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RINGS WITH PROPERTY D

BY

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ABSTRACT. An integral domain is said to have property D if every torsion-free module of finite rank is a direct sum of modules of rank one. In recent papers the author has given partial solutions to the problem of finding all rings with this property. In this paper the author is finally able to show that an integrally closed integral domain has property D if and only if it is the intersection of at most two maximal valuation rings.

1. **Introduction.** An integral domain R is said to have *property* D if every torsion-free R-module of finite rank is a direct sum of modules of rank 1. The main theorem of this paper will prove that an integrally closed domain has property D if and only if it is the intersection of at most two maximal valuation rings. This complete characterization is the end result of a long chain of involved theorems, many of which have appeared in earlier papers. In order to make the final arguments which appear in this paper comprehensible, we will reproduce in this section some of the pertinent definitions and the statements of some of the theorems that have appeared elsewhere.

Definition. An integral domain R is called a *valuation ring* if either x or $1/x$ is in R for every nonzero element x of the quotient field Q of R. A valuation ring R is called a *maximal valuation ring* if Q/R is an injective R-module and R is complete in the R-topology—the topology formed by taking the nonzero ideals of R as a system of neighborhoods of 0 in R.

Definition. An integral domain R is said to be a *ring of type* I if R has exactly two maximal ideals M_1, M_2; R_{M_1}, R_{M_2} are maximal valuation rings; and $M_1 \cap M_2$ contains no nonzero prime ideal of R. An equivalent definition is that $R = V_1 \cap V_2$, where V_1, V_2 are maximal valuation rings with the same quotient field Q; and if W is a valuation ring contained in but not equal to Q, then W contains at most one of the rings V_1, V_2.

Definition. Let R be an integral domain with quotient field Q, and let A be an R-submodule of Q. Then $A^{-1} = \{q \in Q \mid qA \subset R\}$. It is easy to see that $A^{-1} \cong \operatorname{Hom}_R(A, R)$.

Definition. Let R be an integral domain with quotient field Q. We say that Q *is remote from* R if there exists an R-submodule A of Q, $A \neq Q$, such that $A^{-1} = 0$.

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Theorem D1. *Let R be an integral domain with property* D. *If S is a ring extension of R in the quotient field of R, then S also has property* D.

Proof. [6, Lemma 6.2].

Theorem D2. *Let R be a valuation ring. Then R has property* D *if and only if R is a maximal valuation ring.*

Proof. [2, Theorem 12] and [7, Theorem 2].

Theorem D3. *Let R be an integral domain. Then R is a ring of type* I *if and only if R has property* D *and a remote quotient field.*

Proof. [8, Theorem 3].

Theorem D4. *Let R be an integrally closed domain with property* D. *Then R is a Prüfer ring.*

Proof. [9, Corollary 1].

2. Technical lemmas.

Lemma 1. *Let R be an integral domain and A a torsion-free R-module of finite rank n. Suppose that $A = A_1 + \cdots + A_m$ where $\sum_{i=1}^m \operatorname{rank} A_i \leq n$. Then $A = A_1 \oplus \cdots \oplus A_m$.*

Proof. Let $B = A_1 \oplus \cdots \oplus A_m$; then the natural mapping $\phi: B \longrightarrow A$ is surjective. Thus $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank}(\operatorname{Ker} \phi)$. But $\operatorname{rank} B = \sum_{i=1}^m \operatorname{rank} A_i \leq \operatorname{rank} A$, and hence $\operatorname{rank}(\operatorname{Ker} \phi) = 0$. It follows that $\operatorname{Ker} \phi = 0$, and thus ϕ is an isomorphism.

Lemma 2. *Let R be a maximal valuation ring, and A a torsion-free R-module. Let B be a submodule of A such that A/B is a finite direct sum of k cyclic torsion R-modules, where $k < \operatorname{rank} A$. Then there is a nonzero direct summand of B which is also a direct summand of A.*

Proof. There exists a pure submodule F of A of rank $\leq k$ which maps onto A/B. Let $C = F \cap B$; then C is a pure submodule of B. Hence by [4, Theorem 9] C is a direct summand of B, and there is a submodule D of B such that $B = C \oplus D$.

