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SECOND-ORDER TIME DEGENERATE PARABOLIC EQUATIONS

BY

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ABSTRACT. We study the degenerate parabolic operator $Lu = \sum_{i,j=1}^n a^{ij}u_{x_ix_j} + \sum_{i=1}^n b^i u_{x_i} - cu_t + du$ where the coefficients of L are bounded, real-valued functions defined on a domain $D = \Omega \times (0, T] \subset R^{n+1}$. Classically, $c(x, t) \equiv 1$ or, equivalently, $c(x, t) \geq \eta > 0$ for all $(x, t) \in \bar{D}$. We assume only that c is nonnegative. We prove weak maximum principles and Harnack inequalities. Assume that a^{ij} is constant, the coefficients of L and f and their derivatives with respect to time are uniformly Hölder continuous (exponent α) in $\bar{D}$, $\bar{D}$ has sufficiently nice boundary, $c > 0$ on the normal boundary of D, $\psi \in \bar{C}_{z+\alpha}$, and $L\psi = f$ on $\partial B = \partial(\bar{D} \cap \{t = 0\})$. Then there exists a unique solution u of the first initial-boundary value problem $Lu = f$, $u = \psi$ on $\bar{B} + (\partial B \times [0, T])$; and, furthermore, $u \in \bar{C}_{2+\alpha}$. All results require proofs that differ substantially from the classical ones.

1. Notation and preliminary remarks. In this paper we shall study the second-order nonuniformly parabolic operator

$$Lu = \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} - cu_t + du. \tag{1.1}$$

u and all the coefficients of L are real-valued functions defined for $(x, t) = (x_1, \cdots, x_n, t)$ in an $(n+1)$-dimensional, bounded, convex domain D. We are using subscripts to denote differentiation.

We will assume that L is parabolic; that is,

$$\sum_{i,j=1}^{n} a^{ij}\xi_i\xi_i \geq \gamma|\xi|^2 > 0 \tag{1.2}$$

for any real vector $\xi \neq 0$. The coefficient c will be assumed to be nonnegative. However, we will *not* make the usual assumption that c is bounded away from zero. Since $c(x, t) = 0$ for some (x, t) in the closure of D is permissible, L may be a degenerate parabolic operator. We will make assumptions concerning the smoothness of the coefficients as needed.

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(1) These results are contained in the author's doctoral dissertation written under the direction of Professor Wayne T. Ford at Texas Tech University.

We will prove the usual weak maximum principles for the operator (1.1). For the special case in which u satisfies $Lu = f$ with all the a^{ij} constants, estimates are given for max $|u_t|$ and max $|u_{x_i}|$, in terms of their values on a portion of the boundary of D. Furthermore, these estimates are independent of the minimum of c in D.

In addition, we will obtain Harnack inequalities, again independent of the minimum of c in D, for a solution of the equation $Lu = f$ where L is as in (1.1). Classical proofs of both the maximum principles and the Harnack inequalities depend strongly on the assumption that $c(x, t) \geq \lambda > 0$ for some constant λ and for all (x, t) in the closure of D. We have had to modify substantially the methods of proof, making some assumptions concerning the set of zeros of c and, in some instances, requiring that c_t be bounded.

The main result in this paper is a proof of the existence of a unique classical, or strong, solution to the first initial-boundary value problem. The method of proof employed involves perturbing the coefficient c and obtaining a Cauchy sequence of solutions to the perturbed equations which converges in an appropriate Banach space to a solution of the first initial-boundary value problem. Showing that the sequence is Cauchy requires that we obtain somewhat sharper Schauder-type a priori boundary estimates than those of A. Friedman in [6]. That is, we must show exactly how Friedman's estimates depend on the minimum of c. Furthermore, our method of proof entails finding an a priori boundary estimate for u_t which is independent of the minimum of c in D.

We will use the same notation as A. Friedman [6, p. 40] in describing the domain D. As stated above, D is a bounded, convex, $(n+1)$-dimensional domain in R^{n+1}, where $(x, t) = (x_1, \cdots, x_n, t)$ represents a variable point in R^{n+1}. For any set A, let $\bar{A}$ denote the closure of A. The union of two sets, A and B, will be denoted by $A + B$. ∂D denotes the boundary of D; $\partial D = \bar{B} + B_T + S$, where B is a domain in $R^n \times \{0\}$, B_T $(T > 0)$ is a domain in $R^n \times \{T\}$, and S is a manifold, not necessarily connected, in $R^n \times (0, T]$. $S + \bar{B}$ is generally called the parabolic, or normal, boundary of D.

Let $D_\tau = D \cap (R^n \times (0, \tau))$, $B_\tau = D \cap (R^n \times \{\tau\})$, and $S_\tau = S \cap (R^n \times (0, \tau])$. Assume that for each $\tau \in (0, T)$, B_τ is a domain. Then, for every (x, τ) in D, $0 < \tau < T$, if $S(x, \tau) = D_\tau + B_\tau$, then $\overline{S(x, \tau)} - S(x, \tau) = \bar{B} + S_\tau$. Also, assume that D satisfies the following condition: there exists a simple continuous curve α connecting B to B_T along which the t-coordinate is nondecreasing.

For an excellent, as well as comprehensive, discussion of the history and applications of the first initial-boundary value problem and other problems for degenerate quasilinear parabolic equations, the reader is referred to [13], which contains a series of lectures by Professor Olga Oleĭnik. Specific applications to the problem of flows through a porous media can be found in [2] and [4]. The numerical analysis for the case of one space variable has been studied by E. D. Williams [17].

2. **Weak maximum principles.** We now prove a sequence of weak maximum principles which are similar in nature to those proved by A. Friedman in [6]. In some instances, the proof by Friedman can be used with only minor modification. Most of the results, however, require considerable modification because they rely heavily on the fact that $c(x, t) \equiv 1$.

We assume, as does Friedman, that the coefficients of L, defined in (1.1), are continuous in D. Friedman remarks that the reader may observe that the majority of the theorems remain true if the coefficients are only bounded functions and the operators are assumed to be locally uniformly parabolic. The same remark applies for many of the theorems in this section.

Unless otherwise indicated, the operator L is as defined in (1.1), where L is parabolic (1.2) and c is nonnegative. Assume also that u is continuous on the closure of D and that u is of class $C^{2,1}(D)$.

Theorem 2.1. *If $Lu \leq 0$ $(Lu \geq 0)$, $d < 0$ in D, and $u \geq 0$ $(u \leq 0)$ on $\Gamma = \{(x, t) \in \partial D: t < T\}$, then $u \geq 0$ $(u \leq 0)$ in the closure of $D_T = \{(x, t): t < T\}$.*

Proof. Suppose that u has a negative minimum at (x^0, t^0) in D_T. Then,

$$\sum_{i,j=1}^{n} a^{ij}(x^0, t^0)\, u_{x_i x_j}(x^0, t^0) \geq 0.$$

Hence

$$Lu(x^0, t^0) = \sum_{i,j=1}^{n} a^{ij}(x^0, t^0)\, u_{x_i x_j}(x^0, t^0) + d(x^0, t^0)\, u(x^0, t^0)$$
$$\geq d(x^0, t^0)\, u(x^0, t^0) > 0.$$

But, $Lu(x^0, t^0) \leq 0$. This contradiction completes the proof.

Theorem 2.2. *If d is bounded above, $Lu \leq 0$ $(Lu \geq 0)$ in D, and $u \geq 0$ $(u \leq 0)$ on $\Gamma = \{(x, t) \in \partial D: t < T\}$, then $u \geq 0$ $(u \leq 0)$ in the closure of $D_T = \{(x, t): t < T\}$.*

Proof. Suppose that q_i represents the breadth of $\overline{D}$ in the x_i-direction. In particular,

$$q_i = \min\{q > 0: \text{ for some } \overline{x}_i,\ \overline{x}_i - q \leq x_i \leq \overline{x}_i$$
$$\text{for all } x \text{ such that } (x_1, \cdots, x_n, t) \in \overline{D}\}.$$

The minimum of this set exists since $\overline{D}$ is compact.

Without loss of generality, let us assume that $\overline{x}_i = q_i$; for if $\overline{x}_i \neq q_i$, we can perform a simple translation of the domain D in the x_i-direction. We will use

$0 \le x_1 \le q_1$, $q_1 > 0$, in this proof.

Since the matrix (a^{ij}) is positive definite and can be assumed symmetric, $a^{11} \ge \gamma_1 > 0$ for some constant γ_1. If $d(x, t)$ is identically the zero function, then $d(x, t) < \gamma_1/4q_1^2$. Suppose that $d(x, t)$ is positive for some $(x, t) \in D$. Let $K = \max\{d(x, t): (x, t) \in \overline{D}_T\}$.

If $(x_1, \cdots, x_n, t)$ is a point in $\overline{D}$, let $(z_1, x_2, \cdots, x_n, t)$ denote the projection toward zero of (x, t) onto S in the x_1-direction. Note that if y_1 is such that $z_1 < y_1 < x_1$, then $(y_1, x_2, \cdots, x_n, t) \in D$.

Map (x, t) to $(x', t) = (x_1', x_2, \cdots, x_n, t)$, where $x_1' = z_1 + (\gamma_1/8Kq_1^2)^{1/2}(x_1 - z_1) = [1 - (\gamma_1/8Kq_1^2)^{1/2}]z_1 + (\gamma_1/8Kq_1^2)^{1/2}x_1$. Then $x_1 = x_1'(\gamma_1/8Kq_1^2)^{-1/2} - [(\gamma_1/8Kq_1^2)^{-1/2} - 1]z_1$. Let $v(x_1', x_2, \cdots, x_n, t) = u(x, t)$. We may assume that $(\gamma_1/8Kq_1^2)^{1/2} < 1$, for otherwise $d(x, t) < \gamma_1/4q_1^2$.

