁽³⁾ of X_0 if every (formal) deformation of X_0 is induced from (R, X) . A (formal) versal deformation (R, X) of X_0 such that $\dim \text{Der}_k(R, k)$ is minimal will be called a minimally-versal (formal) deformation or a (formal) Schlessinger deformation of X_0 . If X_0 is proper over k or affine with isolated nonsmooth points only, we have the formal existence theorem due to M. Schlessinger, i.e. there exists a minimally-versal formal deformation of X_0 . If X_0 is a projective variety with $H^2(X_0, \underline{O}_{X_0}) = 0$ (for instance X_0 is a complete curve), then it follows from a theorem of A. Grothendieck that every formal deformation is "algebraisable" and in turn there exists a minimally-versal deformation of X_0 .

Besides these existence theorems we have very little knowledge about the deformation of singularities at the present stage. In this paper we study the deformations of curves which is a relative complete-intersection over k , as well as rigid singularities on a surface. A complete-intersection, having no local obstruction for deformations, is the simplest case. Our method is based on a study of

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(2) In general one considers deformations in which the characteristic also changes. Here we restrict ourselves to equicharacteristic case.

(3) One notes here the change of terminology from [8]. A minimally-versal deformation or a Schlessinger deformation in this paper was called a versal deformation in [8]. A change of the terminology comes from a necessity of considering a versal deformation in the sense of this paper which is not necessarily a minimal one.

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torsions and cotorsions of some modules carried out in §1, which is of interest in itself but may be of no use in the case of noncomplete intersections. Nevertheless, our Theorem 2.7 may give an insight towards likely phenomenon in the general case. In a less precise language, we establish the following: Let X_0 be a reduced complete curve which is a relative complete-intersection over k , and let (R, X) be a Schlessinger deformation of X_0 . Then

- (i) $\dim R = 3g - 3 + \dim H^0(X_0, \underline{\Omega}_{X_0}^*)$ where $g = \dim H^1(X_0, \underline{\Omega}_{X_0})$,
- (i)' $\dim R - \dim H^1(X_0, \underline{\Omega}_{X_0}^*) =$ the dimension of the torsion k -differentials of X_0 ,
- (ii) $\Omega_{X|R}$ has no torsion and $\Omega_{X|R}^*$ is an invertible sheaf on X ,
- (iii) $\text{codim}(\text{Sing}_R(X|R) \text{ in } \text{Spec}(R)) = 1$ where $\text{Sing}_R(X|R) = \{z \in \text{Spec}(R) \mid X \rightarrow \text{Spec}(R) \text{ is not smooth at the point } z\}$.

The properties (i) and (iii) are expected to be valid for an arbitrary reduced complete curve. For instance, a complete curve having ordinary multiple points with mutually transversal tangents does enjoy (i) and (iii), and considerably more, even though the parameter scheme $\text{Spec}(R)$ is far from being regular in general [9].

In §3 we are interested in the rigid isolated singularities. It has been known that there exists an affine variety (with an isolated singularity) which cannot be obtained as a specialization of a nonsingular variety. Indeed, H. Grauert and H. Kerner have constructed a rigid isolated singularity of dimension n , provided $n \geq 4$ [4]. On the other hand, every complete reduced curve is conjecturally a specialization of a nonsingular curve, and therefore there cannot exist, conjecturally, a rigid 1-dimensional isolated singularity. Thus it raises the question if there exists a surface with a rigid isolated singularity. We first establish that every reduced surface with isolated singularities only which is obtained as a specialization of a nonsingular surface has to be unibranch at every point. Motivated by this fact, we construct a rigid isolated singularity on an irreducible rational surface. In the last section we have made a few conjectures based on our Theorem 2.7 as well as a number of empirical results. For instance, it will be shown in the forthcoming paper [9] that these conjectures are valid for any curve with ordinary multiple points (with mutually transversal tangents).

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1. **Torsions and cotorsions.** Let A be a commutative noetherian ring. For any generically-free A -module M of finite type, i.e. $K \otimes_A M$ is K -free, where K is the total ring of fractions of A , we set $\text{rg } M =$ the free rank of $K \otimes_A M$ as K -module. Given a generically-free A -module M of finite type with $\text{rg } M = d$, choose a finite presentation

$$0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0$$

where F is A -free of rank m . Since N is a generically-free A -module of rank $m - d$, the natural map $\bigwedge^{m-d} N \otimes \bigwedge^d F \rightarrow \bigwedge^m F$ induces the map $\bigwedge^{m-d} N \otimes \bigwedge^d M \rightarrow \bigwedge^m F$ and hence the map $\alpha_F: \bigwedge^d M \rightarrow \text{Hom}_A(\bigwedge^{m-d} N, \bigwedge^m F)$.

Lemma 1.0. *For any generically-free A -module M of finite type, the map $\alpha_F: \bigwedge^d M \rightarrow \text{Hom}_A(\bigwedge^{m-d} N, \bigwedge^m F)$ does not depend on the choice of presentations of M .*

Proof. Let

$$0 \rightarrow N_1 \rightarrow F_1 \rightarrow M \rightarrow 0,$$

$$0 \rightarrow N_2 \rightarrow F_2 \rightarrow M \rightarrow 0$$

be any two finite presentations of M where $\text{rg } F_i = m_i$. Suppose that there exists an isomorphism

$$\tau: \text{Hom}_A\left(\bigwedge^{m_1-d} N_1, \bigwedge^{m_1} F_1\right) \rightarrow \text{Hom}_A\left(\bigwedge^{m_2-d} N_2, \bigwedge^{m_2} F_2\right)$$

with the following commutative diagram:

$$(*) \quad \begin{array}{ccc} & & \text{Hom}_A\left(\bigwedge^{m_1-d} N_1, \bigwedge^{m_1} F_1\right) \\ & \nearrow \alpha_1 & \downarrow \tau \\ \bigwedge^d M & & \\ & \searrow \alpha_2 & \downarrow \\ & & \text{Hom}_A\left(\bigwedge^{m_2-d} N_2, \bigwedge^{m_2} F_2\right) \end{array}$$

Then such τ must necessarily be unique since α_i ($i = 1, 2$) are generically isomorphisms, and $\text{Hom}_A(_, \bigwedge^m F)$ are torsion-free modules. Therefore it suffices to show an existence of such isomorphism τ . Now, from Schannel's lemma, there exists an automorphism $\sigma \in \text{Aut}_A(F_1 \oplus F_2)$ with the commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & N_1 \oplus F_2 & \rightarrow & F_1 \oplus F_2 & \rightarrow & M \rightarrow 0 \\ & & \downarrow \approx & & \downarrow \sigma & & \parallel \\ 0 & \rightarrow & F_1 \oplus N_2 & \rightarrow & F_1 \oplus F_2 & \rightarrow & M \rightarrow 0 \end{array}$$

In turn, σ induces a commutative diagram

$$\begin{array}{ccc} \bigwedge^{m_1+m_2-d} (N_1 \oplus F_2) \otimes \bigwedge^d M & \xrightarrow{\quad} & \bigwedge^{m_1+m_2} (F_1 \oplus F_2) \\ \downarrow \bigwedge(\sigma) \oplus \text{id} & & \downarrow \bigwedge(\sigma) \\ \bigwedge^{m_1+m_2-d} (F_1 \oplus N_2) \otimes \bigwedge^d M & \xrightarrow{\quad} & \bigwedge^{m_1+m_2} (F_1 \oplus F_2) \end{array}$$

from which an existence of τ with a commutative diagram (*) follows.

In view of the intrinsic nature of the map $\alpha: \bigwedge^d M \rightarrow \text{Hom}_A(\bigwedge^{m-d} N, \bigwedge^m F)$, we shall denote this as $\alpha_M: \bigwedge^d M \rightarrow M_\#$ by setting $M_\# = \text{Hom}_A(\bigwedge^{m-d} N, \bigwedge^m F)$. We note the following simple properties of $M_\#$ which are trivial to verify.

- (1) If we set $M_\#^t = \text{Coker}(\bigwedge^d M \rightarrow M_\#)$, then

$$0 \rightarrow \left(\bigwedge^d M\right)_t \xrightarrow{d} \bigwedge^d M \rightarrow M_\# \rightarrow M_\#^t \rightarrow 0$$

is exact, where $(\bigwedge^d M)_t$ stands for the torsion submodule of $\bigwedge^d M$.

(2) $\alpha_M: \bigwedge^d M \rightarrow M_\#$ commutes with base change $A \rightarrow A'$ provided $\text{Tor}_1^A(M, A') = 0$. In particular, (a) $S^{-1}A_M \rightarrow \alpha_{S^{-1}M}$ is an isomorphism for any multiplicative subset S of A . (b) If A is an R -algebra and M is R -flat, then $\alpha_{R'} \otimes_R M \rightarrow R' \otimes_R \alpha_M$ is an isomorphism for any R -algebra R' .

(3) If M is A -free, then α_M is an isomorphism. Therefore $\text{Supp } M_\#^t \subset \text{Sing}_A(M) = \{x \in \text{Spec}(A) \mid M_x \text{ is not } A_x\text{-free}\}$.

- (4) If $\text{hd}_A M \leq 1$, then $M_\#$ is an invertible A -module, and $\text{Supp } M_\#^t = \text{Sing}_A(M)$.