We will prove that $A = F \oplus D$. Now $A = F + B = F + C + D = F + D$. Let $x \in F \cap D$. Since F/C is isomorphic to A/B, F/C is a torsion R-module. Hence there exists a nonzero element r in R such that $rx \in C$. But then $rx \in C \cap D = 0$, and hence $x = 0$. Thus $F \cap D = 0$, and we have $A = F \oplus D$.

Since $\operatorname{rank} F < \operatorname{rank} A$, D is a nonzero module.

The following lemma was proved in [3, Lemma 1.1], but we reproduce it here for it is an elementary but essential part of the proof of Lemma 4.

Lemma 3. *Let R be a ring, S and T R-modules, and D an injective submodule of $S \oplus T$. Let E be an injective envelope of $D \cap S$ in D, and let F be a comple-*

mentary summand of E in D. Thus $D = E \oplus F$; and E and F project monomorphically into S and T, respectively.

Proof. The kernel of the projection of D into T is $D \cap S$. Hence F projects monomorphically into T. Let f be the projection of E into S. Since $\operatorname{Ker} f \subset T$, we have $\operatorname{Ker} f \cap (D \cap S) = 0$. However, E is an essential extension of $D \cap S$, and thus $\operatorname{Ker} f = 0$.

Lemma 4. *Let R be an integral domain and A a torsion-free, cotorsion R-module of finite rank. Suppose that A has two direct sum decompositions: $A = B \oplus C$ and $A = D_1 \oplus \cdots \oplus D_n$, where* inj $\dim_R C = 1$ *and B and $D_1, \cdots, D_n$ are indecomposable R-modules. Then there exists an integer i, $1 \le i \le n$, such that $B \cong D_i$.*

Proof. Let Q be the quotient field of R, and let $K = Q/R$. Let $K \otimes_R B = G$, $K \otimes_R C = E$, and $K \otimes_R D_i = H_i$ for $i = 1, \cdots, n$. Then we have $G \oplus E = H_1 \oplus \cdots \oplus H_n$. Since $\operatorname{Hom}_R(K, G) \cong B$ and $\operatorname{Hom}_R(K, H_i) \cong D_i$ by [5, Proposition 3.2], it is necessary and sufficient to prove that $G \cong H_i$ for some $i = 1, \cdots, n$. By [5, Corollary 3.3] G and H_i are indecomposable R-modules. Since inj $\dim_R C = 1$, E is an injective R-module.

Suppose that G is an injective R-module. By Lemma 3, $G = G_1 \oplus G_2$ where G_1 is isomorphic to a direct summand of H_1 and G_2 is isomorphic to a direct summand of $H_2 \oplus \cdots \oplus H_n$. Since G is indecomposable, either $G = G_1$ or $G = G_2$. If $G = G_1$, then $G \cong H_1$, since H_1 is indecomposable. Hence we can assume that $G = G_2$. Repeating this argument we see that either $G \cong H_2$ or G is isomorphic to a direct summand of $H_3 \oplus \cdots \oplus H_n$. Continuing in this way we find that G must be isomorphic to H_i for some $i = 1, \cdots, n$. Thus we can assume that G is not an injective R-module.

If every H_i is injective, then their direct sum is injective, and hence G, which is a direct summand of their direct sum, is injective. This contradiction shows that some of the H_i's are not injective. Let L_1 be the direct sum of the H_i's which are injective, and let L_2 be the direct sum of the H_i's which are not injective. Then $L_2 \neq 0$, and we have

$$E \oplus G = L_1 \oplus L_2.$$

Now L_2 has no nonzero injective submodules. For suppose that M is a nonzero injective direct summand of L_2. Then $\operatorname{Hom}_R(K, M)$ is a direct summand of $\operatorname{Hom}_R(K, L_2)$, and thus $\operatorname{Hom}_R(K, M)$ is a torsion-free cotorsion module of finite rank. Thus $\operatorname{Hom}_R(K, M)$ is the direct sum of a finite number of indecomposable modules. Therefore, by the duality of [5, Proposition 3.2], M is the direct sum of a finite number of indecomposable injective modules. Therefore, we can assume that M is an indecomposable injective module. By repeating the process carried out earlier with G we see that M is isomorphic to one of the H_i's making up L_2. This contradiction shows that L_2 has no nonzero injective submodules.