Let D' denote the domain obtained from D by the above transformation. Since $(\gamma_1/8Kq_1^2)^{1/2} < 1$, D' is a subdomain of D. $Lu \le 0$ on D and hence $Lv \le 0$ on D'. $u \ge 0$ on $\Gamma = \{(x, t) \in \partial D: t < T\}$, so $v \ge 0$ on $\Gamma' = \{(x, t) \in \partial D': t < T\}$. q_1 is mapped to $(\gamma_1/8Kq_1^2)^{1/2}q_1 = (\gamma_1/8K)^{1/2} = q_1'$, and hence q_1' is the breadth of D' in the x_1'-direction. A simple computation shows that $\gamma_1/4(q_1')^2 = 2K > d(x, t)$ for all $(x, t) \in D$. Since D' is a subdomain of D, it follows that $\gamma_1/4(q_1')^2 > d(x_1', x_2, \cdots, x_n, t)$ for all $(x_1', x_2, \cdots, x_n, t)$ in D'. If we show that the theorem holds for v and the domain D', we will then have $v \ge 0$ in $\overline{D}_T'$. $(x_1', x_2, \cdots, x_n, t) \in \overline{D}_T'$ implies that

$$(z_1 + (\gamma_1/8Kq_1^2)^{1/2}(x_1' - z_1), x_2, \cdots, x_n, t) \in \overline{D}_T'.$$

Therefore,

$$v(z_1 + (\gamma_1/8Kq_1^2)^{1/2}(x_1' - z_1), x_2, \cdots, x_n, t) \ge 0.$$

But

$$v(z_1 + (\gamma_1/8Kq_1^2)^{1/2}(x_1' - z_1), x_2, \cdots, x_n, t) = u(x_1', x_2, \cdots, x_n, t);$$

hence $u(x_1', x_2, \cdots, x_n, t) \ge 0$.

Consider whether $d(x, t) \le \gamma_1/4(q_1 - q_1')^2$ on $D_T - \overline{D}_T'$. If so, we can simply regard D_T as the union of these two subdomains and prove the theorem for each subdomain separately. If not, we continue with this process until we have written D_T as the union of a finite number N of such domains on each of which $d(x, t) < \gamma_1/4(q_1^{(n)})^2$, $1 \le n \le N$. $q_1^{(n)}$ represents the breadth of the nth domain. We are assured that this can be accomplished in a finite number of steps since D is bounded.

We now return to the proof of the theorem for v and the domain D'. For convenience of notation, let us use D for D', u for v, x_1 for x_1', and q_1 for q_1'.

Let $v = u(e^{\epsilon x_1} + \alpha e^{-\epsilon x_1}) = u \cdot g(x_1)$ where ϵ, $\alpha > 0$ are chosen so that

$1 - 2(e^{\epsilon x_1} - \alpha e^{-\epsilon x_1})^2 (g(x_1))^{-2} > 0$. ϵ, α can always be so chosen since

$$\begin{aligned}
&1 - 2(e^{\epsilon x_1} - \alpha e^{-\epsilon x_1})^2 (g(x_1))^{-2} \\
&\quad = (-e^{2\epsilon x_1} + 6\alpha - \alpha^2 e^{-2\epsilon x_1})(g(x_1))^{-2} \\
&\quad \geq (-e^{2\epsilon q_1} + 6\alpha - \alpha^2)(g(x_1))^{-2} = j(\alpha)(g(x_1))^{-2} \\
&\quad \geq j(\alpha)/16
\end{aligned}$$

for α, ϵ such that $j(\alpha) \geq 0$, $j'(\alpha) = 6 - 2\alpha$, $j''(\alpha) = -2$. Hence, j has a maximum at $\alpha = 3$. Let $\epsilon = (2q_1)^{-1}$. Then $j(3) = 9 - e \geq 1$. Hence $v = u(e^{x_1/2q_1} + 3e^{-x_1/2q_1}) = u \cdot g(x_1)$ can be chosen as the specified transformation.

Some useful observations concerning g are

$$g'(x_1) = (2q_1)^{-1}(e^{x_1/2q_1} - 3e^{-x_1/2q_1}) < 0; \tag{2.1}$$

and

$$1 < g(x_1) \leq 4. \tag{2.2}$$

$v \in C(\bar{D})$ and $C^{2,1}(D)$ since $u \in C(\bar{D})$ and $C^{2,1}(D)$.

A routine computation gives

$$\begin{aligned}
Lv &= g(x_1) Lu \\
&\quad + a^{11}[2g'(x_1)(g(x_1))^{-1}(v_{x_1} - g'(x_1)(g(x_1))^{-1}v) + (4q_1^2)^{-1}v] \\
&\quad + \sum_{j=2}^{n} a^{1j} g'(x_1)(g(x_1))^{-1} v_{x_j} \\
&\quad + \sum_{i=2}^{n} a^{i1} g'(x_1)(g(x_1))^{-1} v_{x_i} + b^1 g'(x_1)(g(x_1))^{-1} v.
\end{aligned}$$

Let

$$\begin{aligned}
Mv &= Lv - \sum_{j=2}^{n} a^{1j} g'(x_1)(g(x_1))^{-1} v_{x_j} - a^{11}(2) g'(x_1)(g(x_1))^{-1} v_{x_1} \\
&\quad - \sum_{i=2}^{n} a^{i1}(2) g'(x_1)(g(x_1))^{-1} v_{x_i} - b^1(2) g'(x_1)(g(x_1))^{-1} v \\
&= g(x_1) Lu + a^{11}(4q_1^2)^{-1}[1 - 2(e^{x_1/2q_1} - 3e^{-x_1/2q_1})^2 (g(x_1))^{-2}]v \\
&\leq a^{11}(4q_1^2)^{-1}[1 - 2(e^{x_1/2q_1} - 3e^{-x_1/2q_1})^2 (g(x_1))^{-2}]v.
\end{aligned}$$

Therefore,

$$Nv = (M - a^{11}(4q_1^2)^{-1}[1 - 2(e^{x_1/2q_1} - 3e^{-x_1/2q_1})^2(g(x_1))^{-2}])v$$

$$\leq d(x, t) - a^{11}(x, t)/4q_1^2 < 0.$$

We have already that $v \geq 0$ on Γ. By Theorem 2.1, $v \geq 0$ on $\overline{D}_T$. $g(x_1) > 0$ and $v \geq 0$ imply $u \geq 0$ on $\overline{D}_T$. This completes the proof of the theorem.

The following are corollaries to the above maximum principle. If the proof is no different from the case $c(x, t) \equiv 1$, it is omitted.

If u is a bounded function on a subset S of R^n, define

$M(u; S) = \sup\{u(x, t): (x, t) \in S\}$ and $m(u; S) = \inf\{u(x, t): (x, t) \in S\}$.

Corollary 2.3. *$Lu > 0$ and $d \equiv 0$ in D imply u has no relative maximum in D.*

Corollary 2.4. *If $Lu = 0$ and d is bounded above, then $M(|u|; \overline{D}_T) \leq M(|u|; \overline{B} + S_T)$.*

Corollary 2.5. *If $d(x, t) \leq 0$, $a^{11}\lambda^2 + b^1\lambda \geq 1$ for some $\lambda > 0$, and if $Lu = f$ in D, then*

$$M(|u|; \overline{D}_T) \leq M(|u|; \overline{B} + S_T) + (e^{\lambda q_1} - 1)M(|f|; \overline{D}).$$

Corollary 2.6. *If $d < a^{11}/4q_1^2$, $a^{11}\lambda^2 + b^1\lambda \geq 1$ for some $\lambda > 0$, and if $Lu = f$ in D, then*

$$M(|u|; \overline{D}_T) \leq 4[M(|u|; \overline{B} + S_T) + (e^{\lambda q_1} - 1)M(|f|; \overline{D}_T)].$$

Proof. Applying Corollary 2.5 to $v = u \cdot g(x_1)$, where $g(x_1)$ is as in the proof of Theorem 2.2, it follows that

$$M(|v|; \overline{D}_T) \leq M(|v|; \overline{B} + S_T) + (e^{\lambda q_1} - 1)M(|g(x_1)f|; \overline{D}_T).$$

$1 \leq g(x_1) \leq 4$ guarantees that

$$M(|u|; \overline{D}_T) \leq M(|v|; \overline{D}_T) \leq 4M(|u|; \overline{B} + S_T) + (e^{\lambda q_1} - 1)M(|f|; \overline{D}_T).$$

Corollary 2.7. *If d is bounded above by $K > 0$, $a^{11}\lambda^2 + b^1\lambda \geq 1$ for some $\lambda > 0$, and if $Lu = f$ in D, then*

$$M(|u|; \overline{D}_T) \leq 4[M(|u|; \overline{B} + S_T) + (\exp\{\lambda(\gamma_1/8K)^{1/2}\} - 1)M(|f|; \overline{D}_T)].$$

Proof. Map x_1 to $x_1' = (\gamma_1/8Kq_1^2)^{1/2}x_1$, and let $v(x', t) = v(x_1', x_2, \cdots, x_n, t) = u(x, t)$. Then q_1 is mapped to $q_1' = (\gamma_1/8Kq_1^2)^{1/2}q_1 = \gamma_1/8K$ and $\gamma_1/4(q_1')^2 = 2K > d$. Let D' be the image of D under the above map.

By Corollary 2.6,

$$M(|v|; \overline{D}_T') \leq 4[M(|v|; \overline{B}' + S_T') + (\exp\{\lambda(q_1')^2\} - 1)M(|\hat{f}(x', t)|; \overline{D}_T')]$$

where $\hat{f}(x', t) = f(x, t)$, and the desired inequality follows.

Theorem 2.8. *Assume that a^{ij} is constant for each i, j, that all coefficients of L are of class $C^{1,1}(\overline{D})$, $u \in C(\overline{D})$, $u \in C^3(D)$, and $a^{11}\lambda^2 + b^1\lambda \geq 1$ for some $\lambda > 0$. Then there is a constant C which depends only on λ, $M(|\alpha|; \overline{D})$, where $\alpha \in \{b_t, c_t, d_{x_i}, b_{x_i}, c_{x_i}\}$, such that*

$$M(|u_t|; \overline{D}_T) \leq C\left[M(|u_t|; \overline{B} + S_T) + \sum_{i=1}^{n} M(|u_{x_i}|; \overline{B} + S_T) + M(|u|; \overline{B} + S_T)\right.$$

$$\left. + M(|f_t|; \overline{D}_T) + \sum_{i=1}^{n} M(|f_{x_i}|; \overline{D}_T) + M(|f|; \overline{D}_T)\right].$$

Proof. We will indicate the proof for the case of one space variable, since in doing so we lose only a good deal of tedium and virtually none of the technique. Suppose that $K > 0$ is an upper bound for $d - c_t$ and $d + b_x$.

