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A NOTE ON THE GEOMETRIC MEANS OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

BY

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ABSTRACT. Let $f(z_1, \cdots, z_n)$ be an entire function of n (≥ 2) complex variables. Recently Agarwal [Trans. Amer. Math. Soc. 151 (1970), 651–657] has obtained certain results involving geometric mean values of f. In this paper we have constructed examples to contradict some of the results of Agarwal and have thereafter given improvements and modifications of his results.

1. **Introduction.** Let

$$f(z_1, z_2) = \sum_{m,n\geq 0} a_{mn} z_1^m z_2^n$$

be an entire function of two complex variables (we consider the two variables case for the sake of simplicity). Let

$$(1.1)\quad \begin{aligned} M(r_1, r_2) &= \max_{|z_1|\leq r_1, |z_2|\leq r_2} |f(z_1, z_2)|; \\ G(r_1, r_2) &= \exp\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log|f(r_1e^{i\theta_1}, r_2e^{i\theta_2})|\, d\theta_1\, d\theta_2\right\}; \end{aligned}$$

$$(1.2)\quad g_{k,\lambda}(r_1, r_2) = \exp\left\{\frac{(k+1)(\lambda+1)}{r_1^{k+1}r_2^{\lambda+1}}\int_0^{r_1}\int_0^{r_2} x_1^k x_2^\lambda \log G(x_1, x_2)\, dx_1\, dx_2\right\},$$

where $0 < k, \lambda < \infty$, be the geometric means of $f(z_1, z_2)$. The term $g_{k,\lambda}(r_1, r_2)$ and its various properties were probably first considered as early as in 1962 by the author [2] in terms of an entire function of a single variable. Recently, Agarwal [1] has generalised some of the results in [3] in terms of $G(r_1, r_2)$ and $g_{k,\lambda}(r_1, r_2)$ when $k = \lambda$, and in addition has also proved the following:

$$(1.3)\quad \lim_{r_1, r_2\to\infty} \left\{\begin{matrix}\sup \\ \\ \inf\end{matrix}\right. \frac{\log\log g_{k,\lambda}(r_1, r_2)}{\log(r_1, r_2)} = \left\{\begin{matrix}\rho \\ \\ \mu\end{matrix}\right. \qquad (k = \lambda)$$

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where(1)

$$(1.4) \qquad \lim_{r_1, r_2 \to \infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\log \log M(r_1, r_2)}{\log (r_1, r_2)} = \begin{Bmatrix} \rho, \\ \mu. \end{Bmatrix}$$

Apart from giving certain growth results involving $G(r_1, r_2)$ and $g_{k,\lambda}(r_1, r_2)$, our chief aim is to present an example which violates (1.3)—an improvement of which (i.e. a correct version of (1.3)) is given in §2 that follows now.

2. **A counterexample for (1.3) and its improvement.** Let $f(z_1, z_2) = e^{z_1 z_2}$. Then $M(r_1, r_2) = e^{r_1 r_2}$. Therefore $\rho = \mu = 1$. Now

$$\log |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| = r_1 r_2 \cos\theta_1 \cos\theta_2 - r_1 r_2 \sin\theta_1 \sin\theta_2$$

$$\Rightarrow G(r_1, r_2) = 1, \quad \text{for all } r_1, r_2 > 0.$$

Hence $\log g_{k,k}(r_1, r_2) = 0$ for all $r_1, r_2 > 0$. Thus if (1.3) is true then $\rho = \mu = -\infty$ which is absurd. We may lead to a similar discussion if $f(z_1, z_2) = \exp(z_1 + z_2)$ and the details are left to the reader. I may point out that the main fault in establishing (1.3) is the following inequality (see line 4 from above, p. 653 of [1]) which Agarwal has proved:

$$\log g_{k,k}(\alpha r_1, \alpha r_2) \geq \{(\alpha - 1)/(\alpha + 1)\}^2 \{1 - 1/\alpha^{k+1}\}^2 \log M(r_1/\alpha, r_2/\alpha), \quad \alpha > 1,$$

and which is also incorrect in view of the above example.