By Lemma 3, $E = E_1 \oplus E_2$, where E_1 projects monomorphically into L_1 and E_2 projects monomorphically into L_2. Since L_2 has no nonzero injective submodules, $E = E_1$ maps monomorphically into L_1. Thus $L_1 = S_1 \oplus S_2$, where S_1 is the image of E and is isomorphic to E. Thus there is a mapping f of $L_1 \oplus L_2$ onto S_1 with kernel $S_2 \oplus L_2$ which sends E isomorphically onto S_1. Hence we have $L_1 \oplus L_2 = E + \operatorname{Ker} f$ and $E \cap \operatorname{Ker} f = 0$. Thus $L_1 \oplus L_2 = E \oplus \operatorname{Ker} f = E \oplus S_2 \oplus L_2$.

Therefore, we have $S_2 \oplus L_2 \cong (L_1 \oplus L_2)/E$. But $(L_1 \oplus L_2)/E \cong (E \oplus G)/E \cong G$. Hence we have proved that $G \cong S_2 \oplus L_2$. Since G is indecomposable and L_2 is a direct sum of H_i's, there is an H_i such that $G \cong H_i$.

Lemma 5. *Let R be an integral domain and P a prime ideal of R such that $P = PR_P$ and R_P is a valuation ring. If A is a torsion-free R-module of finite rank n, then A/PA is a torsion-free R/P-module of rank $\leq n$. Conversely, if B is a torsion-free R/P-module of rank n, then there is a free R_P-module F of rank n and an R-submodule A of F such that $PF \subset A \subset F$ and $B \cong A/PF$.*

Proof. Let A be a torsion-free R-module of rank n. Since $A/PA \subset A_P/PA$, it is sufficient to prove that A_P/PA is a torsion-free R/P-module of R/P rank $\leq n$. Since $R_PPA_P = PR_PA = PA$, we see that $A_P/PA = A_P/R_PPA_P$. Thus A_P/PA is a module over R_P/PR_P and hence is a torsion-free R/P-module. We will prove that rank $A_P/PA \leq n$ over R/P by induction on n, the rank of A over R. Suppose that rank $A = 1$. We may assume that $PA \neq A_P$, and consequently A_P is not the quotient field of R_P. Since R_P is a valuation ring, it follows that A_P is isomorphic to an ideal of R_P. Because the ideals of R_P are linearly ordered, it follows that A_P/PA is isomorphic to R_P/PR_P and hence has rank 1 over R/P.

We assume that rank $A = n > 1$, and the assertion true for modules of rank $n - 1$. Let C be a pure R_P-submodule of A_P of rank $n - 1$, and let $D = A_P/C$. Then D is a torsion-free R_P-module of rank 1. Since D is a flat R_P-module we have an exact sequence:

$$0 \longrightarrow C/PC \longrightarrow A_P/PA_P \longrightarrow D/PD \longrightarrow 0.$$

By the case $n = 1$, D/PD is a torsion-free R/P-module of rank ≤ 1. By induction C/PC is a torsion-free R/P-module of rank $\leq n - 1$. Thus A_P/PA_P is a torsion-free R/P-module of rank $\leq n$.

Conversely, let B be a torsion-free R/P-module of rank n. Let F be a free R_P-module of rank n. Since R_P/P is the quotient field of R/P, F/PF is a direct sum of n copies of the quotient field of R/P. Thus we can assume that B is an

R/P-submodule (and hence an R-submodule) of F/PF. Thus there is an R-submodule A of F such that $PF \subset A \subset F$ and $B \cong A/PF$.

3. **Main theorems.**

Theorem 1. *Let R be an integral domain and P a prime ideal of R such that $P = PR_P$ and R_P is a maximal valuation ring. Then R has property* D *if and only if R/P has property* D.

Proof. Suppose that R has property D. Let B be a torsion-free R/P-module of rank $n > 1$. By Lemma 5,there is a free R_P-module F of rank n and an R-submodule A of F such that $PF \subset A \subset F$ and $A/PF \cong B$. Since R has property D, $A = A_1 \oplus \cdots \oplus A_n$, where A_i has rank 1. Thus $A/PA \cong A_1/PA_1 \oplus \cdots \oplus A_n/PA_n$. By Lemma 5, A_i/PA_i is a torsion-free R/P-module of rank ≤ 1.