Consider $Lu = au_{xx} + bu_x - cu_t + du = f$. If we differentiate both sides of $Lu = f$ with respect to t, it follows that $(L - c_t)u_t = f_t - b_t u_x - d_t u$. Corollary 2.7 implies that

$$\begin{aligned} \tfrac{1}{4} M(|u_t|; \overline{D}_T) \leq M(|u_t|; \overline{B} + S_T) &+ (\exp\{\lambda(a/8K)^{1/2}\} - 1) \\ &\cdot [M(|f_t|; \overline{D}_T) + M(|b_t||u_x|; \overline{D}_T) + M(|d_t||u|; \overline{D}_T)]. \end{aligned} \tag{2.3}$$

Differentiating, instead, with respect to x yields $(L + b_x)u_x = f_x + c_x u_t - d_x u$. Apply Corollary 2.7 again to see that

$$\begin{aligned} \tfrac{1}{4} M(|u_x|; \overline{D}_T) \leq M(|u_x|; \overline{B} + S_T) &+ (\exp\{\lambda(a/8K)^{1/2}\} - 1) \\ &\cdot [M(|f_x|; \overline{D}_T) + M(|c_x||u_t|; \overline{D}_T) + M(|d_x||u|; \overline{D}_T)]. \end{aligned} \tag{2.4}$$

Substitute (2.4) into (2.3), so that

$$\begin{aligned} \tfrac{1}{4} M(|u_t|; \overline{D}_T) &\leq M(|u_t|; \overline{B} + S_T) \\ &+ (\exp\{\lambda(a/8K)^{1/2}\} - 1) \\ &\cdot [M(|f_t|; \overline{D}_T) + 4M(|b_t|; \overline{D}_T)\, M(|u_x|; \overline{B} + S_T) \\ &+ (\exp\{\lambda(a/8K)^{1/2}\} - 1) \\ &\cdot (M(|f_x|; \overline{D}_T) + M(|c_x|\,|u_t|; \overline{D}_T) + M(|d_x|\,|u|; \overline{D}_T)) \\ &+ M(|d_x|\,|u|; \overline{D}_T)]. \end{aligned} \tag{2.5}$$

If $c_x \equiv 0$, we have the desired inequality. If $c_x \not\equiv 0$, then let $K_1 = M(|c_x|; \overline{D}_T) > 0$. Make a change in the time scale so that $M(|b_t|; \overline{D}_T) < [32K_1(\exp\{\lambda(a/8K)^{1/2}\} - 1)^2]^{-1}$. Then

$$\tfrac{1}{4} M(|u_t|;\, \overline{D}_T) \le C[M(|u_t|;\, \overline{B} + S_T) + M(|u_x|;\, \overline{B} + S_T) + M(|u|;\, \overline{B} + S_T) \tag{2.6}$$
$$+ M(|f_t|;\, \overline{D}_T) + M(|f_x|;\, \overline{D}_T) + M(|f|;\, \overline{D}_T) + (1/8) M(|u_t|;\, \overline{D}_T)].$$

Multiply both sides of (2.6) by 8, and then subtract $M(|u_t|;\, \overline{D}_T)$ from both sides. This completes the proof of the theorem.

Corollary 2.9. *Make the same assumptions regarding u and the coefficients of L as in Theorem* 2.8. *Then there is a constant C which depends only on λ, $M(|\alpha|;\, \overline{D})$, where $\alpha \in \{b_t, c_t, d_{x_i}, b_{x_i}, c_{x_i}\}$ such that*

$$M(|u_{x_i}|;\, \overline{D}_T) \le C\Bigg[M(|u_t|;\, \overline{B} + S_T) + \sum_{i=1}^{n} M(|u_{x_i}|;\, \overline{B} + S_T) + M(|u|;\, \overline{B} + S_T)$$
$$+ M(|f_t|;\, \overline{D}_T) + \sum_{i=1}^{n} M(|f_{x_i}|;\, \overline{D}_T) + M(|f|;\, \overline{D}_T)\Bigg].$$

Proof. Use the estimate obtained for $M(|u_t|;\, \overline{D}_T)$ in Theorem 2.8 in (2.5).

3. **Harnack inequalities.** We now derive a Harnack inequality which is similar to the Harnack inequality for nonnegative harmonic functions. For elliptic equations, the inequality gives a bound for the maximum of u over an interior subset of the domain of definition in terms of a minimum of u on the same subset. For parabolic equations it is necessary that we consider two subsets separated by a nonempty time interval. J. Moser in [12] provides an excellent discussion indicating a physical interpretation of this phenomenon as well as giving an example to show why it is a necessary condition.

The proof we shall give is similar in nature to the proof for the special case $u_t = \sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j}$ given by Moser in [12]. It is, in fact, a modification of the case $c \equiv 1$, given by D. G. Aronson and J. Serrin in [3]. In case a portion of the argument is identical to that of Aronson and Serrin, it will be omitted or briefly outlined.

In order to apply this method to the equation $Lu = f$, where L is the operator (1.1), we note that $Lu = f$ can be expressed in the form

$$\sum_{i,j=1}^{n} \{a^{ij}(x, t)\, u_{x_i}\}_{x_j} + \sum_{i=1}^{n} \left(b^i - \sum_{j=1}^{n} a^{ij}_{x_j}\right) u_{x_i} + (1 - c)\, u_t + du - f = u_t. \tag{3.1}$$

For $(x, t) = (x_1, x_2, \cdots, x_n, t) \in D = \Omega \times (0, T]$, (3.1) can be written as

$$\operatorname{div} A(x, t, u_x) + B(x, t, u, u_x, u_t) = u_t \tag{3.2}$$

where $A = (A_1, A_2, \cdots, A_n)$ is a given vector function of (x, t, u_x), B is a given scalar function of (x, t, u, u_x, u_t), and $u_x = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$ denotes the spatial gradient of the dependent variable $u = u(x, t)$. div A is the divergence of $A(x, t, u_x(x, t))$ with respect to the variables $(x_1, \cdots, x_n)$. The structure of (3.2) will be determined by $A(x, t, p)$ and $B(x, t, u, p, q)$.

We now define the classes $L^{p,q}(D)$. Let $u = u(x, t)$ be a measurable function defined on D. We say that $u \in L^{p,q}(D)$ if

$$\|u\|_{p,q} = \left\{\int_0^T \left(\int_\Omega |u|^p \, dx\right)^{q/p} dt\right\}^{1/q} < \infty, \tag{3.3}$$

where $1 \leq p, q \in R$. p or q can be ∞ if the L^∞ norm is used.

The structure inequalities which A and B satisfy are

$$\begin{aligned} &p \cdot A(x, t, p) \geq \gamma |p|^2, \\ &|B(x, t, u, p, q)| \leq M|p| + |1 - c|\,|q| + |d|\,|u| + |f|, \\ &|A(x, t, p)| \leq \mu |p|, \end{aligned} \tag{3.4}$$

where γ, μ, M are positive constants, $M \geq |b^i - \sum_{j=1}^n a^{ij}_{x_j}|$ for all i, j, and the coefficients $|1 - c|$, $|d|$, $|f|$ are each contained in some space $L^{p,q}(D)$, where p and q are nonnegative real numbers (possibly different from coefficient to coefficient) such that

$$p > 1 \quad \text{and} \quad n(2p)^{-1} + (q)^{-1} < 1. \tag{3.5}$$

The norms of $|1 - c|$, $|d|$, $|f|$ in their respective spaces will be denoted by $\|1 - c\|$, $\|d\|$, $\|f\|$.

Suppose that a function $v = v(x, t)$ is locally integrable in D. v is said to be strongly differentiable with respect to x if there exists a locally integrable vector function $v_x(x, t)$ such that $\iint_D v\phi_x \, dx\, dt = - \iint_D \phi v_x \, dx\, dt$ for all functions $\phi \in C_0^1(D) = \{\phi \in C^1(D)\colon \phi \text{ has compact support in } D\}$. Similarly, one defines strong differentiability with respect to t.

If u is a function which is locally of class $L^{2,\infty}$ in Ω and has a strong derivative u_x, which is locally of class $L^{2,2}$ in Ω, we will say that u is a weak solution of (3.2) if

$$\iint_D (-u\phi_t + \phi_x A(x, t, u_x)) \, dx\, dt = \iint_D \phi B(x, t, u, u_x, u_t) \, dx\, dt \tag{3.6}$$

for any $\phi \in C_0^1(D)$.

If u is a strong solution of (3.2), then u not only satisfies (3.6), it satisfies

$$\iint_D (\phi u_t + \phi_x A(x, t, u_x)) \, dx\, dt = \iint_D \phi B(x, t, u, u_x, u_t) \, dx\, dt \tag{3.7}$$

for any $\phi \in C_0^1(D)$.

Recall that $|1 - c|$, $|d|$, and $|f|$ are each contained in some $L^{p,q}(D)$, where p and q satisfy (3.5).

Then there exists a positive constant θ such that

$$p \geq (1-\theta)^{-1} \quad \text{and} \quad n(2p)^{-1} + (q)^{-1} \leq 1-\theta. \tag{3.8}$$

One constant θ works for all of the three possible pairs of (p, q).

We say that a constant *depends on the structure of* (3.2) if it is determined by the constants γ, μ, M, $\|d\|$, $\|1-c\|$, $\|f\|$, θ, and n (and is uniformly bounded whenever these quantities are).

Let $(\bar{x}, \bar{t})$ be a fixed point in D. Denote by $R(\rho)$ the open cube in R^n of edge length $\rho > 0$ centered at $\bar{x}$, and define $D(\rho) = R(\rho) \times (\bar{t} - \rho^2, \bar{t})$. The symbol $\|\cdot\|_{p,q,\rho}$ will denote the $L^{p,q}$ norm of a function over the cylinder $D(\rho)$. Let $D^*(\rho) = R(\rho) \times (\bar{t} - 8\rho^2, \bar{t} - 7\rho^2)$, so that $D^*(\rho)$ is simply the cylinder $D(\rho)$ translated toward zero in the t-direction a distance $7\rho^2$.

In the theorem of Harnack type to follow, we will assume that u is bounded and satisfies $Lu = f$, where L is defined as in (1.1), and the functions u, a^{ij}, b^i, c, d, f satisfy $|u|$, $|a^{ij}|$, $|b^i|$, $|1-c|$, $|d|$, $|f|$, $|b^i - \sum_{j=1}^n a^{ij}_{x_j}| < K$. The structure inequalities (3.4) are then of the form

$$\begin{gathered} p \cdot A(x, t, p) \geq \gamma |p|^2, \\ |B(x, t, u, p, q)| \leq K(|p| + |q| + |u| + 1), \\ |A(x, t, p)| \leq K|p|. \end{gathered} \tag{3.9}$$

The collection of conditions listed in this paragraph shall be referred to as $(*)$.