Let M be a generically-free A -module of rank d . Choose a finite presentation

$$0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0$$

where F is A -projective module of rank m , and we consider the chain complex $K(M)$ defined as follows (see [2]):

$$\cdots \rightarrow \sum \bigwedge^{s_1} N \otimes \bigwedge^{s_2} N \otimes \cdots \bigwedge^{d-s_1-s_2} F \rightarrow \sum \bigwedge^{s_1} N \otimes \bigwedge^{d-s_1} F \xrightarrow{d} \bigwedge^d F \rightarrow M_\#$$

where $s_i > 0$ for all i (and hence it is a complex of length $d+1$), and the boundary operators are given by

$$\begin{aligned} u_1 \otimes u_2 \otimes \cdots \otimes u_r \otimes v &\rightarrow u_1 \wedge u_2 \otimes u_3 \otimes \cdots \otimes u_r \otimes v \\ &\quad - u_1 \otimes u_2 \wedge u_3 \otimes \cdots \otimes u_r \otimes v \\ &\quad + \cdots + (-1)^2 u_1 \otimes \cdots \otimes u_{r-1} \otimes u_r \wedge v. \end{aligned}$$

We note that $H_0(K(M)) = M_\#^t$ and $H_1(K(M)) = \text{Ker}(\bigwedge^d M \rightarrow M_\#) = (\bigwedge^d M)_t$. Furthermore, if M is A -projective (so that the inclusion map $N \rightarrow F$ admits a retract), then $K(M)$ is homotopically trivial, and in particular $\text{Supp } H_*(K(M)) \subset \text{Sing}_A(M)$. We state below the basic facts obtained in [2] concerning the complex $K(M)$. They are Theorem 2.4 and Corollary 4.4 in [2] respectively.

Lemma 1.1. *Let M be a generically-free A -module of finite type of rank d , with $\text{hd}_A M \leq 1$. Then we have*

(a) $d+1 - \text{Sing}(M) - \text{depth } A = \text{the smallest integer } q \text{ such that } H_i(K(M)) = 0 \text{ for all } i > q$.

(b) *Let A be a local ring and assume that M_x is A_x -free for all nonmaximal points x in $\text{Spec}(A)$. If $d+1 > \dim A$, then $\chi(H_*(K(M))) = 0$ where $\chi(H_*(K(M))) = \sum_i (-1)^i \text{length } H_i(K(M))$.*

As an immediate application, we obtain the following:

Theorem 1.2. *Let A be a commutative noetherian ring and M a generically-free A -module of finite type with $\text{hd}_A M \leq 1$. We set $d = \text{rg}_A M$. Then*

(i) *If $\text{Sing}(M) - \text{depth } A \geq d$, then $\text{hd}_A \bigwedge^d M \leq d$. If $\text{Sing}(M) - \text{depth } A > d$, then $\bigwedge^d M$ has no torsion so that $0 \rightarrow \bigwedge^d M \rightarrow M_{\#} \rightarrow M_{\#}^t \rightarrow 0$ is exact, and $M_{\#}^t$ is a perfect⁽⁴⁾ A -module with $\text{hd}_A M_{\#}^t = d + 1$.*

(ii) *Assume that $\text{Sing}(M)$ is a finite set consisting of Cohen-Macaulay maximal points of dimension d . We then have $\text{length}((\bigwedge^d M)_t) = \text{length}(M_{\#}^t)$.*

Proof. Choose a finite presentation of M

$$0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0$$

and consider the complex $K(M)$ as defined above.

(i) Suppose that $\text{Sing}(M) - \text{depth } A \geq d$. By 1.1 (a), we then have $H_i(K(M)) = 0$ for all $i > 1$, and therefore

$$\dots \rightarrow \sum \bigwedge^1 N \otimes \bigwedge^2 N \otimes \bigwedge^{d-s_1-s_2} F \rightarrow \sum \bigwedge^1 N \otimes \bigwedge^{d-s_1} F \rightarrow \bigwedge^d F$$

is an A -projective resolution of $\bigwedge^d M$, which shows that $\text{hd}_A \bigwedge^d M \leq d$. If $\text{Sing}(M) - \text{depth } A > d$, then $H_i(K(M)) = 0$ for all $i > 0$ by 1.1 (a) again, and in particular we have $0 = H_1(K(M)) = (\bigwedge^d M)_t$.

(ii) Assume that $\text{Sing}(M)$ is a finite set consisting of Cohen-Macaulay maximal points of dimension d . We then have

$$\chi(H_*(K(M))) = \sum_{x \in \text{Sing}(M)} \chi(H_*(K(M_x))) = 0$$

by 1.1 (b). On the other hand, we have $H_i(K(M)) = 0$ for all $i > 1$ by 1.1 (a) and therefore $0 = \chi(H_*(K(M))) = \text{length } H_0(K(M)) - \text{length } H_1(K(M)) = \text{length}(M_{\#}^t) - \text{length}((\bigwedge^d M)_t)$.

Let R be a noetherian ring and A a generically smooth R -algebra of finite type (i.e. $A \otimes_R K$ is smooth over K where K is the total ring of fractions of R). Then the module of relative differentials $\Omega_{A|R}$ is a generically-free A -module of finite type such that $\text{rg } \Omega_{A|R} = \dim A \otimes_R K$, and in turn we may consider $\bigwedge^d \Omega_{A|R} \rightarrow (\Omega_{A|R})_{\#}$ where $d = \text{rg } \Omega_{A|R} = \dim A \otimes_R K$. We note that in case when A is a relative complete-intersection over R , then $\text{hd}_A \Omega_{A|R} \leq 1$ and $(\Omega_{A|R})_{\#}$ is nothing but the module of dualizing differentials $\omega_{A|R}$ defined by A. Grothendieck [7], and the canonical map $\bigwedge^d \Omega_{A|R} \rightarrow \omega_{A|R}$ is an isomorphism at each point $x \in \text{Spec}(A)$ at which A is smooth over R , and therefore $\text{Supp}((\bigwedge^d \Omega_{A|R})_t)$ as well as $\text{Supp } \omega_{A|R}^t$ are contained in $\text{Sing}(A|R) = \{x \in \text{Spec}(A) \mid A \text{ is not smooth over } R \text{ at } x\}$. We note furthermore that $\text{Sing}(A|R) = \text{Sing } \Omega_{A|R} = \text{Supp } \omega_{A|R}^t$ since A is a relative complete-intersection over R , where $\omega_{A|R}^t = \text{Coker}(\bigwedge^d \Omega_{A|R} \rightarrow \omega_{A|R})$.

⁽⁴⁾ A -module E of finite type is called perfect if $\text{hd}_A E < \infty$ and $\text{Ext}_A^i(E, A) = 0$ for all $i < \text{hd}_A E$ or equivalently $\text{Supp}(E) - \text{depth } A \geq \text{hd}_A E$.

Corollary 1.3. *Let R be Cohen-Macaulay and A a generically-smooth relative complete-intersection over R , of relative dimension d .*

(i) *If $\dim A_x \geq d$ for all $x \in \text{Sing}(A|R)$, then $\text{hd}_A \bigwedge^d \Omega_{A|R} \leq d$. If $\dim A_x > d$ for all $x \in \text{Sing}(A|R)$, then $0 \rightarrow \bigwedge^d \Omega_{A|R} \rightarrow \omega_{A|R} \rightarrow \omega_{A|R}^t \rightarrow 0$ is exact.*

(ii) *Let R be artinian. If $\text{Sing}(A|R)$ consists of a finite number of maximal points, then we have $\text{length}(\bigwedge^d \Omega_{A|R}) = \text{length}(\omega_{A|R}^t)$.*

Proof. Since R is Cohen-Macaulay and A is a relative complete-intersection over R , A is also Cohen-Macaulay. Thus our statements follow immediately from 1.2.

Before we consider a further application, we recall Fitting ideals of a module: Let E be an R -module of finite type, and choose a finite presentation

$$0 \rightarrow N \rightarrow F \rightarrow E \rightarrow 0$$

where F is R -free module of rank n . For each positive integer p , we set

$$I_R^{(p)}(E) = \text{Im} \left(\det: \bigwedge^{n-p+1} N \otimes \bigwedge^{n-p+1} F \rightarrow R \right).$$

These ideals do not depend on the choice of a presentation of E and hence are the invariants of E . For the sake of simplicity, we set $I_R(e) = I_R^{(1)}(E)$. We note that $\text{Supp } R/I_R(E) = \text{Supp } E$.

Lemma 1.4. *Let R be a local ring with the maximal ideal $\underline{m}$, and A a (noetherian) R -flat algebra, such that $\underline{m}A$ is contained in the Jacobian radical of A . Then for any ideal I in A , we have*

$$I - \text{depth } A \geq (I \cap R) - \text{depth } R + I - \text{depth } R/\underline{m} \otimes_R A.$$

Proof. If $r_1, r_2, \dots, r_k$ is an R -regular sequence in $I \cap R$, then it is an A -regular sequence since A is R -flat. Therefore, replacing R by $R/(r_1, \dots, r_s)$ where $r_1, \dots, r_s$ is a maximal R -regular sequence in $I \cap R$, we may assume that $(I \cap R) - \text{depth } R = 0$. Let f be an element in I which is not a zero-divisor in $R/\underline{m} \otimes_R A$. It suffices to show that f is not a zero-divisor in A and that A/fA is R -flat. Consider the exact sequence $0 \rightarrow fA \rightarrow A \rightarrow A/fA \rightarrow 0$, which induces the exact sequence

$$R/\underline{m} \otimes_R fA \rightarrow R/\underline{m} \otimes_R A \rightarrow R/\underline{m} \otimes_R A/fA \rightarrow 0.$$

Since the composite map $R/\underline{m} \otimes_R A \xrightarrow{1 \otimes f} R/\underline{m} \otimes_R fA \rightarrow R/\underline{m} \otimes_R A$ is injection by hypothesis, and $R/\underline{m} \otimes_R A \xrightarrow{1 \otimes f} R/\underline{m} \otimes_R fA$ is surjection, it follows that $R/\underline{m} \otimes_R A \xrightarrow{1 \otimes f} R/\underline{m} \otimes_R fA$ is an isomorphism, and $R/\underline{m} \otimes_R fA \rightarrow R/\underline{m} \otimes_R A$ is an injection, i.e. $\text{Tor}_1^R(R/\underline{m}, A/fA) = 0$ which shows that A/fA is again A -flat.