Since $PA \subset PF$, we have an R/P-homomorphism of A/PA onto B. Thus $B = B_1 + \cdots + B_n$, where B_i is the image of A_i/PA_i and hence is a torsion-free R/P-module of rank ≤ 1. By Lemma 1, $B = B_1 \oplus \cdots \oplus B_n$. Therefore R/P has property D.

Conversely, assume that R/P has property D. We will assume that R does not have property D and arrive at a contradiction. Let n be the smallest integer greater than one for which there is an indecomposable torsion-free R-module of rank n, and let A be an indecomposable torsion-free R-module of rank n. Then A is a reduced R-module.

By Lemma 5, A/PA is a torsion-free R/P-module whose R/P rank is less than or equal to n. Since PA is an R_P-module, and R_P has property D, PA is a direct sum of n R_P-modules of rank 1. Hence we see that $A \neq PA$.

Since R/P has property D, A/PA is a finite direct sum of torsion-free R/P-modules of rank 1. Therefore $A/PA = B_1/PA \oplus B_2/PA$, where B_1, B_2 are submodules of A such that $A = B_1 + B_2$, $B_1 \cap B_2 = PA$, and rank $B_i/PA = k_i < n$ for $i = 1, 2$.

We will prove that B_1 and B_2 are each direct sums of n R-modules of rank 1. For this we may assume that $B_1 \neq PA$ and $B_2 \neq PA$, since PA is such a direct sum. Let $C = B_1/PA$; then $C = C_1 \oplus \cdots \oplus C_{k_1}$ where C_j is a torsion-free R/P-module of rank 1 and $k_1 < n$. By Lemma 5, $C_{j_P} \cong R_P/P$. Therefore, $C_P = C_{1P} \oplus \cdots \oplus C_{k_1P}$ is the direct sum of k_1 cyclic R_P-modules. But $C_P = B_{1P}/PA$, and B_{1P} is a torsion-free R_P-module of rank n. Thus we can apply Lemma 2 and find a direct sum decomposition $B_{1P} = L \oplus M$, where $L \neq 0$ and $L \subset PA$. It follows that $B_1 = L \oplus (M \cap B_1)$.

Since $B_1 \neq PA$, we see that $M \cap B_1 \neq 0$. But then L and $M \cap B_1$ are nonzero torsion-free R-modules each of rank $< n$. By the minimality of the integer n both L and $M \cap B_1$ are direct sums of R-modules of rank 1. Hence in this case also B_1 is a direct sum of n-modules of rank 1. Similarly B_2 is a direct sum of n-modules of rank 1.

Since $PA = B_1 \cap B_2$ and $B_1 + B_2 = A$, we have an exact sequence

$$0 \longrightarrow PA \longrightarrow B_1 \oplus B_2 \longrightarrow A \longrightarrow 0.$$

Since R_P is a maximal valuation ring and PA is a torsion-free R_P-module, we have by [4, Theorem 9] that inj $\dim_{R_P} PA = 1$. Hence by [1, Chapter VI, Exercise 10] we have inj $\dim_R PA \leq$ inj $\dim_{R_P} PA = 1$.

Let Q be the quotient field R of R. Using [1, Chapter VI, Proposition 4.1.3] and [4, Theorem 9] we have $\operatorname{Ext}_R^1(Q \otimes_R A, PA) \cong \operatorname{Ext}_{R_P}^1(Q \otimes_R A, PA) = 0$. Since inj $\dim_R PA = 1$, we have $\operatorname{Ext}_R^2(K \otimes_R A, PA) = 0$. From this it follows that $\operatorname{Ext}_R^1(A, PA) = 0$. This proves that the preceding exact sequence splits, and we have

$$PA \oplus A \cong B_1 \oplus B_2.$$

From an earlier paragraph we have $B_1 \oplus B_2 = D_1 \oplus \cdots \oplus D_{2n}$, where each D_i is a reduced torsion-free R-module of rank 1.