Theorem 3.1. *Suppose that u is a bounded nonnegative classical solution of* (3.2) *in D and that the conditions* $(*)$, *described immediately preceding this theorem, are satisfied. If $D(3\rho) \subset D$, then*

$$\operatorname*{ess\,max}_{D^*(\rho)} u \leq C \left(\operatorname*{ess\,min}_{D(\rho)} (u + \rho^\theta k) \right)$$

where C is a positive constant depending only on ρ and the structure of (3.2), *θ is defined in* (3.8), *and $k = \|f\| + \|1-c\|$. Furthermore, C is independent of $\|f\|$, $\|1-c\|$.*

Proof. Aronson and Serrin give a normalization argument in [3, p. 97] to show that there is no loss of generality in assuming that $\rho = 1/3$ and that $D = D(3\rho) = R(1) \times (0, 1)$.

For arbitrary $\epsilon > 0$, define $\bar{u} = u + k + \epsilon$. We will show that

$$\operatorname*{ess\,max}_{D^*(1/3)} \bar{u} \leq C \left(\operatorname*{ess\,min}_{D(1/3)} \bar{u} \right) \tag{3.10}$$

which is sufficient to prove the theorem.

Let $G(u) = \bar{u}^\beta$, where β is a fixed nonzero real number. If $H(u)$ is defined by

$$H(u) = \begin{cases} (\beta+1)^{-1}\bar{u}^{\beta+1}, & \text{if } \beta \neq -1, \\ \log \bar{u}, & \text{if } \beta = -1, \end{cases}$$

then $H'(u) = G(u)$.

Let $\chi(t, \tau_1, \tau_2)$ denote the characteristic function of the open interval (τ_1, τ_2), with $0 < \tau_1 < \tau_2 < T$; and let $\phi(x, t) = \eta^2 G(u)\chi(t, \tau_1, \tau_2)$, where $\eta = \eta(x, t)$ is a piecewise-smooth nonnegative function vanishing in a neighborhood of the parabolic boundary. $\bar{u}$ is bounded, and hence ϕ is admissible as a test function in (3.7).

Note that

$$\phi_x = \begin{cases} \eta^2 G'(u)\, u_x + 2\eta\eta_x G(u), & t \in (\tau_1, \tau_2), \\ 0, & t \notin (\tau_1, \tau_2). \end{cases}$$

Also, $(H(u))_t = G(u)u_t$ and $\eta^2 G(u)u_t = (\eta^2 H(u))_t - 2\eta\eta_t H(u)$. Then

$$\text{(3.11)} \quad \iint_D \phi u_t \, dx\, dt = \int_{\tau_1}^{\tau_2}\int_\Omega \eta^2 G(u)\, u_t \, dx\, dt = \int_\Omega (\eta^2 H)\Big|_{t=\tau_1}^{t=\tau_2} - \int_{\tau_1}^{\tau_2}\int_\Omega 2\eta\eta_t H \, dx\, dt$$

Thus, if $\phi = \eta^2 G(u)\chi(t, \tau_1, \tau_2)$ is used as a test function in (3.7), (3.11) implies

$$\text{(3.12)} \quad \int_\Omega \eta^2 H(u)\Big|_{t=\tau_1}^{t=\tau_2} dx + \iint_D (\phi_x A - \phi B)\, dx\, dt = \int_{\tau_1}^{\tau_2}\int_\Omega 2\eta\eta_t H(u)\, dx\, dt.$$

Observe that $u_x = \bar{u}_x$, $u_t = \bar{u}_t$, and that $u < \bar{u}$. If $\beta < 0$, $G'(u) = \beta\bar{u}^{\beta-1} < 0$. Thus,

$$\text{(3.13)} \quad \begin{aligned} &(\operatorname{sgn}\beta)(\phi_x A - \phi B) \\ &\geq \eta^2\gamma|G'|\,|\bar{u}_x|^2 - 2\eta\,|\eta_x|\,G|\bar{u}_x| - \eta^2 GK(|u_x| + |u_t| + |u| + 1). \end{aligned}$$

A simple computation shows that $G^2 = \bar{u}^{\beta+1}G'\beta^{-1}$. The next inequality is trivial to prove:

$$\text{(3.14)} \quad \begin{aligned} 2\eta\,|\eta_x|\,G|\bar{u}_x| &\leq |\beta|\,\eta^2 G^2|u_x|^2(4\bar{u}^{\beta+1})^{-1} + 4|\eta_x|^2\bar{u}^{\beta+1}|\beta|^{-1} \\ &\leq |\beta|\,\eta^2 G'|\bar{u}_x|^2(4\beta)^{-1} + 4|\eta_x|^2\bar{u}^{\beta+1}|\beta|^{-1}. \end{aligned}$$

Similarly,

$$\text{(3.15)} \quad KG|\bar{u}_x| \leq |\beta|\, G'|\bar{u}_x|^2(4\beta)^{-1} + K^2\bar{u}^{\beta+1}|\beta|^{-1}.$$

$G = \bar{u}^\beta$ and $k \leq \bar{u}$ imply

$$\text{(3.16)} \quad KG(\bar{u} + 1) \leq K(1 + k^{-1})\,\bar{u}^{\beta+1}.$$

Combining inequalities (3.13)–(3.16) yields

$$(\operatorname{sgn}\beta)(\phi_x A - \phi B) \geq \eta^2\gamma|G'|\,|\bar u_x|^2 - |\beta|\,\eta^2 G'|\bar u_x|^2(4\beta)^{-1}$$

$$- 4|\eta_x|^2\bar u^{\beta+1}|\beta|^{-1} - \eta^2|\beta|\,G'|\bar u_x|^2(4\beta)^{-1}$$

$$- \eta^2K^2\bar u^{\beta+1}|\beta|^{-1} - KG|\bar u_t| - \eta^2K(1+k^{-1})\bar u^{\beta+1}$$

$$= \tfrac12|\beta|\,\gamma\eta^2\bar u^{\beta-1}|\bar u_x|^2 - \eta^2KG|\bar u_t|$$

$$- \bar u^{\beta+1}(4|\eta_x|^2|\beta|^{-1} + \eta^2K^2|\beta|^{-1} + \eta^2K(1-k^{-1})).$$

Hence,

$$\eta^2KG|u_t| + (F\eta^2 + E|\eta_x|^2)\bar u^{\beta+1} + (\operatorname{sgn}\beta)(\phi_x A - \phi B) \geq \tfrac12|\beta|\,\gamma\eta^2\bar u^{\beta-1}|\bar u_x|^2,$$

where $F = K^2|\beta|^{-1} + K(1-k^{-1})$ and $E = 4|\beta|^{-1}$.

It follows that

$$(3.17)\quad (\operatorname{sgn}\beta)\left\{\int_\Omega \eta^2H\Big|_{t=\tau_1}^{t=\tau_2}dx + (\beta/2)\iint_D \eta^2\bar u^{\beta-1}|\bar u_x|^2\,dx\,dt\right\}$$
$$\leq \iint_D \eta^2KG|\bar u_t|\,dx\,dt + \iint_D J\bar u^{\beta+1}\,dx\,dt + 2\iint_D \eta|\eta_t|\,|H|\,dx\,dt,$$

where $J = F\eta^2 + E|\eta_x|^2$. But, if $\beta \neq -1$,

$$\iint_D \eta^2KG|\bar u_t|\,dx\,dt = \iint_D \eta^2K\bar u^\beta|\bar u_t|\,dx\,dt$$

$$= (\operatorname{sgn}\bar u_t)\iint_D \eta^2K(\beta+1)^{-1}(\bar u^{\beta+1})_t\,dx\,dt$$

$$= -(\operatorname{sgn}\bar u_t)(\beta+1)^{-1}\iint_D(\eta^2K)_t\bar u^{\beta+1}\,dx\,dt$$

$$= -(\operatorname{sgn}\bar u_t)(\beta+1)^{-1}\iint_D 2\eta\eta_tK\bar u^{\beta+1}\,dx\,dt, \quad \text{since } \eta \in C_0^1(D).$$

Now

$$-(\operatorname{sgn}\bar u_t)(\beta+1)^{-1}\iint_D 2\eta\eta_tK\bar u^{\beta+1}\,dx\,dt \leq |\beta+1|^{-1}\iint_D 2\eta|\eta_t|\,K\bar u^{\beta+1}.$$

We have thus proved the basic inequality

$$(3.18)\quad (\operatorname{sgn}\beta)\left\{\int_\Omega \eta^2H\Big|_{t=\tau_1}^{t=\tau_2}dx + (\beta/2)\iint_D \eta^2\bar u^{\beta-1}|\bar u_x|^2\,dx\,dt\right\}$$
$$\leq \iint_D J\bar u^{\beta+1}\,dx\,dt + (2+2K)|\beta+1|^{-1}\iint_D \eta|\eta_t|\,\bar u^{\beta+1}\,dx\,dt,$$

where $J = F\eta^2 + E|\eta_x|^2$, which holds for $\beta \neq -1$.

Let us now consider the case $\beta = -1$. Recall that (3.17) holds for all β. If $\beta = -1$, then $H = \log\bar u$, $G = \bar u^{-1}$. Then (3.17) becomes

$$(-1)\left\{\int_\Omega \eta^2 \log \bar u \Big|_{t=\tau_1}^{t=\tau_2} dx - \frac{1}{2}\iint_D \eta^2 \bar u^{-2}|\bar u_x|^2\,dx\,dt\right\}$$
$$\leq \iint_D J\,dx\,dt + 2\iint_D \eta\,|\eta_t|\,\log \bar u\,dx\,dt + \iint_D \eta^2 K\bar u^{-1}|\bar u_t|\,dx\,dt.$$

But

$$\iint_D \eta^2 K\bar u^{-1}|\bar u_t|\,dx\,dt = (\operatorname{sgn} \bar u_t)\iint_D \eta^2 K(\log \bar u)_t\,dx\,dt$$
$$= -(\operatorname{sgn} \bar u_t)\iint_D 2\eta\eta_t K \log \bar u\,dx\,dt$$
$$\leq \iint_D 2\eta\,|\eta_t|\,K\,|\log \bar u|\,dx\,dt,$$

since $\eta \in C_0^1(D)$, and hence

$$-\int_\Omega \eta^2 \log \bar u \Big|_{t=\tau_1}^{t=\tau_2} dx + \frac{1}{2}\iint_D \eta^2 \bar u^{-2}|\bar u_x|^2\,dx\,dt \tag{3.19}$$
$$\leq \iint_D J\,dx\,dt + (2+2K)\iint_D \eta\,|\eta_t|\,\log \bar u\,dx\,dt.$$

Suppose that $\beta \neq -1$. Then if $v = \bar u^r$, $r = (\beta+1)/2$, (3.18) becomes

$$(\operatorname{sgn}\beta)\left\{(\beta+1)^{-1}\int \eta^2 v^2 \Big|_{t=\tau_1}^{t=\tau_2} dx + \gamma\beta(2r^2)^{-1}\iint \eta^2|v_x|^2\,dx\,dt\right\} \tag{3.20}$$
$$\leq \|[J + (2+2K)\,|\beta+1|^{-1}\eta\,|\eta_t|]\,v^2\|_{1,1},$$

where the (x, t) integration is over $R(1) \times (\tau_1, \tau_2)$, and $J = F\eta^2 + E|\eta_x|^2$, $F = K^2|\beta|^{-1} + K(1-k^{-1})$, and $E = 4|\beta|^{-1}$.