Now consider the exact sequence $0 \rightarrow \text{Ker } f \rightarrow A \xrightarrow{f} fA \rightarrow 0$. Since A and A/fA are R -flat, we have $\text{Tor}_1^R(R/\underline{m}, fA) = 0$ and hence we get the exact sequence

$$0 \rightarrow R/\underline{m} \otimes_R \text{Ker } f \rightarrow R/\underline{m} \otimes_R A \xrightarrow{1 \otimes f} R/\underline{m} \otimes_R fA \rightarrow 0.$$

However, $R/\underline{m} \otimes_R A \xrightarrow{1 \otimes f} R/\underline{m} \otimes_R fA$ is an isomorphism and therefore $R/\underline{m} \otimes_R \text{Ker } f = 0$. Since $\underline{m}A$ is contained in the Jacobson radical of A , it follows that $\text{Ker } f = 0$ by Nakayama's lemma.

Theorem 1.5. *Let R be a complete local ring with the maximal ideal $\underline{m}$, and A a (noetherian) R -flat algebra. Let M be a generically-free R -flat A -module of finite type with $\text{hd}_A M \leq 1$. Assume that*

(i) $\text{Sing}(M_0)$ consists of a finite number of Cohen-Macaulay maximal points in $\text{Spec}(A_0)$ of dimension $\geq \text{rg } M$, where $M_0 = M/\underline{m}M$,

(ii) $K \otimes_R M$ is $K \otimes_R A$ -projective, where K = the total ring of fractions of R ,

(iii) $M_\#^t$ is of finite type as an R -module.

Then $\bigwedge^d M$ has no torsion, $M_\#^t$ is a perfect A -module with $\text{hd}_A M_\#^t = d + 1$, and $I_R(M_\#^t)$ is an invertible ideal in R .

Proof. Set $I = \text{Ann}_A M_\#^t$. Then (iii) entails that A/I is of finite type as an R -module, i.e. A/I is an integral R -algebra of finite type, and hence there are only a finite number of maximal ideals $\underline{m}_1, \underline{m}_2, \dots, \underline{m}_r$ of A containing I , and all of them dominate the maximal ideal $\underline{m}$ of R . Set $A' = S^{-1}A$ where $S = A - \underline{m}_1 \cup \dots \cup \underline{m}_r$, and set $M' = A' \otimes_A M$. Then A' is an R -flat algebra such that $\underline{m}A'$ is contained in the Jacobson radical of A' , and M' is a generically-free R -flat A' -module of finite type with $\text{hd}_{A'} M' \leq 1$, and the hypotheses (i), (ii), (iii) are carried over to A' -module M' . Furthermore, $A' \otimes_A A/I = A/I$ entails that $(M')_\#^t \simeq A' \otimes_A M_\#^t \simeq M_\#^t$ and hence $I_R((M')_\#^t) = I_R(M_\#^t)$. Consequently, replacing A by A' , we may assume that $\underline{m}A$ is contained in the Jacobson radical of A . The hypothesis (ii) entails that $\text{Ass } R \cap \text{Supp } R/I \cap R = \emptyset$ and hence $(I \cap R) - \text{depth } R > 0$. Therefore it follows from 1.4 that $\text{Sing}(M) - \text{depth } A > \text{Sing}(M_0) - \text{depth } A_0$. Since $\text{Sing}(M_0) - \text{depth } A_0 \geq d (= \text{rg } M)$ by hypothesis (i), we get $\text{Sing}(M) - \text{depth } A > d$, and therefore it follows from 1.2 (i) that

$$0 \rightarrow \bigwedge^d M \rightarrow M_\# \rightarrow M_\#^t \rightarrow 0$$

is exact and $M_\#^t$ is a perfect A -module with $\text{hd}_A M_\#^t = d + 1$. Since M as well as A are R -flat, so are $\bigwedge^d M$ and $M_\#$. Consequently, $\text{flat-dim}_R M_\#^t \leq 1$, and hence $\text{hd}_R M_\#^t \leq 1$ since $M_\#^t$ is an R -module of finite type. Let

$$0 \rightarrow R^m \xrightarrow{b} R^n \rightarrow M_\#^t \rightarrow 0$$

be a finite presentation of $M_\#^t$ as an R -module. Since $M_\#^t$ is a torsion module by

the hypothesis (ii), we must have $m = n$, and therefore $I_R(M_\#^t) = (\det b)$ and $\det b$ is not a zero-divisor in R .

Remark. If A is complete under the $\underline{m}$ -adic topology, then the hypothesis (iii) is a consequence of (i). Indeed, $M_\#^t$, being an A -module of finite type, is complete under the $\underline{m}$ -adic topology and $R/\underline{m} \otimes_R M_\#^t$ is a finite-dimensional vector space over $R/\underline{m}$ by (i), and therefore must be of finite type as an R -module.

2. Torsion differentials and deformations. We briefly recall here the notion of a (formal) Schlessinger deformation: Let k be a fixed perfect field, and X_0 a scheme over k . A deformation of X_0 is meant by a pair (R, X) where R is a complete noetherian local k -algebra with the residue field k , and X a flat R -scheme together with a Cartesian diagram:

$$\begin{array}{ccc} X_0 & \xrightarrow{i_X} & X \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \hookrightarrow & \text{Spec}(R) \end{array}$$

A morphism $(R, X) \rightarrow (R', X')$ of deformations of X_0 is a pair (b, ϕ) where $b: R \rightarrow R'$ is a local k -algebra map and $\phi: X' \rightarrow X$ is a morphism of schemes with a commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ \text{Spec}(R') & \xrightarrow{\text{Spec}(b)} & \text{Spec}(R) \end{array}$$

A formal deformation of X_0 is a pair $(R, \mathcal{X})$ where R is a complete noetherian local k -algebra with the maximal ideal $\underline{m}$, and $\mathcal{X}$ a R -flat $\underline{m}$ -adic formal scheme together with a Cartesian diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{i_{\mathcal{X}}} & \mathcal{X} \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \hookrightarrow & \text{Spf}(R) \end{array}$$

in the category of formal schemes. Thus if we set by $\underline{M}_{X_0}$, $\hat{\underline{M}}_{X_0}$ the category of deformations and formal deformations of X_0 respectively, then we get a canonical functor $\sim: \underline{M}_{X_0} \rightarrow \hat{\underline{M}}_{X_0}$ via $(R, X) \rightarrow (R, \hat{X})$ where $\hat{X}$ denotes the formal $\underline{m}$ -adic completion of X . We note that if X_0 is proper over k then the functor $\sim$ is a full-faithful imbedding, and that $\sim$ is an equivalence of categories in case when X_0 is a projec-

tive variety with $H^2(X_0, \underline{O}_{X_0}) = 0$ (for instance the case when X_0 is a complete curve) [5].

For each complete noetherian local k -algebra S , we set $\hat{M}_{X_0}(S)$ = the category of formal deformations of X_0 over S , the morphisms being the one which is the identity on S . We may note that every morphism in the category $\hat{M}_{X_0}(S)$ is necessarily an isomorphism, i.e. $\hat{M}_{X_0}(S)$ is a groupoid. Let $(R, \mathfrak{X})$ be a fixed formal deformation of X_0 . For any local map $R \rightarrow R'$, $(R', \mathfrak{X} \hat{\otimes}_R R')$ is a formal deformation of X_0 over R' , and therefore $(R, \mathfrak{X})$ induces a canonical map

$$(\widetilde{R, \mathfrak{X}}): \text{Hom}_{\text{local } k\text{-alg}}(R, R') \rightarrow [\hat{M}_{X_0}(R')]$$

where $[\hat{M}_{X_0}(R')]$ denotes the set of isomorphism classes of objects in $\hat{M}_{X_0}(R')$.

Definition 2.1⁽⁵⁾. A (formal) deformation (R, X) of X_0 is called a versal (formal) deformation of X_0 if every (formal) deformation of X_0 is induced from (R, X) . In other words, a versal deformation or a versal formal deformation of X_0 is a quasi-initial object in the category $\underline{M}_{X_0}, \hat{M}_{X_0}$ respectively. A versal (formal) deformation (R, X) such that

$$(R, X): \text{Hom}_{\text{local } k\text{-alg}}(R, k[\epsilon]) \rightarrow [\underline{M}_{X_0}(k[\epsilon])]$$

is a bijection is called a minimally versal (formal) deformation or a Schlessinger (formal) deformation of X_0 .

Any two (formal) Schlessinger deformations of X_0 are easily seen to be (noncanonically) isomorphic. A basic theorem on deformations is the following existence theorem due to M. Schlessinger [10] supplemented by a theorem of A. Grothendieck [5].

Theorem 2.2. *Let X_0 be a scheme over k . If $\text{Sing}(X_0) = \{x \in X_0 \mid X_0 \text{ is not smooth over } k \text{ at } x\}$ is proper over k , then there exists a formal Schlessinger deformation $(R, \mathfrak{X})$ of X_0 . If, furthermore, X_0 is projective over k and $H^2(X_0, \underline{O}_{X_0}) = 0$, then there exists a Schlessinger deformation of X_0 .*

Definition 2.3. Let $(R, \mathfrak{X})$ be a formal Schlessinger deformation of X_0 . We set $s_{X_0} = \dim R$. It is an invariant of X_0 .