To offer an improvement of (1.3), let us define first

$$G^+(r_1, r_2) = \exp\left\{\frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} \log^+ |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|\, d\theta_1\, d\theta_2\right\},$$

where

$$\log^+ |f| = \max(\log |f|, 0);$$

also let

$$\lim_{r_1, r_2 \to \infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\log \log G(r_1, r_2)}{\log (r_1, r_2)} = \begin{Bmatrix} \rho_G, \\ \mu_G. \end{Bmatrix}$$

We have then the following result:

Theorem 2.1. *If $f(z_1, z_2)$ is an entire function, then for $R_1 > r_1$, $R_2 > r_2$,*

$$(2.1) \qquad \log G^+(r_1, r_2) \leq \log^+ M(r_1, r_2) \leq \frac{R_1 + r_1}{R_2 + r_2}\, \frac{R_2 + r_2}{R_2 - r_2} \log G^+(R_1, R_2),$$

(1) Agarwal's claim that ρ and μ are nonintegral is irrelevant as far as the proof of (1.3) goes.

and

$$\lim_{r_1, r_2 \to \infty} \left\{ \begin{matrix} \sup \\ \\ \inf \end{matrix} \right. \frac{\log \log g_{k,\lambda}(r_1, r_2)}{\log (r_1, r_2)} = \left\{ \begin{matrix} \rho_G, \\ \\ \mu_G, \end{matrix} \right. \tag{2.2}$$

for any k, λ *such that* $0 < k, \lambda < \infty$.

Proof. (2.1) immediately follows from Poisson's inequality in two variables. For (2.2), we observe

$$\log g_{k,\lambda}(r_1, r_2) \le \left\{ \frac{(k+1)(\lambda+1)}{r_1^{k+1} r_2^{\lambda+1}} \int_0^{r_1} \int_0^{r_2} x_1^k x_2^\lambda dx_1 dx_2 \right\} \log G(r_1, r_2)$$
$$= \log G(r_2, r_2). \tag{2.3}$$

Moreover

$$\log g_{k,\lambda}(R_1, R_2) \ge \frac{(k+1)(\lambda+1)}{R_1^{k+1} R_2^{\lambda+1}} \int_{r_1}^{R_2} \int_{r_2}^{R_2} x_1^k x_2^\lambda \log G(x_1, x_2)\, dx_1\, dx_2$$

$$\ge \frac{(R_1^{k+1} - r_1^{k+1})(R_2^{\lambda+1} - r_2^{\lambda+1})}{R_1^{k+1} R_2^{\lambda+1}} \log G(r_1, r_2).$$

Hence, putting $R_1 = \alpha r_1$, $R_2 = \beta r_2$; $\alpha, \beta > 1$,

$$\log g_{k,\lambda}(\alpha r_1, \beta r_2) \ge \frac{(\alpha^{k+1} - 1)(\beta^{\lambda+1} - 1)}{\alpha^{k+1} \beta^{\lambda+1}} \log G(r_1, r_2). \tag{2.4}$$

The inequalities (2.3) and (2.4) result in (2.2).

In this section we offer improvements of Theorem 2, (3.3) and Theorem 3(ii) and (iii) of Agarwal [1].

$$\lim_{r_1, r_2 \to \infty} \left\{ \begin{matrix} \sup \\ \\ \inf \end{matrix} \right. \frac{\log g_{k,\lambda}(r_1, r_2)}{(r_1 r_2)^{\rho_G} \phi(r_1, r_2)} = \left\{ \begin{matrix} p \\ \\ q \end{matrix} \right. \quad (0 < q \le p < \infty), \tag{3.1}$$

$$\lim_{r_1, r_2 \to \infty} \left\{ \begin{matrix} \sup \\ \\ \inf \end{matrix} \right. \frac{\log G(r_1, r_2)}{(r_1 r_2)^{\rho_G} \phi(r_1, r_2)} = \left\{ \begin{matrix} c \\ \\ d \end{matrix} \right. \quad (0 < d \le c < \infty), \tag{3.2}$$

where $\phi(r_1, r_2)$ is as mentioned by Agarwal. Then we have

Theorem 3.1. *If* $f(z_1, z_2)$ *is an entire function having finite nonzero value* ρ_G, *i.e.* $0 < \rho_G < \infty$, *then*

$$d(k+1)(\lambda+1)/\{(k+\rho_G+1)(\lambda+\rho_G+1)\} \le q \le p$$
$$\le c(k+1)(\lambda+1)/\{(k+\rho_G+1)(\lambda+\rho_G+1)\}. \tag{3.3}$$