The remainder of the proof is identical to that of Aronson and Serrin. The only difference between our fundamental inequalities (3.18), (3.19) and the corresponding ones of Aronson and Serrin is that inequalities (3.18) and (3.19) have the constant $2 + 2K$ as a coefficient of the second term on the right-hand side instead of the constant 2. (See [3, pp. 103–110].)

4. **The $\bar C_{2+\alpha}$ spaces.** We will assume the notation and description of D given in §1. We introduce the metric d defined by

$$d(P, Q) = [|x - \bar x|^2 + |t - \bar t|]^{1/2}, \tag{4.1}$$

where $P = (x, t)$, $Q = (\bar x, \bar t)$, and $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. Hölder continuity of a function f will be defined with respect to the metric d.

Suppose $0 < \alpha < 1$, and let

$$|u|_0^D = \sup_D |u|, \tag{4.2}$$

$$\overline{H}_\alpha^D(u) = \sup_{P,Q\in D} \frac{|u(P) - u(Q)|}{d(P, Q)^\alpha}, \tag{4.3}$$

$$|\overline{u}|_\alpha^D = |u|_0^D + \overline{H}_\alpha^D(u), \tag{4.4}$$

$$\overline{C}_\alpha(D) = \{u\colon\ u\colon\overline{D} \to R,\ |\overline{u}|_\alpha^D < \infty\}. \tag{4.5}$$

Theorem 4.1. *$\overline{C}_\alpha(D)$ is a Banach space with norm $|\overline{\cdot}|_\alpha^D$.*

Proof. (See [6, pp. 62–63].)

Denote by D_x^m any partial derivative of order m with respect to the variables $x_1, \cdots, x_n$ and let $D_t = \partial/\partial t$. If $D_x u$, $D_x^2 u$, $D_t u$ exist in D, then we define

$$|\overline{u}|_{2+\alpha}^D = |\overline{u}|_\alpha^D + \sum |\overline{D_x u}|_\alpha^D + \sum |\overline{D_x^2 u}|_\alpha^D + |\overline{D_t u}|_\alpha^D, \tag{4.6}$$

where the sums are taken over all partial derivatives of the indicated order. Let

$$\overline{C}_{2+\alpha}(D) = \{u\colon\ u\colon\overline{D} \to R,\ |\overline{u}|_{2+\alpha}^D < \infty\}. \tag{4.7}$$

Theorem 4.2. *$\overline{C}_{2+\alpha}(D)$ is a Banach space with norm $|\overline{\cdot}|_{2+\alpha}^D$.*

Proof. (See [6].)

Definition 4.3. We say that D has property $(\overline{\mathrm{E}})$ if for every point Q of $\overline{S}$, there exists an $(n+1)$-dimensional neighborhood V such that $V \cap \overline{S}$ can be represented, for some i $(1 \le i \le n)$, in the form $x_i = h(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t)$, where h, $D_x h$, $D_x^2 h$, $D_t h$ are Hölder continuous of exponent α.

Definition 4.4. If D has property $(\overline{\mathrm{E}})$ and if the functions $D_x D_t h$, $D_t^2 h$ of the local representations of $\overline{S}$ exist and are continuous functions, then we say that D has the property $(\overline{\overline{\mathrm{E}}})$.

Definition 4.5. A function ψ defined on $\overline{B} + S$ is said to belong to $\overline{C}_{2+\alpha}(D)$ if there exist functions Ψ in $\overline{C}_{2+\alpha}(D)$ such that $\Psi = \psi$ on $\overline{B} + S$. Then $|\overline{\psi}|_{2+\alpha}^D$ is defined by

$$|\overline{\psi}|_{2+\alpha}^D = \inf_{\Psi} |\overline{\Psi}|_{2+\alpha}^D, \tag{4.8}$$

where the infimum is taken over all $\Psi \in \overline{C}_{2+\alpha}(D)$ which coincide with ψ on $\overline{B} + S$.

5. **Refinement of the Friedman boundary estimates.** For the case $c \equiv 1$, A. Friedman [6] obtains a priori boundary estimates for u in terms of the $|\overline{u}|_{2+\alpha}$ norms. We will state the theorem which he proves. Assume that the coefficients of L, defined in (1.1), are uniformly Hölder continuous (exponent α) in D, that there exists a constant $\overline{K}_1$ such that $|\overline{a^{ij}}|_\alpha^D \le \overline{K}_1$, $|\overline{b^i}|_\alpha^D \le \overline{K}_1$, $|\overline{d}|_\alpha^D \le \overline{K}_1$, and $c \equiv 1$. Assume also that $|\overline{f}|_\alpha^D < \infty$ and that $u = \psi$ on $\overline{B} + S$. Furthermore, for any $(x, t) \in D$ and any real vector ξ,

$$\sum_{i=1}^{n} a^{ij}(x, t)\,\xi_i\xi_j \geq K_2|\xi|^2, \qquad K_2 > 0.$$

We will refer to all the above conditions as conditions (*).

Theorem 5.1 **(Friedman).** *Let conditions* (*) *hold and assume that* D *has the property* $(\overline{\mathrm{E}})$ *and that* $\psi \in \overline{C}_{2+\alpha}(D)$. *There exists a constant* $\overline{K}$ *depending only on* $\overline{K}_1$, K_2, α, *and* D *such that if* u *is a solution of the first initial-boundary value problem,* $Lu = f$ *on* $D + B_T$, $u = \psi$ *on* $\overline{B} + S$, *and if* $u \in \overline{C}_{2+\alpha}(D)$, *then*

$$|\overline{u}|^D_{2+\alpha} \leq \overline{K}(|\overline{\overline{\psi}}|_{2+\alpha} + |\overline{f}|_\alpha). \tag{5.1}$$

We remark that $\overline{K}$ also depends on $c(x, t)$ being identically one.

Friedman then proves the following theorem.

Theorem 5.2. *Let conditions* (*) *hold and assume that* D *has property* $(\overline{\overline{\mathrm{E}}})$, *that* $\psi \in \overline{C}_{2+\alpha}(D)$, *and that* $L\psi = f$ *on* ∂B. *Then there exists a unique solution* u *of the first initial-boundary value problem,* $Lu = f$ *on* $D + B_T^*$, $u = \psi$ *on* $\overline{B} + S$, *and, furthermore,* $u \in \overline{C}_{2+\alpha}(D)$.

We shall consider the case $c(x, t) \geq \mu > 0$ on $\overline{D}$, $\mu > 1$, which is clearly equivalent to the case $c(x, t) \equiv 1$, and determine exactly how the constant μ appears in the estimate corresponding to (5.1). The estimates of Friedman are obtained via the study of a fundamental solution of the adjoint of the equation

$$Lu = \sum_{i=1}^{n} a^{ij}(x, t)\, u_{x_i x_j} - c(x, t)\, u_t = 0.$$

Definition 5.3. A fundamental solution of $Lu = 0$ (in $\overline{D}$), L defined in (1.1), is a function $\Gamma(x, t; \xi, \tau)$ defined for all $(x, t) \in \overline{D}$, $(\xi, \tau) \in \overline{D}$, $t > \tau$, which satisfies the following conditions:

$$\begin{aligned} &\text{for fixed } (\xi, \tau) \in \overline{D},\ L(\Gamma(x, t; \xi, \tau)) = 0, \\ &\text{for each } (x, t) \text{ such that } x \in \Omega,\ t \in (\tau, T]; \end{aligned} \tag{5.2}$$

$$\text{if } f \in C(\overline{\Omega}) \text{ and } x \in \Omega, \text{ then } \lim_{t \to \tau^+} \int_\Omega \Gamma(x, t; \xi, \tau) f(\xi)\, d\xi = f(x). \tag{5.3}$$

$d\xi = d\xi_1 \cdots d\xi_n$ and Ω is Lebesgue measurable.

We will state the theorems analogous to the case $c(x, t) \equiv 1$ which are proved in [6]. The proofs are straightforward and generally the same as the case $c(x, t) \equiv 1$.

The procedure which Friedman uses to establish a fundamental solution is the parametrix method due to E. E. Levi [11]. Assume as previously that L is the operator (1.1) where

$$\lambda_0|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x, t)\xi_i\xi_j \leq \lambda_1|\xi|^2 \tag{5.4}$$

for some positive constants λ_0 and λ_1, where $(x, t) \in \overline{D}$ and $\xi = (\xi_1, \cdots, \xi_n)$ is a nonzero real vector.

Assume that the coefficients of L are continuous and satisfy, for all (x, t), $(x^0, t^0) \in \overline{D}$,

$$\text{(5.5)} \qquad \begin{cases} |a^{ij}(x, t) - a^{ij}(x^0, t^0)| \le C(|x - x^0|^\alpha + |t - t^0|^{\alpha/2}), \\ |b^i(x, t) - b^i(x^0, t^0)| \le C|x - x^0|^\alpha, \\ |d(x, t) - d(x^0, t^0)| \le C|x - x^0|^\alpha, \\ |c(x, t) - c(x^0, t^0)| \le C(|x - x^0|^\alpha + |t - t^0|^{\alpha/2}), \end{cases}$$

$c(x, t) \ge \mu > 0$, where $\mu > 1$.

Denote the inverse of the matrix $(a^{ij}(x, t))$ by $(\overline{a}^{ij}(x, t))$; and let

$$\text{(5.6)} \qquad Q^{y,\sigma}(x, \xi) = \sum_{i,j=1}^{n} \overline{a}^{ij}(y, \sigma)(x_i - \xi_i)(x_j - \xi_j),$$

where $y = (y_1, \cdots, y_n)$. Then

$$\text{(5.7)} \qquad \overline{\lambda}_0|x - \xi|^2 \le Q^{y,\sigma}(x, \xi) \le \overline{\lambda}_1|x - \xi|^2,$$

and

$$\text{(5.8)} \qquad |\overline{a}^{ij}(x, t) - \overline{a}^{ij}(x^0, t^0)| \le \overline{C}(|x - x^0|^\alpha + |t - t^0|^{\alpha/2})$$

where $\overline{\lambda}_0$, $\overline{\lambda}_1$, $\overline{C}$ are positive constants depending only on λ_0, λ_1, and C.