As we have just seen, $\operatorname{Ext}_R^1(Q, PA) = 0$. Therefore PA is a cotorsion R-module. Since A/PA is a torsion module of bounded order, we have $\operatorname{Ext}_R^1(Q, A/PA) = 0$. From these facts it follows that $\operatorname{Ext}_R^1(Q, A) = 0$. Thus A is also a cotorsion R-module. Hence $PA \oplus A$ is a cotorsion R-module. We can now apply Lemma 4 and conclude that $A \cong D_i$ for some i, $1 \leq i \leq 2n$. Thus A has rank 1. This contradiction shows that R has property D.

Corollary 1. *Let R be an integrally closed ring and P a prime ideal of R such that $P = PR_P$. Then R has property* D *if and only if both R_P and R/P have property* D.

Proof. Assume that R has property D. Then R_P has property D by Theorem D1. Since R_P is integrally closed, R_P is a Prüfer ring by Theorem D4. Therefore, R_P is a valuation ring. By Theorem D2, R_P is a maximal valuation ring. Hence by Theorem 1, R/P also has property D.

Conversely assume that both R/P and R_P have property D. As in the preceding paragraph R_P is a maximal valuation ring. Therefore, R has property D by Theorem 1.

Corollary 2. *Let R be a valuation ring and P a prime ideal of R. Then R is a maximal valuation ring if and only if both R_P and R/P are maximal valuation rings.*

Proof. By Theorem D2, a valuation ring has property D if and only if it is a maximal valuation ring. Thus Corollary 2 is an immediate consequence of Corollary 1.

This is certainly a very roundabout way of proving Corollary 2, and there is a relatively direct homological proof involving theorems on change of rings, but there seems to be no point in introducing that proof here.

Lemma 6. *Let R be a nonlocal Prüfer ring whose quotient field Q is not remote, and let J be the Jacobson radical of R. If M is a maximal ideal of R, then $R_M^{-1} = P$ is a nonzero prime ideal of R contained in J and P contains every prime ideal of R that is contained in J. We have $P^{-1} = R_P$ and $PR_P = P$. If N is any other maximal ideal of R, then $R_N^{-1} = P$ also.*

Proof. Since Q is not remote from R, we have $P \neq 0$. Clearly, P is a proper ideal of R_M, and thus $P \subset MR_M \cap R = M$. Let N be another maximal ideal of R, and suppose that $P \not\subset N$. Then $R = P + N$, and hence $1 = a + b$, where $a \in P$ and $b \in N$. Since $a \in M$, we have $b \in R - M$. Therefore, $1/b \in R_M$, and $a/b \in PR_M \subset R$. Thus $1/b = a/b + 1 \in R$, which contradicts $b \in N$. This contradiction shows that $P \subset N$. We have proved that $P \subset J$.

Let x be an element of R_M such that $x^n \in P$ for some integer $n > 0$. If $y \in R_M$, then $(xy)^n = x^n y^n \in PR_M \subset R$, and xy is integral over R. But R is integrally closed, and thus $xy \in R$. Hence we have $xR_M \subset R$, which shows that $x \in P$. Thus P is a radical ideal of R_M. Since R_M is a valuation ring, P is a prime ideal of R_M. A fortiori P is a prime ideal of R.

Since $P \subset M$, we have $R_M \subset R_P$. Thus R_P is a valuation ring and $PR_P \subset MR_M$. Hence PR_P is a prime ideal of R_M. Now $P \cap R = P = PR_P \cap R$; therefore, by the one-to-one correspondence between the prime ideals of R_M and the prime ideals of R contained in M, we have $P = PR_P$. Thus $P \subset R_P^{-1}$; but R_P^{-1} is a proper ideal of R_P, and hence $R_P^{-1} \subset PR_P = P$. Thus we see that $P = R_P^{-1}$.

Now $(P^{-1}P)R_M = P^{-1}(PR_M) = P^{-1}P \subset R$. Thus $P^{-1}P \subset R_M^{-1} = P$. This shows that P^{-1} is a ring. Since $R_P \subset P^{-1}$, P^{-1} is a valuation ring with maximal ideal $m(P^{-1}) \subset PR_P = P$. But P is a proper ideal of P^{-1}, and thus $P = m(P^{-1})$. Therefore we have $P^{-1} = R_P$.