Proof. The proof is sketched as follows: Let $0 < \alpha, \beta < 1$, $0 < r_1^0 < r_1$, $0 < r_2^0 < r_2$. Then

$$
\begin{aligned}
&\log g_{k,\lambda}(r_1 + \alpha r_1, r_2 + \beta r_2) \\
&< \frac{A}{r_1^{k+1} r_2^{\lambda+1}} \\
&+ \frac{1}{(\alpha+1)^{k+1}(\beta+1)^{\lambda+1}} \left\{ \left(\frac{r_1^0}{r_1}\right)^{k+1} \left[\left\{ 1 - \left(\frac{\beta r_2^0}{r_2}\right)^{\lambda+1} \right\} \log G(r_1^0, r_2) \right.\right. \\
&\qquad \left. + \{(1+\beta)^{\lambda+1} - 1\} \log G(r_1^0, r_2 + \beta r_2) \right] \\
&\qquad + \left(\frac{r_2^0}{r_2}\right)^{\lambda+1} \left[\left\{ 1 - \left(\frac{\alpha r_1^0}{r_1}\right)^{k+1} \right\} \log G(r_1, r_2^0) \right. \\
&\qquad \left. + \{(1+\alpha)^{k+1} - 1\} \log G(r_1 + \alpha r_1, r_2^0) \right] \\
&\qquad + \frac{(c+\epsilon)(k+1)(\lambda+1)}{r_1^{k+1} r_2^{\lambda+1}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} x_1^{\rho_G + k} x_2^{\rho_G + \lambda} \phi(x_1, x_2)\, dx_1\, dx_2 \\
&\qquad + \frac{(k+1)((1+\beta)^{\lambda+1} - 1)}{r_1^{k+1}} \int_{r_1^0}^{r_1} x_1^k \log G(x_1, r_2 + \beta r_2)\, dx_1 \\
&\qquad + \frac{(\lambda+1)((1+\alpha)^{k+1} - 1)}{r_2^{\lambda+1}} \int_{r_2^0}^{r_2} x_2^{\lambda} \log G(r_1 + \alpha r_1, x_2)\, dx_2 \\
&\qquad \left. + \{(1+\alpha)^{k+1} - 1\}\{1+\beta)^{\lambda+1} - 1\} \log G(r_1 + \alpha r_1, r_2 + \beta r_2) \right\}.
\end{aligned}
$$

Next, observe that the seventh, eighth, and ninth lines of the foregoing inequality are respectively equal at most to the following estimates:

(i) $$\frac{(c+\epsilon)(k+1)(\lambda+1)}{(1+\alpha)^{k+1}(1+\beta)^{\lambda+1}(\rho_G + k + 1)(\rho_G + \lambda + 1)} \phi(r_1, r_2)(r_1 r_2)^{\rho_G},$$

(ii) $$\frac{(k+1)(1+\beta)^{\rho_G}((1+\beta)^{\lambda+1} - 1)}{(1+\alpha)^{k+1}(1+\beta)^{\lambda+1}(\rho_G + k + 1)} \phi(r_1, r_2 + \beta r_2)(r_1 r_2)^{\rho_G},$$

(iii) $$\frac{(\lambda+1)(1+\alpha)^{\rho_G}((1+\alpha)^{k+1} - 1)}{(1+\alpha)^{k+1}(1+\beta)^{\lambda+1}(\rho_G + \lambda + 1)} \phi(r_1 + \alpha r_1, r_2)(r_1 r_2)^{\rho_G}.$$

Making use of these estimates in the corresponding terms of the above inequality, then dividing the complete expression by

$$(r_1 + \alpha r_1)^{\rho_G}(r_2 + \beta r_2)^{\rho_G}\phi(r_1 + \alpha r_1, r_2 + \beta r_2)$$

and finally proceeding to the limit as $r_1, r_2 \to \infty$, one gets the following: namely,

$$p \le \frac{c}{(1+\alpha)^{k+1}(1+\beta)^{\lambda+1}}\left\{\frac{(k+1)(1+\beta)^{\rho_G}((1+\beta)^{\lambda+1}-1)}{\rho_G + k + 1} - \frac{(\lambda+1)(1+\alpha)^{\rho_G}((1+\alpha)^{k+1}-1)}{\rho_G + \lambda + 1} + \frac{(k+1)(\lambda+1)}{(\rho_G + k + 1)(\rho_G + \lambda + 1)}\right\}.$$

But, α, β are arbitrary and so making $\alpha, \beta \to 0$, we find that the right-hand inequality in (3.3) is established.