Now suppose that $t > \tau$ and define

$$\text{(5.9)} \qquad w^{y,\sigma}(x, t; \xi, \tau) = (t - \tau)^{-n/2} \exp\left\{-\frac{c(y, \sigma)\, Q^{y,\sigma}(x, \xi)}{4(t - \tau)}\right\},$$

and

$$\text{(5.10)} \qquad Z(x, t; \xi, \tau) = C(\xi, \tau)\, w^{\xi,\tau}(x, t; \xi, \tau),$$

where

$$\text{(5.11)} \qquad C(x, t) = [c(x, t)(4\pi)^{-n} \det(\overline{a}^{ij}(x, t))]^{1/2}.$$

It is then routine to show that for each fixed (ξ, τ), $Z(x, t; \xi, \tau)$ satisfies the equation $L_0 u = 0$ as a function of (x, t) for L_0 given by

$$\text{(5.12)} \qquad L_0 u = \sum_{i,j=1}^{n} a^{ij}(\xi, \tau)\, u_{x_i x_j} - c(\xi, \tau)\, u_t.$$

Theorem 5.4. *Let $f \in C(\overline{D})$. Then*

$$J(x, t, \tau) = \int_\Omega Z(x, t; \xi, \tau) f(\xi, \tau)\, d\xi$$

is a continuous function in (x, t, τ), $x \in \overline{\Omega}$, $0 \le \tau < t \le T$, and

$$\lim_{\tau \to t} J(x, t, \tau) = f(x, t)$$

uniformly with respect to (x, t), $x \in B$, $0 < t \le T$, *where* B *is any closed subset of* Ω.

Proof. The proof is a slight modification of [6, pp. 4–6].

Then Z satisfies (5.3) and is therefore a fundamental solution of $L_0 u = 0$. L_0 is regarded as a first approximation to L, and regarding Z as a principal part of the fundamental solution Γ to $Lu = 0$, we seek to determine Γ in the form

$$\Gamma(x, t; \xi, \tau) = Z(x, t; \xi, \tau) + \int_\tau^t \int_\Omega Z(x, t; \eta, \sigma)\Phi(\eta, \sigma; \xi, \tau)\, d\eta\, d\sigma. \tag{5.13}$$

Φ will be determined by the requirement that $L\Gamma = 0$.

By using volume potentials, it can be shown that (see [6]) solving the equation $L\Gamma = 0$ is equivalent to solving the Volterra integral equation

$$\Phi(x, t; \xi, \tau) = LZ(x, t; \xi, \tau) + \int_\tau^t \int_\Omega LZ(x, t; y, \sigma)\Phi(y, \sigma; \xi, \tau)\, dy\, d\sigma. \tag{5.14}$$

One shows that there exists a solution $\Phi(x, t; \xi, \tau)$ to (5.14) given by

$$\Phi(x, t; \xi, \tau) = \sum_{\nu=1}^{\infty} (LZ)_\nu(x, t; \xi, \tau),$$

where

$$(LZ)_1 = LZ$$

and

$$LZ_{\nu+1}(x, t; \xi, \tau) = \int_\tau^t \int_\Omega [LZ(x, t; y, \sigma)](LZ)_\nu(y, \sigma; \xi, \tau)\, dy\, d\sigma.$$

An argument similar to that of Friedman shows that the series converges and that

$$\begin{aligned} |\Phi(x, t; \xi, \tau)| &\le \mu^{-1/2} C(t-\tau)^{-\beta}|x-\xi|^{-(n+2-2\beta-\alpha)} \exp\left[-\frac{\mu^{1/2}\lambda_0^*|x-\xi|^2}{4(t-\tau)}\right] \\ &\le \mu^{-1/2} C(t-\tau)^{-\beta}|x-\xi|^{-n-2+2\beta+\alpha}, \end{aligned} \tag{5.15}$$

for $0 \le \beta \le (n+2-\alpha)/2$, where λ_0^* is any real number less than λ_0, $t - \tau > 0$, and C is a constant independent of μ.

Theorem 5.5. $\Phi(x, t; \xi, \tau)$ *is Hölder continuous in* x. *For any* $0 < \beta < \alpha$,

$$\begin{aligned} &|\Phi(x, t; \xi, \tau) - \Phi(y, \sigma; \xi, \tau)| \\ &\le \mu^{-1/2} C|x-y|^\beta (t-\tau)^{-(n+2-\gamma)/2} \\ &\quad \cdot \left\{\exp\left[-\frac{\mu^{1/2}\lambda^*|x-\xi|^2}{t-\tau}\right] + \exp\left[-\frac{\mu^{1/2}\lambda^*|y-\xi|^2}{t-\tau}\right]\right\}, \end{aligned} \tag{5.16}$$

where $\gamma = \alpha - \beta$, λ^* *is a positive constant, and* $t - \tau > 0$.

Proof. (See [6, p. 17].) One must, of course, keep track of $\mu^{1/2}$.

For the equation $Lu = 0$, where L is as defined in (1.1), one can write the adjoint equation

$$(5.17) \qquad L^*v = \sum_{i,j=1}^{n} a^{ij}(x, t)\, v_{x_i x_j} + \sum_{i=1}^{n} b^{i*} v_{x_i} + c^* v_t + d^* v = 0,$$

where $b^{i*} = -b^i + 2\sum_{j=1}^n a^{ij}_{x_j}$, $c^* = c - 2\sum_{j=1}^n a^{ij}_{x_j}$, $d^* = d - \sum_{i=1}^n b^i_{x_i} + \sum_{i,j=1}^n a^{ij}_{x_i x_j} + c_t$.

For $u, v \in C^{2,1}(\overline{D})$, Green's identity,

$$(5.18) \qquad vLu - uL^*v = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} (va^{ij}u_{x_j} - ua^{ij}v_{x_j} - uva^{ij}_{x_j}) + b^i uv \right]_{x_j} - (cuv)_t,$$

can be easily verified.

If u and v have compact support in D, the integration of both sides of (5.18) yields

$$(5.19) \qquad \iint_{\overline{D}} (vLu - uL^*v)\, dx\, dt = 0.$$

Definition 5.6. A fundamental solution of $L^*v = 0$ is a function $\Gamma^*(x, t; \xi, \tau)$ defined for all (x, t), $(\xi, \tau) \in \overline{D}$, $t < \tau$, satisfying

(5.20) if (ξ, τ) is fixed, $L^*\Gamma^* = 0$ as a function of (x, t), $x \in \Omega$, $t \in [0, \tau)$;

and

(5.21) if $f \in C(\overline{D})$, for each $x \in \overline{\Omega}$,

$$\lim_{t \to \tau^-} \int_\Omega \Gamma^*(x, t; \xi, \tau) f(\xi)\, d\xi = f(x).$$

Using the same method as for $Lu = 0$, we can construct a fundamental solution Γ^* of $L^*v = 0$. The parametrix $Z^*(x, t; \xi, \tau)$ is analogous to $Z(x, t; \xi, \tau)$, namely,

$$(5.22) \qquad Z^*(x, t; \xi, \tau) = C(x, t)(\tau - t)^{-n/2} \exp\left\{ -\frac{c(x, t)\, Q^{x,t}(x, \xi)}{4(\tau - t)} \right\}, \qquad t < \tau.$$

Z^* satisfies $L_0^* v = 0$, where L_0 is defined in (5.12).

The estimates

$$(5.23) \qquad |\Gamma^*(x, t; \xi, \tau)| \le \mu^{-1/2} C(\tau - t)^{-n/2} \exp\left\{ -\frac{\mu\lambda_0^*|x - \xi|^2}{4(\tau - t)} \right\}$$

and

(5.24) $|D_\tau^k D_\xi^j \Gamma^*(x, t; \xi, \tau)| \le \mu^{-1/2} C(\tau - t)^{-(n+2k+j)/2} \exp\left\{-\dfrac{K\mu|x-\xi|^2}{\tau - t}\right\}, \quad \mu < 1,$

can be verified in the usual manner.

Suppose that N is a fixed semicube with top $P = (x^0, t^0)$ and edge d; that is, N is defined by the inequalities

$$x_i^0 - d \le x_i \le x_i^0 + d, \qquad i = 1, \cdots, n,$$
$$t^0 - d^2 \le t \le t^0.$$

Note that if $Q \in N$, $d(P, Q) \le (n+1)^{1/2} d$.

Let $H_{Q,N}[g]$ be defined by

$$H_{Q,N}[g] = \sup_{R \in N} \frac{|g(Q) - g(R)|}{d(Q, R)^\alpha}. \tag{5.25}$$

Using the preceding estimates for $\Gamma^*(x, t; \xi, \tau)$, we can prove the following theorem.

Theorem 5.7. *Let f and c, $c \ge \mu > 0$, be Hölder continuous (exponent α) in N. Suppose that a^{ij} is constant for each i, j, that $|a^{ij}| \le k_1$, and that $\sum_{i,j=1}^n a^{ij}\xi_i\xi_j \ge k_2|\xi|^2$, $k_2 > 0$. Furthermore, let u, u_{x_i}, $u_{x_ix_j}$ be Hölder continuous (of exponent α) in N. If u satisfies*

$$Lu = \sum_{i,j=1}^n a^{ij} u_{x_ix_j} - cu_t = f \tag{5.26}$$

in N, then there is a constant K depending only on n, α, k_1, k_2 such that for $i = 0, 1, 2$,

$$|D_x^i u(P)| \le \mu^{-1/2} K[d^{-i} \sup_N |u| + d^{2-i} \sup_N |f| + d^{2-i+\alpha} H_{P,N}[f]] \equiv \mu^{-1/2} K I_i \tag{5.27}$$

and

$$\frac{d^\alpha |D_x^i u(P) - D_x^i u(Q)|}{d(P, Q)^\alpha} \le \mu^{-1/2} K[I_i + d^{2-i+\alpha} H_{Q,N}[f]] \tag{5.28}$$

if $d(P, Q) \le d/4$.

Proof. The proof is given in [6]. We need be concerned with the constant only.

The main theorem which we shall need is the following.