Remark 2.4. Let X_0 be an affine scheme over k , and set $\Gamma(X_0, \underline{O}_{X_0}) = A_0$. We then have a canonical isomorphism $[\underline{M}_{X_0}(k[\epsilon])] \simeq D^1(A_0|k, A_0)$ via

$$(k[\epsilon], \text{Spec}(A)) \rightarrow (0 \rightarrow A_0 \xrightarrow{\epsilon} A \rightarrow A_0 \rightarrow 0)$$

where $D^1(A_0|k, A_0)$ = the isomorphism classes of commutative k -algebra exten-

(5) See footnote (3).

sions of A_0 by A_0 . Now assume that X_0 is a complete-intersection over k of dimension r with isolated nonsmooth points only, and choose a presentation

$$0 \rightarrow I \rightarrow P \rightarrow A_0 \rightarrow 0$$

where $P = k[x_1, \dots, x_n]$, and $I = (f_1, f_2, \dots, f_{n-r})$ is generated by a P -regular sequence. We then have the exact sequence

$$\mathrm{Der}_k(P, A_0) \rightarrow \mathrm{Hom}_{A_0}(I/I^2, A_0) \rightarrow D^1(A_0|k, A_0) \rightarrow 0$$

and $D^1(A_0|k, A_0)$ is a finite-dimensional vector space over k since A_0 has isolated nonsmooth points only. We set $s = \dim_k D^1(A_0|k, A_0)$, and choose $\phi_i: I/I^2 \rightarrow A_0$ ($i = 1, 2, \dots, s$) representing k -basis elements of $D^1(A_0|k, A_0)$, and then choose M_{ij} in P such that $\phi_i(f_j) = M_{ij} \pmod{I}$. If we set $R = k[[t_1, \dots, t_s]]$ and $A = R[X_1, \dots, X_n]/(F_1, F_2, \dots, F_{n-r})$ where

$$\left\{ \begin{array}{l} F_i = f_i + \sum_{j=1}^s t_j M_{ij} \\ R[X_1, \dots, X_n] = \text{the restricted formal power-series ring over the adic-ring } R \end{array} \right.$$

then A is R -flat, and indeed $(R, \mathrm{Spf}(A))$ is a formal Schlessinger deformation of X_0 (see [8] for its detail). In particular, we have $s_{X_0} = \dim_k D^1(A_0|k, A_0)$. We note that $F_1, F_2, \dots, F_{n-r}$ are all polynomials with coefficients in R and hence $\mathrm{Spf}(A)$ is "algebraic", i.e. $\mathrm{Spf}(A) = \widehat{\mathrm{Spec}(B)}$ where $B = R[X_1, \dots, X_n]/(F_1, \dots, F_{n-r})$. Now assume that A_0 is reduced so that A_0 is, since k is a perfect field, generically smooth over k . If we set $N = \mathrm{Im}(I/I^2 \xrightarrow{d} A_0 \otimes_P \Omega_P|_k)$, then $I/I^2 \xrightarrow{d} N$ is an isomorphism at each generic point of $\mathrm{Spec}(A_0)$, and therefore $\mathrm{Supp}(\mathrm{Ker} d)$ contains none of the generic points of $\mathrm{Spec}(A_0)$, and consequently $\mathrm{Hom}_{A_0}(N, A_0) \rightarrow \mathrm{Hom}_{A_0}(I/I^2, A_0)$ is an isomorphism. Then the exact sequence

$$0 \rightarrow N \rightarrow A_0 \otimes_P \Omega_P|_k \rightarrow \Omega_{A_0|k} \rightarrow 0$$

together with the fact that $A_0 \otimes_P \Omega_P|_k$ is A_0 -free module entails that $\mathrm{Ext}_{A_0}^1(\Omega_{A_0|k}, A_0) \rightarrow D^1(A_0|k, A_0)$ is an isomorphism. We thus conclude that if $X_0 = \mathrm{Spec}(A_0)$ is a reduced complete-intersection over k with isolated nonsmooth points only, then $s_{X_0} = \dim_k \mathrm{Ext}_{A_0}^1(\Omega_{A_0|k}, A_0)$.

Remark 2.5. Let $X_0 = \mathrm{Spec}(A_0)$ be a complete-intersection over k . Then we set $A_0 = k[x_1, x_2, \dots, x_n]/(f_1, f_2, \dots, f_r)$ where $f_1, f_2, \dots, f_r$ is a $k[x_1, \dots, x_n]$ -regular sequence. Then X_0 admits a generically-smooth deformation, i.e. X_0 can be "desingularized via deformation". Indeed, set $S = k[[t]]$ and $A = S[X_1, \dots, X_n]/(F_1, \dots, F_r)$ where $F_j = (1-t)f_j + tX_j$ for $j = 1, 2, \dots, r$. Since $F_j \equiv f_j \pmod{tA}$ and $f_1, f_2, \dots, f_r$ is an A/tA -regular sequence,

it follows that A is S -flat, i.e. $(S, \text{Spec}(A))$ is a deformation of X_0 over S . Now $A = S \otimes_{k[t]} A'$, where $A' = k[t, X_1, \dots, X_n]/(F_1, \dots, F_r)$, and $A'/(t-1)A' \simeq k[X_1, \dots, X_n]/(X_1, \dots, X_r) \simeq k[X_{r+1}, \dots, X_n]$. Therefore, $k(t) \otimes_{k[t]} A'$ is smooth over $k(t)$ and hence $k((t)) \otimes_{k[[t]]} A$ is smooth over $k((t))$, i.e. $(S, \text{Spec}(A))$ is a generically-smooth deformation of X_0 .

In this section we are interested in the deformation of complete curves. For this purpose we need the following lemma.

Lemma 2.6. *Let X_0 be a k -scheme of finite type with isolated nonsmooth points only. Assume that $H^2(X_0, \underline{\Omega}_{X_0}^*) = 0$ where*

$$\underline{\Omega}_{X_0}^* = \underline{\text{Hom}}_{\underline{\mathcal{O}}_{X_0}}(\underline{\Omega}_{X_0}, \underline{\mathcal{O}}_{X_0}).$$

Then

(i) $s_{X_0} = \dim_k H^1(X_0, \underline{\Omega}_{X_0}^*) + s_{U_0}$ where U_0 is any affine open subscheme of X_0 containing $\text{Sing}(X_0|k) = \{x \in X_0 \mid X_0 \text{ is not smooth over } k \text{ at } x\}$.

(ii) Let $(R, \mathfrak{X})$ be a formal Schlessinger deformation of X_0 . Then for any affine open $V_0 \subset X_0$, $(R, \mathfrak{X}|_{V_0})$ is a versal formal deformation of V_0 .

The above lemma is not difficult to prove, the main reason being that $H^2(X_0, \underline{\Omega}_{X_0}^*) = 0$ entails the vanishing of the global obstructions. In any case, a detailed argument can be found in [8] and thus we omit its proof here. Let X_0 be a complete curve over k , and let (R, X) be a Schlessinger deformation of X_0 . We note that the functor $\wedge: \underline{M}_{X_0} \rightarrow \hat{\underline{M}}_{X_0}$ is an equivalence of categories since X_0 is projective over k and $H^2(X_0, \underline{\mathcal{O}}_{X_0}) = 0$, and therefore a Schlessinger deformation of X_0 does exist. Assume that X_0 is a relative complete intersection over k . Then X , being R -flat, is a relative complete-intersection over R and hence $\text{Sing}(X|R) = \text{Supp } \omega_{X|R}^t$ where $\text{Sing}(X|R) = \{x \in X \mid X \text{ is not smooth over } R \text{ at } x\}$, $\omega_{X|R}^t = \text{Coker}(\underline{\Omega}_X|_R \rightarrow \omega_X|_R)$, and $\omega_X|_R$ is the dualizing sheaf of A . Grothendieck [7]. In view of this, we may provide the closed subset $\text{Sing}(X|R)$ with the closed subscheme structure defined by the ideal sheaf $\underline{I}_X(\omega_X^t)$, where $\underline{I}_X(E)$, for any coherent X -module E , stands for the ideal sheaf of $\underline{\mathcal{O}}_X$ given locally by the 1st Fitting ideal of E , i.e. for each affine open $U = \text{Spec}(A) \subset X$,

$$\Gamma(U, \underline{I}_X(E)) = \text{Im} \left(\det: \bigwedge^m N \otimes \bigwedge^m F \rightarrow A \right)$$

where $0 \rightarrow N \rightarrow F \rightarrow \Gamma(U, E) \rightarrow 0$ is a finite presentation of A -module $\Gamma(U, E)$ in which F is A -free, and m is the free rank of A -module F . We note that, for any coherent X -module E , the ideal sheaf $\underline{I}_X(E)$ annihilates E , and in particular ω_X^t is a coherent $\text{Sing}(X|R)$ -module. We are also interested in "the image of $\text{Sing}(X|R)$ under f ". Since $f: X \rightarrow \text{Spec}(R)$ is proper, $\Gamma(X, \omega_X^t|_R)$ is an R -module of finite type and $\text{Supp } \Gamma(X, \omega_X^t|_R) = f(\text{Sing}(X|R))$. We thus note that $f(\text{Sing}(X|R))$

$= \text{Supp } R/I_R(\Gamma(X, \omega_X^t|_R))$ where $I_R(\Gamma(X, \omega_X^t|_R))$ is the 1st Fitting ideal of the R -module $\Gamma(X, \omega_X^t|_R)$. The main purpose of this section is to establish the following.

Theorem 2.7(6). *Let X_0 be a reduced complete curve defined over k , and let (R, X) be a Schlessinger deformation of X_0 . Assume that X_0 is a relative complete intersection over k . We then have*

- (i) $s_{X_0} = 3g - 3 + \dim_k H^0(X_0, \underline{\Omega}_{X_0}^*)$ where $g = \dim_k H^1(X_0, \underline{\mathcal{O}}_{X_0})$.
- (ii) $\text{Sing}(X|R) \rightarrow \text{Spec}(R)$ is a finite morphism, and $\text{Sing}(X|R) = \text{Sing}_{x_1}(X|R) \cup \dots \cup \text{Sing}_{x_m}(X|R)$ is a disjoint union, where $\{x_1, x_2, \dots, x_m\} = \text{Sing}(X_0|k)$, and $\text{Sing}_x(X|R)$ stands for the connected component of $\text{Sing}(X|R)$ containing the point x .
- (iii) $\underline{\Omega}_{X|R}$ has no torsion, the canonical map $\omega_X^*|_R \rightarrow \underline{\Omega}_{X|R}^*$ is an isomorphism, and $I_R(\Gamma(X, \omega_X^t|_R))$ is an invertible ideal in R . In particular, $\underline{\Omega}_{X|R}^*$ is an invertible sheaf and $\text{codim}(f(\text{Sing}(X|R)) \text{ in } \text{Spec}(R)) = 1$.