Let P' be a prime ideal of R contained in J. Then $R_N \subset R_{P'}$, and hence $P'R_{P'} \subset R_N$ for all maximal ideals N of R. Thus $P'R_{P'} \subset \bigcap_N R_N = R$, and $P' \subset R_{P'}^{-1} \subset R_M^{-1} = P$. Therefore, P contains all prime ideals of R that are contained in J. From this it follows immediately that $P = R_N^{-1}$ for all maximal ideals N of R.

We are now ready to prove the main theorem of this paper.

Theorem 2. *Let R be an integrally closed ring. Then R has property* D *if and only if R is the intersection of at most two maximal valuations rings; i.e. if and only if R has only two maximal ideals M_1, M_2 (not necessarily distinct) and R_{M_1}, R_{M_2} are maximal valuation rings.*

Proof. Assume that R has property D. By Theorem D4, R is a Prüfer ring. If M is a maximal ideal of R, then R_M is a valuation ring, and hence by Theorem D2, R_M is a maximal valuation ring. Since $R = \bigcap_M R_M$, where M ranges over all maximal ideals of R, we must prove that R has at most two maximal ideals. We will suppose that R has more than two maximal ideals, and arrive at a contradiction.

Since R is not a ring of type I, we have by Theorem D3 that R does not have a remote quotient field. Thus by Lemma 6, there is a nonzero prime ideal P contained in the Jacobson radical J of R such that $PR_P = P$. Let $\overline{R} = R/P$; then by Theorem 1, $\overline{R}$ has property D.

Let $\overline{M}$ be a maximal ideal of $\overline{R}$. We have $\overline{M} = M/P$, where M is a maximal ideal of R. Then $\overline{R}_{\overline{M}} \cong R_M/PR_M$ is a valuation ring, and thus $\overline{R}$ is a Prüfer ring. Since $P \subset J$, $\overline{R}$ has the same number of maximal ideals as R. Therefore, $\overline{R}$ is not a ring of type I, and thus $\overline{R}$ does not have a remote quotient field by Theorem D3. Hence we can apply Lemma 6 again and obtain a nonzero prime ideal P^* of $\overline{R}$ contained in the radical $\overline{J}$ of $\overline{R}$. Now $\overline{J} = J/P$; and $P^* = P'/P$, where P' is a prime ideal of R such that $P \subsetneq P' \subset J$. But this contradicts Lemma 6 which asserts that P contains every prime ideal of R contained in J. Therefore, R is the intersection of one or two maximal valuation rings.

Conversely, assume that R is the intersection of one or two maximal valuation rings. If R is a maximal valuation ring or a ring of type I, then R has property D either by Theorem D2 or by Theorem D3. Thus we can assume that R is the intersection of two independent maximal valuation rings and is not a ring of type I.

Let M_1 and M_2 be the two maximal ideals of R. Then R_{M_1} and R_{M_2} are maximal valuation rings by [10, Theorem 11.11]. Since R is not a ring of type I, there is a nonzero prime ideal P of R such that $P \subset M_1 \cap M_2$. Then $R_{M_1} \subset R_P$, and hence R_P is a maximal valuation ring by Corollary 2. We have $PR_P \subset R_{M_1} \cap R_{M_2} = R$, and thus $PR_P = P$.

By Zorn's lemma (or by Lemma 6) we can assume that P contains every prime ideal of R contained in $M_1 \cap M_2$. Thus $\overline{R} = R/P$ has exactly two maximal ideals $\overline{M}_1 = M_1/P$ and $\overline{M}_2 = M_2/P$; and $\overline{M}_1 \cap \overline{M}_2$ contains no nonzero prime ideals of R. By Corollary 2, $\overline{R}_{\overline{M}_i} \cong R_{M_i}/P_{M_i}$ is a maximal valuation ring for $i = 1, 2$. Therefore, $\overline{R}$ is a ring of type I; and hence R/P has property D by Theorem D3. Thus R has property D by Theorem 1.