Theorem 5.8. *Suppose that the coefficients of L, defined in (1.1), are uniformly Hölder continuous (exponent α) in D and $\overline{|a^{ij}|}_\alpha \le \overline{K}_1$, $\overline{|b^i|}_\alpha \le \overline{K}_1$, $\overline{|c|}_\alpha \le \overline{K}_1$, $\overline{|d|}_\alpha \le \overline{K}_1$, that $c \ge \mu > 0$, that $\overline{|f|}_\alpha < \infty$, and that $\sum_{i,j=1}^n a^{ij}\xi_i\xi_j \ge K_2|\xi|^2$, $K_2 > 0$. Assume that D has property $(\overline{\mathrm{E}})$ and that $\psi \in \overline{C}_{2+\alpha}(D)$.*

Then there exists a constant $\overline{K}$ depending only on $\overline{K}_1$, K_2, α, and D such *that if* u *is a solution to* $Lu = f$, *with* $u = \psi$ *on* $\overline{B} + S$, *and if* $u \in \overline{C}_{2+\alpha}(D)$, *then*

(5.29) $$|\overline{u}|_{2+\alpha} \leq \mu^{-1/2}\overline{K}(|\overline{\psi}|_{2+\alpha} + |\overline{f}|_{\alpha}).$$

Proof. The proof follows directly from the proofs in [6], and the more refined estimates (5.24).

6. **Uniform estimates for** $|\overline{u_t}|_{\alpha}$. First, we will show that u_t is uniformly bounded, independent of the minimum of c on D. Then, knowing that u_t is uniformly bounded, we can estimate the Hölder coefficient of u_t, using the Harnack inequality derived in §3.

Theorem 6.1. *Assume that* a^{ij} *is constant for each* i, j, *that all coefficients of* L, *defined in* (1.1), *are of class* $C^{1,1}(\overline{D})$, $u \in C(\overline{D})$, $u \in C^3(D)$, *and* $a^{11}\lambda^2 + b^1\lambda \geq 1$ *for some* $\lambda > 0$. *Suppose, further, that the coefficients of* L *are uniformly Hölder continuous (exponent* α) *in* D, $|\overline{a^{ij}}|_{\alpha} \leq \overline{K}_1$, $|\overline{b^i}|_{\alpha} \leq \overline{K}_1$, $|\overline{c}|_{\alpha} \leq \overline{K}_1$, $|\overline{d}|_{\alpha} \leq \overline{K}_1$, *that* $m(c; \overline{B} + S) \geq \mu > 0$, *that* $|\overline{f}|_{\alpha} < \infty$, *and that* $\sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \geq K_2|\xi|^2$, $K_2 > 0$. *Assume that* D *has property* (E), *and that* $\psi \in \overline{C}_{2+\alpha}(D)$.

Then there exist positive constants C *and* $\overline{K}$, *where* C *depends only on* λ, $M(|\beta|; \overline{D})$, $\beta \in \{b_t, c_t, d_{x_i}, b_{x_i}, c_{x_i}\}$ *and* $\overline{K}$ *depends only on* $\overline{K}_1$, K_2, α, *and* D, *such that if* u *is a solution to* $Lu = f$, *with* $u = \psi$ *on* $\overline{B} + S$, *and if* $u \in \overline{C}_{2+\alpha}(D)$, *then*

$$M(|u_t|; \overline{D}) \leq C\left[\hat{\mu}^{-3/2} K(|\overline{\psi}|_{2+\alpha} + |\overline{f}|_{\alpha}) + M(|f_t|; \overline{D}) + \sum_{i=1}^{n} M(|f_{x_i}|; \overline{D}) + M(|f|; \overline{D})\right],$$

for some $\hat{\mu}$ *which is independent of* $m(c; D)$.

Proof. Since $m(c; \overline{B} + S) \geq \mu > 0$, there exists an ϵ-strip, $(\overline{B} + S)_{\epsilon}$, about $\overline{B} + S$ such that $(\overline{B} + S)_{\epsilon} \subset \overline{D}$ and $m(c; \overline{(\overline{B} + S)_{\epsilon}}) > 0$. Thus, $m(c; \overline{(\overline{B} + S)_{\epsilon}}) \geq \hat{\mu} > 0$. Also, u satisfies $Lu = f$ on $\overline{(\overline{B} + S)}_{\epsilon}$. If one mimics the proof of [6, Theorem 4, p. 121], it can be shown that

$$|\overline{u}|_{\alpha}^{(\overline{B}+S)_{\epsilon}} + \sum |\overline{D_x u}|_{\alpha}^{(\overline{B}+S)_{\epsilon}} + \sum |\overline{D_x^2 u}|_{\alpha}^{(\overline{B}+S)_{\epsilon}} \leq \hat{\mu}^{-1/2} K(|u|_0^D + |\overline{\psi}|_{2+\alpha}^D + |\overline{f}|_{\alpha}^D).$$

Since u satisfies $Lu = f$ and c is bounded away from zero, one can solve the differential equation over $(\overline{B} + S)_{\epsilon}$ for $D_t u$ in terms of u, $D_x u$ and $D_x^2 u$ and obtain a similar bound on $D_t u$. Solving the differential equation for $D_t u$ gives $\hat{\mu}^{-3/2}$ in the estimate. Corollary 2.7 guarantees that $|u|_0^D \leq C(M(|\psi|; D) + |f|_0)$ for some constant C. Hence we obtain an estimate for $|u_t|_{\alpha}^{(\overline{B}+S)_{\epsilon}}$.

It is clear that

$$M(|u|; \overline{B} + S) + M(|u_t|; \overline{B} + S) + \sum_{i=1}^{n} M(|u_{x_i}|; \overline{B} + S)$$
$$\leq M(|u|; (\overline{B} + S)_{\epsilon}) + M(|u_t|; (\overline{B} + S)_{\epsilon}) + \sum_{i=1}^{n} M(|u_{x_i}|; (\overline{B} + S)_{\epsilon}).$$

We have already proved that the right-hand side of the above

$$\leq \hat{\mu}^{-3/2} K(|u|_0^D + |\overline{\psi}|_{2+\alpha}^D + |\overline{f}|_\alpha^D),$$

which is

$$\leq \hat{\mu}^{-3/2} \overline{K}(|\overline{\psi}|_{2+\alpha} + |\overline{f}|_\alpha).$$

Applying Theorem 2.8, the desired inequality follows.

We have now shown that for constant a^{ij} and $c > 0$ on $\overline{B} + S$, $|u_t|$ is uniformly bounded on $\overline{D}$. Since $|u_t|$ is bounded, independent of $m(c; D)$, we can proceed with estimating the Hölder coefficient (exponent α) of u_t. Recall that if a^{ij} is constant for each i, j, and u satisfies $Lu = f$, where L is defined as in (1.1), then u_t satisfies

$$(L - c_t)u_t = f_t - \sum_{i=1}^n b_t^i u_{x_i} - d_t u. \tag{6.1}$$

Theorem 6.2. *Suppose that u_t is a bounded nonnegative classical solution of (6.1) in D, where a^{ij} is constant for each i, j. Suppose that all coefficients of L are of class $C^{1,1}(\overline{D})$, $u \in C(\overline{D})$, $u \in C^3(D)$, and $a^{11}\lambda^2 + b^1\lambda \geq 1$ for some $\lambda > 0$. Suppose that the functions u, u_t, a^{ij}, b^i, c, d, f_t, $\sum_{i=1}^n b_{x_i}^i$, c_t, and d_t satisfy the property that their absolute values are uniformly bounded by some constant. If $D(3\rho) \subset D$, then*

$$\operatorname*{ess\,max}_{D^*(\rho)} u_t \leq C \left(\operatorname*{ess\,min}_{D(\rho)} (u + \rho^\theta k) \right),$$

where C is a positive constant depending only on ρ and the structure of (6.1), θ is defined in (3.8), and $k = \|f_t\| + \|d_t u\| + \|1 - c\|$. Furthermore, C is independent of $\|f_t\|$, $\|d_t\|$, $\|1 - c\|$.

Proof. The equation (6.1) can be rewritten in the form

$$\sum_{i,j=1}^n \{a^{ij}(u_t)_{x_i}\}_{x_j} + \sum_{i=1}^n b^i(u_t)_{x_i} + (1 - c)(u_t)_t$$
$$+ (d - c_t)\, u_t - \sum_{i=1}^n b_t^i u_{x_i} - (d_t u + f_t) = (u_t)_t.$$

The structure inequalities

$$\begin{aligned} &p \cdot A(x, t, p) \geq \gamma |p|^2, \\ &|B(x, t, u_t, p, q)| \leq C(|p| + |q| + |u_t| + 1), \\ &|A(x, t, p)| \leq C|p| \end{aligned} \tag{6.2}$$

are obviously satisfied, with $A(x, t, p)$ defined as before in (3.1) and

$B(x, t, u_t, (u_t)_x, (u_t)_t) = b(u_t)_x + (1-c)(u_t)_t + (d - c_t)u_t + b_t u_x - (d_t u + f_t)$. Now apply Theorem 3.1.

Theorem 6.3. *Suppose that u satisfies the hypotheses of Theorems 6.1 and 6.2. Suppose that $Q = (x, t)$, $P = (y, s)$ are points in D with $s \leq t$. Then*

$$|u_t(P) - u_t(Q)| \leq H\left(\max_{\overline{B}+S} |\psi| + k\right)(d(P, Q)/R)$$

where H and α $(0 < \alpha < 1)$ are positive constants depending only on the structure of (6.1), $k = \|f_t\| + \|d_t u\| + \|1 - c\|$, *and* $R = \min\{d(Q, \overline{B} + S), 1\}$.

Proof. Since u is a solution to the first initial-boundary value problem, a corollary to the weak maximum principle, Corollary 2.7, implies that $|u|$ is uniformly bounded on $\overline{D}$, independent of the minimum of c on D. Theorem 6.1 assures that $|u_t|$ is uniformly bounded on $\overline{D}$, independent of $m(c; D)$. Corollary 2.9 together with the argument of Theorem 6.1 implies that $|u_x|$ is uniformly bounded on $\overline{D}$, likewise independent of $m(c; D)$. This theorem is analogous to Theorem 4 of [3, p. 110] of Aronşon and Serrin, which holds for $c(x, t) \equiv 1$. We use Theorem 6.2 in place of their analogue, Theorem 3, and the proof is identical.

We have thus established interior Hölder continuity for u_t and have shown that on a compact set in the interior of D, the Hölder coefficient is uniformly bounded independent of $m(c; D)$. We now establish uniform Hölder continuity of u_t in $\overline{D}$.