Proof. (i) Let $U_0 = \text{Spec}(A_0)$ be an affine open subscheme of X_0 containing $\text{Sing}(X_0|k)$. Since X_0 is a relative complete-intersection over k , we may assume that $A_0 = k[x_1, x_2, \dots, x_{n+1}]/(f_1, f_2, \dots, f_n)$ where $f_1, f_2, \dots, f_n$ is a $k[x_1, \dots, x_{n+1}]$ -regular sequence. Since A_0 is reduced by hypothesis, we have

$$s_{U_0} = \dim_k \text{Ext}_{A_0}^1(\underline{\Omega}_{A_0|k}, A_0) = \dim_k H^0(U_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{\mathcal{O}}_{X_0}))$$

by 2.4. However, $\text{Sing}(X_0|k) \subset U_0$ and hence $\text{Supp } \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{\mathcal{O}}_{X_0}) \subset U_0$ and therefore

$$H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{\mathcal{O}}_{X_0})) = H^0(U_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{\mathcal{O}}_{X_0})).$$

It follows from 2.6 that

$$s_{X_0} = \dim_k H^1(X_0, \underline{\Omega}_{X_0}^*|_k) + \dim_k H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{\mathcal{O}}_{X_0})).$$

Thus it suffices to show that

$$\dim H^1(X_0, \underline{\Omega}_{X_0}^*|_k) + \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{\mathcal{O}}_{X_0})) = 3g - 3 + \dim H^0(X_0, \underline{\Omega}_{X_0}^*|_k)$$

i.e. $\chi(\underline{\Omega}_{X_0}^*|_k) = 3 - 3g + \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{\mathcal{O}}_{X_0}))$ where $\chi(\underline{E}) = \sum_i (-1)^i \dim H^i(X_0, \underline{E})$ for any coherent X_0 -module $\underline{E}$. Now consider the exact sequence

$$0 \rightarrow (\underline{\Omega}_{X_0|k})_t \rightarrow \underline{\Omega}_{X_0|k} \xrightarrow{i} \omega_{X_0|k} \rightarrow \omega_{X_0}^t|_k \rightarrow 0.$$

Since

(6) P. Deligne has proven recently a stronger version of 2.7(i). He has communicated to me that the formula 2.7(i) is valid for any curve which is a specialization of a nonsingular curve.

$$\underline{\text{Hom}}_{\underline{O}_{X_0}}((\underline{\Omega}_{X_0|k})_t, \underline{O}_{X_0}) = 0,$$

we get an isomorphism $i(\underline{\Omega}_{X_0|k})^* \simeq \underline{\Omega}_{X_0}^*$. Since X_0 is a Gorenstein scheme, $\omega_{X_0|k}$ is an invertible X_0 -module and it follows immediately $\underline{\text{Ext}}_{X_0}^i(i(\underline{\Omega}_{X_0|k}), \underline{O}_{X_0}) = 0$ for all $i > 0$. Consequently we get

$$(*) \quad \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{O}_{X_0}) \simeq \underline{\text{Ext}}_{X_0}^1((\underline{\Omega}_{X_0|k})_t, \underline{O}_{X_0}),$$

$$(**) \quad 0 \rightarrow \omega_{X_0|k}^* \rightarrow \underline{\Omega}_{X_0|k}^* \rightarrow \underline{\text{Ext}}_{X_0}^1(\omega_{X_0|k}^t, \underline{O}_{X_0}) \rightarrow 0.$$

The exact sequence (**) entails

$$\begin{aligned} \chi(\underline{\Omega}_{X_0|k}^*) &= \chi(\omega_{X_0|k}^*) + \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\omega_{X_0|k}^t, \underline{O}_{X_0})) \\ &= 3 - 3g + \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\omega_{X_0|k}^t, \underline{O}_{X_0})). \end{aligned}$$

On the other hand, the local duality over Gorenstein scheme together with the isomorphic (*) entails

$$\begin{aligned} \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\omega_{X_0|k}^t, \underline{O}_{X_0})) &= \dim H^0(X_0, \omega_{X_0|k}^t), \\ \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{O}_{X_0})) \\ &= \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1((\underline{\Omega}_{X_0|k})_t, \underline{O}_{X_0})) = \dim H^0(X_0, (\underline{\Omega}_{X_0|k})_t). \end{aligned}$$

However, X_0 is a relative complete-intersection over k and hence it follows from 1.3(ii) that $\dim H^0(X_0, (\underline{\Omega}_{X_0|k})_t) = \dim H^0(X_0, \omega_{X_0|k}^t)$. Consequently we obtain that

$$\dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{O}_{X_0})) = \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\omega_{X_0|k}^t, \underline{O}_{X_0}))$$

and hence $\chi(\underline{\Omega}_{X_0|k}^*) = 3 - 3g + \dim H^0(X_0, \underline{\text{Ext}}_{X_0}^1(\underline{\Omega}_{X_0|k}, \underline{O}_{X_0}))$.

(ii) $\text{Sing}(X|R)$ being a closed subscheme of X , $\text{Sing}(X|R) \rightarrow \text{Spec}(R)$ is proper. Since $\text{Sing}(X|R) \cap X_0 = \text{Sing}(X_0|k)$ is a finite set, i.e. $k \otimes_R \text{Sing}(X|R)$ consists of a finite number of points, it follows that $\text{Sing}(X|R) \rightarrow \text{Spec}(R)$ is a finite morphism, and in particular $\text{Sing}(X|R)$ is affine. Set $\text{Sing}(X|R) = \text{Spec}(R')$. Since R is a complete local ring and R' is integral over R of finite type, we conclude that R' is a finite direct product of local rings, i.e. $R' = R_1 \times \cdots \times R_r$, and therefore $\text{Sing}(X|R) = V_1 \cup \cdots \cup V_r$ (disjoint union) where $V_i = \text{Spec}(R_i)$ and R_i is a local ring. Since the underlying space of $k \otimes_R \text{Sing}(X|R)$ is $\text{Sing}(X_0|k) = \{x_1, x_2, \dots, x_m\}$, we conclude that $r = m$ and $V_i = \text{Sing}_{x_i}(X|R)$.

(iii) Since X_0 is a relative complete-intersection over k and X is R -flat, it

follows that X is a relative complete-intersection over R . Since X_0 is 1-dimensional, we have $H^2(X_0, \underline{\Omega}_{X_0}^*) = 0$ and therefore, for each affine open $U_0 \subset X_0$, $\hat{X}|_{U_0}$ is a versal formal deformation of U_0 . On the other hand, since U_0 is a relative complete-intersection over k , U_0 admits a generically smooth deformation by 2.5, and consequently we conclude that a Schlessinger deformation X is generically smooth over R since R is a formal power-series ring so that $\text{Spec}(R)$ is irreducible. Since $\text{Sing}(X_0|k)$ is a finite set, we may choose an affine open $U \subset X$ containing $\text{Sing}(X_0|k)$. Let $y \in \text{Sing}(X|R)$. Since $\{\bar{y}\} \cap \text{Sing}(X_0|k) \neq \emptyset$ so that $\{\bar{y}\} \cap U \neq \emptyset$, we find that $y \in U$. Therefore $\text{Sing}(X|R) \subset U$, i.e. $\text{Supp } \omega_{X|R}^t \subset U$. Consequently we have that $\Gamma(X, \omega_{X|R}^t) = \Gamma(U, \omega_{X|R}^t)$ and therefore we may replace X by an affine open $U = \text{Spec}(A)$ containing $\text{Sing}(X_0|k)$. Then A -module $\Omega_{A|R}$ has the following properties:

(a) $\text{Sing}(k \otimes_R \Omega_{A|R}) = \text{Sing}(\Omega_{A_0|k}) = \text{Sing}(X_0|k)$ consists of a finite number of Cohen-Macaulay maximal points in $\text{Spec}(k \otimes_R A) = \text{Spec}(A_0)$ of dimension $\geq 1 = \text{rg } \Omega_{A|R}$.

(b) $\Omega_{A|R}$ is R -flat (since X is R -flat and is a relative complete-intersection over R), and $K \otimes_R \Omega_{A|R}$ is $K \otimes_R A$ -projective (since X is generically smooth over R).

(c) $\omega_{A|R}^t (= \Gamma(U, \omega_{X|R}^t) = \Gamma(X, \omega_{X|R}^t))$ is an R -module of finite type.

It follows from 1.5 that $\Omega_{A|R}$ has no torsion, $\omega_{A|R}^t$ is a perfect A -module with $\text{hd}_A \omega_{A|R}^t = 2$, and $I_R(\Gamma(X, \omega_{X|R}^t)) = I_R(\omega_{A|R}^t)$ is an invertible ideal in R . Since $\text{Sing}(X|R) \subset U$ and $\Omega_{A|R} = \Gamma(U, \Omega_{X|R})$ has no torsion, it follows that $\Omega_{X|R}$ has no torsion, i.e.

$$0 \rightarrow \Omega_{X|R} \rightarrow \omega_{X|R} \rightarrow \omega_{X|R}^t \rightarrow 0$$

is exact. Since $\text{depth } \omega_{A|R}^t = 2$, we have $\underline{\text{Ext}}_X^1(\omega_{X|R}^t, \mathcal{O}_X) = 0$ for $i = 0, 1$, and therefore $\omega_{X|R}^* \rightarrow \Omega_{X|R}^*$ is an isomorphism. This completes our proof.