Corollary 3. *Let R be a ring with property* D. *Then R has at most two maximal ideals.*

Proof. Let F be the integral closure of R. Then F has property D by Theorem D1. Hence by Theorem 2, F has at most two maximal ideals. Therefore, R has at most two maximal ideals.

Remark. We note that it follows easily from Theorem 2 that if R is a ring with property D, then its lattice of prime ideals can be represented symbolically by $|$, $\|$, or by $\curlyvee$.

4. Examples. (1) The first example is of a non-Noetherian local ring with property D which is not a maximal valuation ring. It is easily seen that this ring is isomorphic to the example of [9]. However, the construction is different and presents this example in a new light. We will use Theorem 1 to prove that it has property D.

Let k be a field and X and Y indeterminates over k. Let A be the ring of formal power series in Y with coefficients in k and nonnegative integer exponents: $A = k[[Y]]$. Let B be the quotient field of A, and let F be the ring of formal power series in X with coefficients in B and nonnegative integer exponents, but with constant term in A:

$$F = \left\{ \sum_{i=0}^{\infty} b_i X^i \mid b_0 \in A,\ b_i \in B \text{ for } i > 0 \right\}.$$

It is easily seen that F is a valuation ring. Let P be the prime ideal of F consisting of power series with constant term $b_0 = 0$. Then $F_P = B[[X]]$ is a complete discrete valuation ring and $F/P \cong A = k[[Y]]$ is also a complete discrete valuation ring. Therefore, by Corollary 2, F is a maximal valuation ring. F is not a Noetherian ring, since it is a rank two valuation ring.

Let A' be the subring of A consisting of power series in Y with linear term missing. Then A' is a ring of type II; i.e. A' is a complete Noetherian local ring of Krull dimension one such that every ideal of A' can be generated by two elements. Thus A' has property D by [7, Theorem 4].

Let R be the subring of F of power series in X with constant term $b_0 \in A'$. Let P be the same prime ideal as in F. Then $R_P = F_P = B[[X]]$ is a complete discrete valuation ring; $R/P \cong A'$ has property D; and $PR_P = P$. Thus by Theorem 1, R has property D. R is a local ring with maximal ideal consisting of those power series in X with constant term in the maximal ideal of A'. F can be generated over R by two elements, and hence R is not a Noetherian ring, since F is not Noetherian.

(2) The second example is of a ring which is the intersection of two independent maximal valuation rings, but is not a ring of type I. Mrs. Osofsky has communicated to me an example of this type of ring, but our example illustrates more easily the methods of this paper. Together with the example of Barbara Osofsky of a ring of type I (see [8]) this example proves the existence of rings which are intersections of two maximal valuation rings whether of type I or not.

Let A be a ring of type I (in particular A could be the example of Barbara Osofsky [8]). Let B be the quotient field of A, and let X be an indeterminate over B. Let R be the ring of formal power series in X with coefficients in B and nonnegative integer exponents, but with constant term in A:

$$R = \left\{ \sum_{i=0}^{\infty} b_i X^i \mid b_0 \in A,\ b_i \in B \text{ for } i > 0 \right\}.$$

Let P be the prime ideal of R consisting of power series with constant term $b_0 = 0$. Then $R/P \cong A$ has property D by Theorem D3; and $R_P \cong B[[X]]$ is a complete discrete valuation ring. Clearly $PR_P = P$. Thus R has property D by Theorem 1.

Let M_1 and M_2 be the two maximal ideals of A. Then $N_1 = M_1 + P$ and $N_2 = M_2 + P$ are the only two maximal ideals of R, and $P \subset N_1 \cap N_2$. Thus R is not a maximal valuation ring or a ring of type I.

R_{N_1} is the ring of power series in X with constant term $b_0 \in A_{M_1}$. Clearly R_{N_1} is a valuation ring, since A_{M_1} is a valuation ring. $(R_{N_1})_P \cong B[[X]]$ is a complete discrete valuation ring; and $R_{N_1}/P \cong A_{M_1}$ is a maximal valuation ring. Thus by Corollary 2, R_{N_1} is a maximal valuation ring. Similarly, R_{N_2} is a maximal valuation ring. Thus $R = R_{N_1} \cap R_{N_2}$ is the intersection of two independent maximal valuation rings, and R is not a ring of type I.

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