Theorem 6.4. *Assume that the conditions of Theorem 6.3 hold. Then u_t is uniformly Hölder continuous (exponent α) in $\overline{D}$ and the Hölder coefficient of u_t is bounded independent of $m(c; D)$.*

Proof. The proof is similar to that of Theorem 6.1. Since $m(c; \overline{B} + S) \geq \mu > 0$, there exists an ϵ-strip (measured in the metric d), $(\overline{B} + S)_\epsilon$, about $\overline{B} + S$, $(\overline{B} + S)_\epsilon \subset \overline{D}$, such that $m(c; (\overline{B} + S)_\epsilon) \geq \mu > 0$. Again, the proof by Friedman [6, Theorem 4, p. 121] holds since c is bounded away from zero on $(\overline{B} + S)_\epsilon$. As in the proof of Theorem 6.1, we see that

$$|u|_\alpha^{(\overline{B}+S)_\epsilon} + |\overline{D_x u}|_\alpha^{(\overline{B}+S)_\epsilon} + |\overline{D_x^2 u}|_\alpha^{(\overline{B}+S)_\epsilon} \leq \hat{\mu}^{-3/2} K(|u|_0^D + |\overline{\psi}|_{2+\alpha} + |\overline{f}|_\alpha).$$

Solve $Lu = f$ for u_t in $(\overline{B} + S)_\epsilon$. Then

$$|\overline{u_t}|_\alpha^{(\overline{B}+S)_\epsilon} \leq \hat{\mu}^{-3/2} K(M(|\psi|; D) + |f|_0 + |\overline{\psi}|_{2+\alpha} + |\overline{f}|_\alpha).$$

Therefore,

$$\overline{H}_\alpha^{(\overline{B}+S)_\epsilon}(u_t) \leq \hat{\mu}^{-3/2} \overline{K}(|\overline{\psi}|_{2+\alpha} + |\overline{f}|_\alpha).$$

Let D_ϵ be the complement of $(\overline{B} + S)_\epsilon$ in $\overline{D}$. Theorem 6.3 implies $\overline{H}_\alpha^{D_\epsilon}(u_t) \leq C/\epsilon^\alpha$ for some constant C. Hence, u_t is uniformly Hölder continuous (exponent α) in $\overline{D}$ and the Hölder coefficient is bounded independent of $m(c; D)$.

We now proceed to show existence of a solution to the first initial-boundary value problem in case the conditions of Theorems 6.3–6.4 hold.

7. **A classical (strong) solution.** This section shall be devoted to proving the existence of a unique solution to the first initial-boundary value problem for the nonuniformly parabolic operator (1.1) where a^{ij} is constant for each i, j. Assume that all the conditions mentioned in Theorems 6.3–6.4 are satisfied. Assume also that D has property $(\overline{\overline{\mathrm{E}}})$; see Definition 4.4.

The method of proof involves perturbing the coefficient c. We shall use the sharper Schauder-type estimates given in §5 as well as the estimate for $|\overline{u_t}|^D$, which is independent of $m(c; D)$, to obtain a Cauchy sequence of solutions $\{u_n\}$ which converges to a classical solution. The following existence theorem for $c(x, t) \equiv 1$ is due to Friedman [6, p. 65].

Theorem 7.1. *Assume that the conditions* (*) *of Theorem* 5.1 *are satisfied, that D has property* $(\overline{\overline{\mathrm{E}}})$, *that* $\psi \in \overline{C}_{2+\alpha}(D)$, *and that* $L\psi = f$ *on* ∂B. *Then there exists a unique solution u of the first initial-boundary value problem* $Lu = f$ *on* $D + B_T$, $u = \psi$ *on* $\overline{B} + S$ *and, furthermore,* $u \in \overline{C}_{2+\alpha}(D)$.

Remark. Theorem 7.1 holds for $c(x, t) \geq \mu > 0$, which is clearly equivalent to the case $c(x, t) \equiv 1$.

We now solve the first initial-boundary value problem

$$Lu = \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i(x, t)\, u_{x_i} - c(x, t)\, u_t + d(x, t)\, u \tag{7.1}$$
$$= f \quad \text{on } D + B_T, \qquad u = \psi \quad \text{on } \overline{B} + S,$$

without assuming that c has a positive minimum in D.

Perturb the coefficient c by $1/k$ and consider the problem

$$L^k u = Lu - (1/k)\, u_t = f \quad \text{on } D + B_T, \qquad u = \psi \quad \text{on } \overline{B} + S. \tag{7.2}$$

By Theorem 7.1, there exists a unique solution $u_k \in \overline{C}_{2+\alpha}(D)$ of (7.2).

We wish to show that the sequence $\{u_k\}$ obtained in this manner is Cauchy in the Banach space $\overline{C}_{2+\alpha}(D)$. Consider, for $m > n$, $u_n - u_m$. $u_n - u_m$ satisfies

$$\sum_{i,j=1}^{n} a^{ij}(u_n - u_m)_{x_i x_j} + \sum_{i=1}^{n} b^i(x, t)(u_n - u_m)_{x_i}$$
$$+ d(x, t)(u_n - u_m) - [c(x, t) + (1/n)](u_n - u_m)_t$$
$$= [(1/n) - (1/m)](u_m)_t \quad \text{on } D + B_T,$$

with $u_n - u_m = 0$ on $\overline{B} + S$.

Applying Theorem 5.8, we have the estimate

$$\overline{|u_n - u_m|}^D_{2+\alpha} \le n^{1/2}\overline{K}|(1/n) - (1/m)|\,\overline{|(u_m)_t|}_\alpha \le n^{1/2}|(1/n) - (1/m)|\,\overline{K}M,$$

for some constant M which is independent of n and m. The latter inequality follows from §6. Since $m > n$,

$$n^{1/2}|(1/n) - (1/m)| \le n^{-1/2} + m^{-1/2}.$$

Hence, $\lim_{m,n\to\infty}\overline{|u_n - u_m|}^D_{2+\alpha} = 0$. We have shown, therefore, that the sequence $\{u_k\}$ is Cauchy. Since $\overline{C}_{2+\alpha}(D)$ is a Banach space, $\{u_k\} \to u$ in $|\cdot|^D_{2+\alpha}$ for some $u \in \overline{C}_{2+\alpha}(D)$. That is, $\{u_n\}$ converges to u and the sequences of derivatives converge to the corresponding derivatives of u.

Since u_n satisfies $L^n u_n = f$ in $D + B_T$, $u_n = \psi$ in $\overline{B} + S$, we let $n \to \infty$ and obtain that u satisfies (7.1).

Uniqueness follows in the usual manner. Apply Corollary 2.7 to the difference $u - v$ of u and v where u and v both satisfy (7.1).

We have thus proved the main theorem.

Theorem 7.2. *Assume that the conditions of Theorems 6.3 and 6.4 are satisfied, that D has property* $(\overline{\overline{\text{E}}})$, *that $\psi \in \overline{C}_{2+\alpha}(D)$, and that $L\psi = f$ on ∂B. Then there exists a unique solution u of the first initial-boundary value problem* (7.1) *and, furthermore, $u \in \overline{C}_{2+\alpha}(D)$.*

Certain problems can be reduced to the case a^{ij} constant. If one can show that u_t is bounded, independent of the minimum of c in D, then the techniques we have for estimating the Hölder coefficient of u_t do not require the a^{ij} to be constant.

Friedman derives estimates for the case a^{ij} constant and then uses a perturbation technique to derive estimates for the general case. We cannot do the same because our estimates are not in such a usable form. It seems reasonable to expect that the case a^{ij} constant can be used to obtain the more general result.

In the special case of one space variable, $Lu = f$, where L is defined by (1.1), becomes

$$a(x,t)\,u_{xx} + b(x,t)\,u_x - c(x,t)\,u_t + d(x,t)\,u = f(x,t).$$

Since $a(x, t)$ is bounded away from zero, the above equation is equivalent to

$$u_{xx} + \frac{b(x,t)}{a(x,t)}u_x - \frac{c(x,t)}{a(x,t)}u_t + \frac{d(x,t)}{a(x,t)}u = \frac{f(x,t)}{a(x,t)},$$

and Theorem 7.2 guarantees existence of a unique solution to the first initial-boundary value problem.

REFERENCES

1. D. G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa 29 (1968), 607–694.

2. ———, *Regularity properties of flows through porous media*, SIAM J. Appl. Math. 17 (1969), 461–467. MR **40** #571.

3. D. G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal. **25** (1967), 81–122. MR 39 #5952.

4. W. T. Ford, *Elements of simulation of fluid flow in porous media*, Texas Tech University Press, Lubbock, Tex., 1971.

5. ———, *The first initial boundary value problem for a nonuniform parabolic equation*, J. Math. Anal. Appl. 3 (1972).

6. A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, N. J., 1964. MR 31 #6062.

7. C. D. Hill, *A sharp maximum principle for degenerate elliptic-parabolic equations* (to appear).

8. J. J. Kohn and L. Nirenberg, *Degenerate elliptic-parabolic equations of second order*, Comm. Pure Appl. Math. **20** (1967), 797–872. MR **38** #2437.

9. O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural′ceva, *Linear and quasi-linear equations of parabolic type*, "Nauka", Moscow, 1967; English transl., Transl. Math. Monographs, vol. 23, Amer. Math. Soc., Providence, R.I., 1968. MR **39** #3159a, b.

10. O. A. Ladyženskaja and N. N. Ural′ceva, *On Hölder continuity of solutions and their derivatives of linear and quasilinear elliptic and parabolic equations*, Trudy Mat. Inst. Steklov. 73 (1964), 172–220; English transl., Amer. Math. Soc. Transl. (2) **61** (1967), 207–269. MR **30** #4063.

11. E. E. Levi, *Sulle equazioni lineari totalmente ellittiche alle derivate parziali*, Rend. Circ. Mat. Palermo **24** (1907), 275–317.

12. J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134; correction, ibid. **20** (1967), 231–236. MR **28** #2357; MR **34** #3121.

13. O. A. Oleĭnik, *Quasi-linear second-order parabolic equations with many independent variables*, Seminari 1962/63 Anal. Alg. Geom. e Topol., vol. 1, Ist. Naz. Alta Mat., Edizioni Cremonese, Rome, 1965, pp. 332–354. MR 33 #2969.

14. O. A. Oleĭnik and S. N. Kružkov, *Quasi-linear parabolic second-order equations with several independent variables*, Uspehi Mat. Nauk **16** (1961), no. 5 (101), 115–155 = Russian Math. Surveys **16** (1961), no. 5, 105–146. MR **25** #5289.

15. M. H. Protter and H. F. W. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N. J., 1967. MR 36 #2935.

16. N. S. Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math. 21 (1968), 205–226. MR 37 #1758.

17. E. D. Williams, *The numerical solution of degenerate parabolic equations*, Doctoral Dissertation, Texas Tech University, Lubbock, Tex., 1971.

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