Remark. For any reduced complete curve X_0 , we define $l_{X_0} = s_{X_0} - \dim H^1(X_0, \underline{\Omega}_{X_0}^*)$ which measures the local contribution to the deformations of X_0 . We note, as a consequence of 2.7(i), that if X_0 is a relative complete-intersection over k , then $l_{X_0} = \dim H^0(X_0, (\Omega_{X_0|k})_t) = \dim H^0(X_0, \omega_{X_0|k}^t)$.

3. Local connection and rigid singularities. Let k be a fixed perfect field as before, and X_0 a reduced k -scheme.

Definition 3.1. X_0 is said to be a "limit of smooth k -schemes" or X_0 is a specialization of a smooth k -scheme if there exists a deformation (R, X) of X_0 such that $X \otimes_R K$ is smooth over K , where K is the total ring of fractions of R . We say that X_0 is rigid if every deformation of X_0 over $k[\epsilon]$ is trivial, i.e. for any deformation $(k[\epsilon], X)$ of X_0 we have $X \simeq X_0 \otimes_k k[\epsilon]$. We note that if $X_0 = \text{Spec}(A_0)$ is a reduced affine k -scheme, X_0 is rigid if and only if $\text{Ext}_{A_0|k}^1(\Omega_{A_0|k}, A_0)$

$= 0$ (see 2.4). We also note that a rigid k -scheme X_0 can never be a specialization of a smooth k -scheme unless X_0 itself is smooth over k .

Every reduced curve is conjecturally a specialization of a smooth curve [3]. We have also seen in 2.5 that every affine k -scheme which is a complete-intersection is a specialization of a smooth k -scheme. On the other hand, there exists a 4-dimensional affine variety which can never be a specialization of a smooth variety ([4], [11]). The main purpose of this section is to provide an example of rigid 2-dimensional singularity. To clarify our motivation, we consider the phenomena of local connection under specialization, for which the author is indebted to A. Grothendieck.

For any local ring A , we denote by $\text{Spex}(A)$ the open subscheme of $\text{Spec}(A)$ deleting the maximal point. A variance of the following lemma is contained in [6]. However, we give a complete proof for the convenience of the readers.

Lemma 3.2. *Let A be a noetherian local ring with the maximal ideal $\underline{m}$. If $\text{depth } A \geq 2$ and $H_{\underline{m}}^2(A)$ is coherent, then for any A -regular element $t \in \underline{m}$, $\text{Spex}(A/tA)$ is connected.*

Proof. Since $\text{depth } A \geq 2$, we have $H_{\underline{m}}^i(A) = 0$ for $i = 0, 1$, and hence $A \rightarrow H^0(\text{Spex}(A), A)$ is an isomorphism. The exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{t^m} & A & \longrightarrow & A/t^m A \longrightarrow 0 \\ & & \downarrow t^{m-n} & & \parallel & & \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{t^n} & A & \longrightarrow & A/t^n A \longrightarrow 0 \end{array} \quad (m \geq n)$$

entails the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & A/t^m A & \rightarrow & H^0(\text{Spex}(A), A/t^m A) & \rightarrow & {}_m H^1(\text{Spex}(A), A) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow t^{m-n} \\ 0 & \rightarrow & A/t^n A & \rightarrow & H^0(\text{Spex}(A), A/t^n A) & \rightarrow & {}_n H^1(\text{Spex}(A), A) \rightarrow 0 \end{array}$$

where for any A -module E we set ${}_n E = \{x \in E \mid t^n x = 0\} = \text{Hom}_A(A/t^n A, E)$. Sending to limit we get the exact sequence

$$0 \rightarrow \varprojlim_v A/t^v A \rightarrow \varprojlim_v H^0(\text{Spex}(A), A/t^v A) \rightarrow \varprojlim_v {}_v H^1(\text{Spex}(A), A).$$

Now $H_{\underline{m}}^2(A) = H^1(\text{Spex}(A), A)$ is coherent by hypothesis and hence there exists an integer n such that ${}_n H^1(\text{Spex}(A), A) = {}_m H^1(\text{Spex}(A), A)$ for all $m \geq n$. Consequently,

$${}_m H^1(\operatorname{Spex}(A), A) \xrightarrow{t^n} {}_{m-n} H^1(\operatorname{Spex}(A), A)$$

is a zero-map for all $m \geq n$, and hence

$$\varprojlim_v {}_v H^1(\operatorname{Spex}(A), A) = 0,$$

and therefore $\varprojlim_v A/t^v A \rightarrow \varprojlim_v H^0(\operatorname{Spex}(A), A/t^v A)$ is an isomorphism. In other words, $\hat{A} \rightarrow \Gamma(\widehat{\operatorname{Spex}(A)})$ is an isomorphism, where $\widehat{\operatorname{Spex}(A)}$ is the formal completion of $\operatorname{Spex}(A)$ along the closed subscheme $\operatorname{Spex}(A/tA)$. Consequently, $\Gamma(\widehat{\operatorname{Spex}(A)})$ and therefore $\Gamma(\operatorname{Spex}(A/tA))$ has no nontrivial idempotent elements and hence $\operatorname{Spex}(A/tA)$ is connected.

Proposition 3.3. *Let A_0 be a reduced noetherian local ring with $\dim A_0 \geq 2$. If there exists a noetherian local ring A which is a homomorphic image of a regular local ring, and an A -regular element t in the maximal ideal of A such that $A/tA \simeq A_0$ and A_t is regular, then A_0 is equidimensional and $\operatorname{Spex}(\hat{A}_0)$ is connected, where $\hat{A}_0$ stands for the completion of A_0 with respect to the maximal ideal of A_0 .*

Proof. Assume that A is normal. Then tA is an unmixed ideal in A and hence $A_0 \simeq A/tA$ is equidimensional. Let us denote by $\hat{A}$ the completion of A with respect to the maximal ideal $\underline{m}$ of A , so that t is $\hat{A}$ -regular and $\hat{A}/t\hat{A} \simeq \hat{A}_0$. We note that $\operatorname{depth} \hat{A} = \operatorname{depth} A \geq 2$ since A is normal and $\dim A \geq 2$. If $H_{\underline{m}\hat{A}}^2(\hat{A})$ is coherent, it follows from the above lemma that $\operatorname{Spex}(\hat{A}_0) = \operatorname{Spex}(\hat{A}/t\hat{A})$ is connected. We note that, for every $i > 0$,

$$\begin{aligned} H_{\underline{m}\hat{A}}^i(\hat{A}) &= \varprojlim_v \operatorname{Ext}_A^i(\hat{A}/\underline{m}^v \hat{A}, \hat{A}) \\ &= \varprojlim_v \operatorname{Ext}_A^i(A/\underline{m}^v, A) \otimes_{\hat{A}} \hat{A} = \varprojlim_v \operatorname{Ext}_A^i(A/\underline{m}^v, A) = H_{\underline{m}}^i(A), \end{aligned}$$

and thus it suffices to show that $H_{\underline{m}}^2(A)$ is coherent. Now let x be a point of $\operatorname{Spec}(A)$ such that $\dim \{\bar{x}\} \leq 1$. Since $\dim A \geq 3$ we must have $\dim A_x \geq 2$ and hence $\operatorname{depth} A_x \geq 2$ since A is normal. It follows from Grothendieck's criteria [6] that $H_{\underline{m}}^2(A)$ is coherent. Thus it suffices to show that A is normal. We use Serre's criteria. Let x be a point in $\operatorname{Spec}(A)$ such that $\dim A_x = 1$. If $x \notin V(t)$, then A_x is regular by hypothesis. If $x \in V(t)$, then $\dim (A_0)_x = \dim A_x/tA_x = 0$ so that $(A_0)_x$ is a field since $(A_0)_x$ is reduced, which entails that A_x is regular. Now let x be a point in $\operatorname{Spec}(A)$ such that $\dim A_x \geq 2$. If $x \notin V(t)$, then A_x is regular and hence $\operatorname{depth} A_x = \dim A_x \geq 2$. Let $x \in V(t)$. Then $t \in \underline{m}_x$ and it is an A_x -regular element such that $\operatorname{depth} A_x/tA_x = \operatorname{depth} (A_0)_x > 0$ since A_0 is reduced, and therefore $\operatorname{depth} A_x \geq 2$. This completes our proof.

Corollary 3.4. *Let X_0 be a reduced algebraic k -scheme which is a specialization of a smooth k -scheme, where k is an algebraically closed field. Then for*

$x \in X_0$ with $\dim \underline{O}_{X_0, x} \geq 2$, $\text{Spex}(\underline{O}_{X_0, x})$ is connected. In particular, every point $x \in X_0$ such that $\dim \underline{O}_{X_0, x} \geq 2$ and $\text{Spex}(\underline{O}_{X_0, x})$ is normal is unibranch.