Next, we have from (1.2) for all sufficiently large values of r_1 and r_2,

$$\begin{aligned}\log g_{k,\lambda}(r_1 + \alpha r_1, r_2 + \beta r_2) &> \frac{(d-\epsilon)(k+1)(\lambda+1)}{r_1^{k+1} r_2^{\lambda+1}(1+\alpha)^{k+1}(1+\beta)^{\lambda+1}} \int_{r_1^0}^{r_1}\int_{r_2^0}^{r_2} x_1^{\rho_G+k} x_2^{\rho_G+\lambda}\phi(x_1, x_2)\,dx_1\,dx_2 \\ &+ \frac{1}{(1+\alpha)^{k+1}(1+\beta)^{\lambda+1}}\left\{\frac{(k+1)((1+\beta)^{\lambda+1}-1)}{r_1^{k+1}}\int_{r_1^0}^{r_1} x_1^k \log G(x_1, r_2)\,dx_1\right. \\ &+ \frac{(\lambda+1)((1+\alpha)^{k+1}-1)}{r_2^{\lambda+1}}\int_{r_2^0}^{r_2} x_2^\lambda \log G(r_1, x_2)\,dx_2 \\ &\left. + \{(1+\alpha)^{k+1}-1\}\{(1+\beta)^{\lambda+1}-1\}\log G(r_1, r_2)\right\}.\end{aligned}$$

Observe that

$$\log G(x_1, r_2) > (d-\epsilon)(x_1 r_2)^{\rho_G}\phi(x_1, r_2), \quad \text{for } x_1 > r_1^0,$$

$$\log G(r_1, x_2) > (d-\epsilon)(r_1 x_2)^{\rho_G}\phi(r_1, x_2), \quad \text{for } x_2 > r_2^0,$$

$$\log G(r_1, r_2) > (d-\epsilon)(r_1 r_2)^{\rho_G}\phi(r_1, r_2), \quad \text{for } r_1 > r_1^0,\ r_2 > r_2^0.$$

Hence

$$(1+\alpha)^{\rho_G+k+1}(1+\beta)^{\rho_G+\lambda+1}q \geq \frac{(k+1)(\lambda+1)d}{(\rho_G+k+1)(\rho_G+\lambda+1)} + \frac{(k+1)((1+\beta)^{\lambda+1}-1)d}{k+\rho_G+1}$$

$$+ \frac{(\lambda+1)((1+\alpha)^{k+1}-1)d}{\lambda+\rho_G+1}$$

$$+ \{(1+\alpha)^{k+1}-1\}\{(1+\beta)^{\lambda+1}-1\}d,$$

and making now $\alpha, \beta \longrightarrow 0$, the left-hand inequality in (3.3) is obtained.

Invoking Theorem 2 and the technique of its proof as envisaged in [3] together with the method adopted in the proof of the above theorem, one may now easily prove the following:

Theorem 3.2. *If $f(z_1, z_2)$ is an entire function, such that $c = d$, then $p = q = (k+1)(\lambda+1)c/\{k+\rho_G+1\}\{\lambda+\rho_G+1\}$, and*

$$\lim_{r_1, r_2 \to \infty} \frac{\log g_{k,\lambda}(r_1, r_2)}{\log G(r_1, r_2)} = \frac{(k+1)(\lambda+1)}{(k+\rho_G+1)(\lambda+\rho_G+1)}.$$

Remark. The author is of the view that the results (3.4) and (3.5) of Agarwal may not be generalised in terms of $\log g_{k,\lambda}(r_1, r_2)$ when $k \neq \lambda$ and are arbitrary. Attempts towards these generalisations involve enormous calculations without yielding any solid solution.

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