Proof. We may and shall assume that X_0 is not smooth over k . Let (R, X) be a deformation of X_0 such that $K \otimes_R X$ is smooth over K where K is the total ring of fractions of R . We may assume that $R = k[[t]]$. Indeed, $D = \{s \in \text{Spec}(R) \mid \kappa(s) \otimes_R X \text{ is smooth over } \kappa(s)\}$ is nonempty open in $\text{Spec}(R)$ and does not contain the maximal point of $\text{Spec}(R)$, and therefore there exists $s \in D$ such that $\dim \{\bar{s}\} = 1$. Thus if we set $\underline{p}$ to be the prime ideal in R corresponding to s , then $\dim R/\underline{p} = 1$ and $\kappa(s) \otimes_R X$ is smooth over $\kappa(s)$. Let R' be the normalization of $R/\underline{p}$. Since $R/\underline{p}$ is complete with $\dim R/\underline{p} = 1$ and k is algebraically closed, we conclude that $R' = k[[t]]$. Then $(R', R' \otimes_R X)$ is a deformation of X_0 such that $k[[t]] \otimes_{R'} (R' \otimes_R X)$ is smooth over $k((t))$. We thus assume that $R = k[[t]]$. Now let $x \in X_0 \subset X$ be a point such that $\dim \underline{O}_{X_0, x} \geq 2$. Since t is $\underline{O}_{X, x}$ -regular such that $\underline{O}_{X, x}/t\underline{O}_{X, x} \simeq \underline{O}_{X_0, x}$ and $(\underline{O}_{X, x})_t \simeq k((t)) \otimes_{k[[t]]} \underline{O}_{X, x}$ is smooth over $k((t))$ and hence is regular, it follows from 3.3 that $\text{Spex}(\underline{\hat{O}}_{X_0, x})$ is connected. As for the second statement, let $\dim \underline{O}_{X_0, x} \geq 2$ and assume that $\text{Spex}(\underline{O}_{X_0, x})$ is normal. It suffices to show that $\text{Spex}(\underline{O}_{X_0, x})$ is connected if and only if $\underline{O}_{X_0, x}$ is unibranch. We set $A = \underline{O}_{X_0, x}$ and A' the normalization of A . Since A'/A as an A -module is annihilated by some power of the maximal ideal of A , we get the exact sequence

$$0 \rightarrow \hat{A} \rightarrow \hat{A}' \rightarrow A'/A \rightarrow 0$$

and hence $(\hat{A})_f \rightarrow (\hat{A}')_f$ is an isomorphism for every f in the maximal ideal of A . Consequently the morphism $\pi: \text{Spec}(\hat{A}') \rightarrow \text{Spec}(\hat{A})$ induces the isomorphism $\text{Spex}(\hat{A}') \xrightarrow{\sim} \text{Spex}(\hat{A})$ where $\text{Spex}(\hat{A}') = \text{Spec}(A) - \pi^{-1}\{\hat{m}\}$, $\hat{m}$ = the maximal ideal of $\hat{A}$. However, A' is semilocal and hence $\hat{A}' = A'_1 \times \cdots \times A'_n$, where A'_i is a complete local normal domain, and hence $\text{Spex}(\hat{A}') = \text{Spex}(A'_1) \cup \cdots \cup \text{Spex}(A'_n)$ (disjoint union). We note that $\text{depth } A'_i \geq 2$ since it is normal with dimension ≥ 2 , and therefore $\text{Spex}(A'_i)$ is connected by Hartshorne's lemma. It follows that $\text{Spex}(\hat{A}) \simeq \text{Spex}(A'_1) \cup \cdots \cup \text{Spex}(A'_n)$ is connected if and only if $n = 1$. This completes our proof.

It is now clear how to construct an algebraic variety of dimension ≥ 2 which can never be a limit of smooth varieties: Let X be a normal variety. Given a 0-dimensional closed subscheme $Y \subset X$, there exists an algebraic variety $X_{\{Y\}}$ together with the birational morphism $\pi: X \rightarrow X_{\{Y\}}$ such that

(i) $\pi(Y)$ consists of one point z , and $X - Y \xrightarrow{\pi} X_{\{Y\}} - \{z\}$ is an isomorphism.

(ii) The ideal sheaf I_Y of Y in $\underline{O}_X$ is equal to the conductor ideal sheaf of π .

If $\dim X \geq 2$ and Y consists of more than one point, then $\dim \underline{O}_{X_{\{Y\}}, z} \geq 2$ and z is of multiple branches, and consequently $X_{\{Y\}}$ cannot be a specialization of a

smooth variety. Motivated by this fact, we can now construct, for any integer $n \geq 2$, a rigid n -dimensional affine variety with an isolated singularity.

Lemma 3.5. *Set*

$$A = K[[X_1, \dots, X_n]] \times_k k[[Y_1, \dots, Y_n]] \\ \cong K[[X_1, \dots, X_n, Y_1, \dots, Y_n]] / (X_i Y_j \mid 1 \leq i, j \leq n).$$

If E is an A -module of finite type such that $\text{depth } E > 0$ and $\text{depth } E/(x_i, y_j)E > 0$ for all i, j , then $\text{Ext}_A^1(\hat{\Omega}_A|_k, E) = 0$, where $\hat{\Omega}_A|_k$ stands for the A -module of continuous k -differentials of A . In particular, $\text{Ext}_A^1(\hat{\Omega}_A|_k, A) = 0$ provided $n \geq 2$.

Proof. We have the exact sequence

$$0 \rightarrow N \rightarrow A dX_1 \oplus \dots \oplus A dX_n \oplus A dY_1 \oplus \dots \oplus A dY_n \rightarrow \hat{\Omega}_A|_k \rightarrow 0$$

where N as A -module is generated by the set $\{\omega_{ij} (= d(X_i Y_j)) = y_j dX_i + x_i dY_j \mid 1 \leq i, j \leq n\}$. Let $\lambda: N \rightarrow E$ be any A -linear map. Since we have the relations

$$(*) \quad \begin{cases} x_k \omega_{ij} = x_i \omega_{kj} \\ y_k \omega_{ij} = y_j \omega_{ik} \end{cases} \quad \text{for all } i, j, k,$$

we must have that

$$\begin{cases} (x_1, \dots, x_n) \lambda(\omega_{ij}) \subset x_i E \\ (y_1, \dots, y_n) \lambda(\omega_{ij}) \subset y_j E, \end{cases}$$

i.e. $\underline{m} \lambda(\omega_{ij}) \subset (x_i, y_j)E$ where $\underline{m} = (x_1, \dots, x_n, y_1, \dots, y_n)$. Since $\text{depth } E/(x_i, y_j)E > 0$ by hypothesis, we have $\underline{m} \notin \text{Ass } E/(x_i, y_j)E$ and therefore $\lambda(\omega_{ij}) \in (x_i, y_j)E$.

Hence we set

$$(**) \quad \lambda(\omega_{ij}) = y_j \alpha_{ij} + x_i \beta_{ij}$$

where α_{ij}, β_{ij} are the elements in E . Then $(*)$ entails again that

$$\begin{cases} x_k (y_j \alpha_{ij} + x_i \beta_{ij}) = x_i (y_j \alpha_{kj} + x_k \beta_{kj}) \\ y_k (y_j \alpha_{ij} + x_i \beta_{ij}) = y_j (y_k \alpha_{ik} + x_i \beta_{ik}), \end{cases}$$

i.e.

$$\begin{cases} x_k x_i \beta_{ij} = x_i x_k \beta_{kj} \\ y_j y_k \alpha_{ij} = y_j y_k \alpha_{ik}, \end{cases}$$

i.e.

$$\begin{cases} x_k x_i (\beta_{ij} - \beta_{kj}) = 0 \\ y_j y_k (\alpha_{ij} - \alpha_{ik}) = 0 \end{cases} \quad \text{for all } i, j, k.$$

If $(x_1, \dots, x_n)E \neq 0$, then $\text{depth}(x_1, \dots, x_n)E > 0$ since $\text{depth } E > 0$ by hypothesis, and therefore there exists an $(x_1, \dots, x_n)E$ -regular element in $\underline{m}$, say let it be x_1 . Then $x_1 x_i (\beta_{ij} - \beta_{kj}) = 0$ for all i entails that $x_i (\beta_{ij} - \beta_{kj}) = 0$ for all i , i.e. $x_i \beta_{ij} = x_i \beta_{kj}$ and therefore we may assume that $\beta_{ij} = 0$ for all i, j . Therefore, in either situation, we may assume in the expression (**) that β_{ij} depends only on j . Likewise we may assume that α_{ij} depends only on i . We thus conclude that there exist $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ in E such that $\lambda(\omega_{ij}) = y_j \alpha_i + x_i \beta_j$. This simply means that

$$\text{Ext}_A^1(\hat{\Omega}_{A|k}, E) = \text{Coker}(\text{Hom}_A(F, A) \rightarrow \text{Hom}_A(N, A)) = 0$$

where $F = A dX_1 \oplus \dots \oplus A dX_n \oplus A dY_1 \oplus \dots \oplus A dY_n$. The second statement is immediate since $\text{depth } A > 0$ provided $n \geq 1$ and $\text{depth } A/(x_i, y_j) > 0$ provided $n \geq 2$.

Corollary 3.6. *Let $X = A_k^n$ (= the affine n -space over k) where $n \geq 2$, and let $Y = \{z_1, z_2\}$ be a reduced closed subscheme of X consisting of two distinct k -rational points z_1, z_2 . Then $X_{\{Y\}} = \text{Spec}(k + \underline{m}_{z_1} \cap \underline{m}_{z_2})$ is a rigid irreducible n -dimensional affine variety with the isolated nonsmooth point.*

Proof. The morphism $\pi: X \rightarrow X_{\{Y\}}$ induces by definition an isomorphism $X - Y \rightarrow X_{\{Y\}} - \{z\}$ where $z = \pi(Y)$, and therefore $X_{\{Y\}}$ is smooth over k except at the point z . Set $X_{\{Y\}} = \text{Spec}(A)$. Since A is smooth over k except at the point z , $\Omega_{A|k}$ is locally-free everywhere except at the point z and therefore $\text{Ext}_A^1(\Omega_{A|k}, A)$ is an A -module of finite type annihilated by some power of $\underline{m}_z$. Consequently the canonical maps

$$\text{Ext}_A^1(\Omega_{A|k}, A) \rightarrow \text{Ext}_{A_z}^1(\Omega_{A_z|k}, A_z) \rightarrow \text{Ext}_{\hat{A}_z}^1(\widehat{\Omega_{A_z|k}}, \hat{A}_z)$$

are all isomorphisms, where $\widehat{\Omega_{A_z|k}}$ stands for the module of continuous k -differentials of $\hat{A}_z$. However, we have by definition the exact sequence

$$0 \rightarrow A_z \rightarrow \underline{Q}_{X,z} \xrightarrow{z_1 - z_2} k \rightarrow 0$$

where $\underline{Q}_{X,z} = \varinjlim_{U \supset Y} \Gamma(U, \underline{Q}_X)$ is the semilocal ring with two distinct maximal ideals corresponding to the points z_1, z_2 . Passing to the completion we get the exact sequence

$$0 \rightarrow \hat{A}_z \rightarrow \hat{\underline{Q}}_{X,z_1} \times \hat{\underline{Q}}_{X,z_2} \xrightarrow{z_1 - z_2} k \rightarrow 0$$

and therefore $\hat{A}_z \simeq \hat{O}_{X,z_1} \times_k \hat{O}_{X,z_2} \simeq k[[X_1, \dots, X_n]] \times_k k[[Y_1, \dots, Y_n]]$. It follows from 3.5 that $\text{Ext}_A^1(\Omega_A|_k, A) \simeq \text{Ext}_{\hat{A}_z}^1(\widehat{\Omega_{\hat{A}_z}|_k}, \hat{A}_z) = 0$ provided $n \geq 2$, which completes our proof.

4. **Comments.** In this paper we have studied the deformations of projective curves which is locally a complete-intersection. Admittedly it is the simplest situation since the local obstructions for the prolongations of deformation do vanish by virtue of a property of complete-intersection [1]. However, we have examined the deformation problems of several curves which is not a complete-intersection including the ordinary multiple point with mutually transversal tangents. Based on 2.7 and a few empirical results, we make the following conjectures.

Let k be a fixed perfect field (or an algebraically closed field if necessary) of any characteristic, and let X_0 be an irreducible reduced complete curve over k , and let (R, X) be a Schlessinger deformation of X_0 . The complete local k -algebra R is in general not a formal power-series ring even when X_0 has ordinary multiple points (with mutually transversal tangents) only (see [9]).

$$(I) \quad s_{X_0} = 3g - 3 + \dim_k H^0(X_0, \underline{\Omega}_{X_0}^*) \quad \text{where } g = \dim_k H^1(X_0, \underline{O}_{X_0}).$$

One notes that the above formula coincides with the original conjecture of P. Deligne [3] in the case when the base field k is of characteristic zero. In view of 2.6, the conjecture (I) is entirely of local nature. Indeed, we set, for each point $x \in X_0$,

$$s_x = s_{\text{Spec}(\underline{O}_{X_0, x})}.$$

Then $s_x = 0$ at every smooth point x , and 2.6 entails that $s_{X_0} = \dim_k H^1(X_0, \underline{\Omega}_{X_0}^*) + \sum_x s_x$. Therefore,

$$\begin{aligned} s_{X_0} &= 3g - 3 + \dim_k H^0(X_0, \underline{\Omega}_{X_0}^*) \\ &\Leftrightarrow 3g - 3 + \dim_k H^0(X_0, \underline{\Omega}_{X_0}^*) = \dim_k H^1(X_0, \underline{\Omega}_{X_0}^*) + \sum_x s_x \\ &\Leftrightarrow \chi(\underline{\Omega}_{X_0}^*) = 3 - 3g + \sum_{x \in X_0} s_x. \end{aligned}$$

If we denote by $\pi: X'_0 \rightarrow X_0$ the normalization of X_0 , it follows from the Riemann-Roch theorem that $\chi(\underline{\Omega}_{X'_0}^*) = 3 - 3(g - \delta_{X_0})$ where

$$\delta_{X_0} = \dim_k H^0(X_0, \pi_* \underline{O}_{X'_0} / \underline{O}_{X_0}), \quad \underline{\Omega}_{X'_0}^* = \underline{\text{Hom}}_{\underline{O}_{X'_0}}(\underline{\Omega}_{X'_0}, \underline{O}_{X'_0}).$$

Now let

$$\mathcal{C}_{X_0} = \underline{\text{Hom}}_{\underline{O}_{X_0}}(\pi_* \underline{O}_{X'_0}, \underline{O}_{X_0})$$

be the conductor ideal sheaf of X_0 . We then have canonical inclusion maps

$$(*) \quad \begin{cases} 0 \rightarrow \underline{\text{Hom}}_{\underline{\mathcal{O}}_{X_0}}(\pi_* \underline{\Omega}_{X'_0}, \underline{\mathcal{C}}_{X_0}) \rightarrow \pi_*(\underline{\Omega}_{X'_0}^*) \\ 0 \rightarrow \underline{\text{Hom}}_{\underline{\mathcal{O}}_{X_0}}(\pi_* \underline{\Omega}_{X'_0}, \underline{\mathcal{C}}_{X_0}) \rightarrow \underline{\Omega}_{X_0}^*. \end{cases}$$

For each point $x \in X_0$, we set

$$d_x = \dim_k [\text{Coker}(\underline{\text{Hom}}_{\underline{\mathcal{O}}_{X_0}}(\pi_* \underline{\Omega}_{X'_0}, \underline{\mathcal{C}}_{X_0}) \rightarrow \underline{\Omega}_{X_0}^*)]_x,$$

$$c_x = \dim_k (\pi_* \underline{\mathcal{O}}_{X'_0} / \underline{\mathcal{C}}_{X_0})_x.$$

We note that, for each point $x \in X_0$,

$$\dim_k [\text{Coker}(\underline{\text{Hom}}_{\underline{\mathcal{O}}_{X_0}}(\pi_* \underline{\Omega}_{X'_0}, \underline{\mathcal{C}}_{X_0}) \rightarrow \pi_*(\underline{\Omega}_{X'_0}^*))]_x = \dim_k (\pi_* \underline{\mathcal{O}}_{X'_0} / \underline{\mathcal{C}}_{X_0})_x = c_x,$$

and that $d_x = 0 = c_x$ at every smooth point x . Now the inclusion maps $(*)$ entails that $\chi(\underline{\Omega}_{X'_0}^*) = \chi(\underline{\Omega}_{X_0}^*) + \sum_x c_x - \sum_x d_x$, and therefore we get $3 - 3(g - \delta_{X_0}) = \chi(\underline{\Omega}_{X_0}^*) + \sum_x (c_x - d_x)$, i.e. $\chi(\underline{\Omega}_{X_0}^*) = 3 - 3g + 3\delta_{X_0} + d_{X_0} - c_{X_0}$ where $d_{X_0} = \sum_x d_x$, $c_{X_0} = \sum_x c_x = \deg \underline{\mathcal{C}}_{X_0}$. Therefore, $\chi(\underline{\Omega}_{X_0}^*) = 3 - 3g + \sum_x s_x \Leftrightarrow \sum_x s_x = 3\delta_{X_0} + d_{X_0} - c_{X_0}$, i.e. the conjecture (I) holds if and only if $\sum_x s_x = \sum_x (3\delta_x + d_x - c_x)$. Taking complete curves with a single singular point, we conclude that the conjecture (I) holds if and only if $s_x = 3\delta_x + d_x - c_x$ for every (singular) point x . Therefore one can give the local reformulation of the above conjecture as follows.

(I)' Let A_0 be any 1-dimensional geometric local domain over k . Set A'_0 to be the normalization of A_0 , and $\mathcal{C}$ the conductor ideal of A . Then $s_{\text{Spec}(A_0)} = 3\delta + d - c$ where

$$\delta = \dim_k A'_0 / A_0, \quad c = \dim_k A'_0 / \mathcal{C},$$

$$d = \dim_k \text{Coker}(\text{Der}_k(A'_0, \mathcal{C}) \rightarrow \text{Der}_k(A_0, A_0)).$$

Remark. It follows from a well-known inequality $\delta \leq c \leq 2\delta$ that $\delta + d \leq 3\delta + d - c \leq 2\delta + d$, and in particular $3\delta + d - c = 0$ if and only if $\delta = 0$, i.e. A_0 is smooth over k . We also note that if A_0 is Gorenstein then $3\delta + d - c = \delta + \dim \Omega_{A'_0/A_0}$.

(II) Let X_0 be Gorenstein. Then for each integer $0 \leq p \leq \dim \omega_{X_0}^t|_k$, $I_R(\wedge^p \omega_{X|_R}^t)$ is an unmixed ideal of codimension p , where the exterior power is taken over R .

Needless to say, (II) is equally a local problem. (II) implies in particular that for every integer $0 \leq m \leq \dim \omega_{X_0}^t|_k$, $\{z \in \text{Spec}(R) \mid \dim \omega_{X(z)}^t|_{\kappa(z)} = m\}$ is a nonempty locally closed subset of $\text{Spec}(R)$. In case when (II) is correct, one can make the following conjecture: We set, for each complete curve Y , $l_Y = s_Y - \dim H^1(Y, \Omega_Y^*)$. One may note that if Y is Gorenstein and the conjecture (I) is valid, then $l_Y = \dim \omega_Y^t$.

(III) $l: \text{Spec}(R) \rightarrow N$ given by $z \rightarrow l_{X(z)}$ is an upper semicontinuous function, and for each integer $0 \leq p \leq l_{X_0}$, $\Sigma(p) = \{z \in \text{Spec}(R) \mid l_{X(z)} \geq p\}$ is an equidimensional closed subset of codimension p , where $X(z) = \kappa(z) \otimes_R X$.

One may note that (III) certainly would imply that every complete curve is a specialization of a nonsingular curve. The above conjecture (III) has been verified for curves with ordinary multiple points with mutually transversal tangents (see [9]).

(IV) Assume that X_0 is affine. Given a point $z \in \text{Spec}(R)$, $(R_z, R_z \otimes_R X)$ is a versal deformation of the curve $X(z) = \kappa(z) \otimes_R X$.

This conjecture is also correct for curves with ordinary multiple points with mutually transversal tangents. We remark that $(R_z, R_z \otimes_R X)$ is not in general a Schlessinger deformation of $X(z)$ and that (IV) is false for complete curves. The conjecture (IV) will have the following significant consequence:

(IV)' If $X_1 \rightarrow X_0$ is a nontrivial deformation of complete curves, then $l_{X_1} < l_{X_0}$.

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