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CONVERSE THEOREMS AND EXTENSIONS IN CHEBYSHEV RATIONAL APPROXIMATION TO CERTAIN ENTIRE FUNCTIONS IN $[0, +\infty)$

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ABSTRACT. Recent interest in rational approximations to e^{-x} in $[0, +\infty)$, arising naturally in numerical methods for approximating solutions of heat-conduction-type parabolic differential equations, has generated results showing that the best Chebyshev rational approximations to e^{-x}, and to reciprocals of certain entire functions, have errors for the interval $[0, +\infty)$ which converge geometrically to zero. We present here some related converse results in the spirit of the work of S. N. Bernstein.

1. **Introduction.** In numerical methods for approximating solutions of heat-conduction-type equations, one necessarily considers matrix approximations of

$$e^{-tA}, \tag{1.1}$$

where A is an $n \times n$ Hermitian and positive definite matrix, and where t, the time parameter, belongs to $[0, +\infty)$. Though rational Padé matrix approximations of e^{-tA} are familiar (cf. [7, Chapter 8]), rational Chebyshev matrix approximations of e^{-tA} have only recently been considered ([2], [8]), and are associated with rational approximations of e^{-x} in $[0, +\infty)$. Specifically, if for any nonnegative integer m, π_m denotes the collection of real polynomials of degree at most m, and if $\pi_{m,n}$ for any nonnegative integers m and n similarly denotes the collection of all real rational functions $r_{m,n}(x) \equiv p_m(x)/q_n(x)$, $p_m \in \pi_m$, $q_n \in \pi_n$, then let

$$\lambda_{m,n} \equiv \inf_{\pi_{m,n}} \|e^{-x} - r_{m,n}(x)\|_{L_\infty[0,\infty]}, \qquad 0 \le m \le n, \tag{1.2}$$

denote the *Chebyshev constants* for e^{-x} on $[0, +\infty)$.

In [2], the following was established.

Theorem 1. *Let $\{m(n)\}_{n=0}^{\infty}$ be any sequence of nonnegative integers with $0 \le m(n) \le n$ for each $n \ge 0$. Then the Chebyshev constants $\lambda_{m(n),n}$ for e^{-x} in*

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$[0, +\infty)$ *converge geometrically to zero, i.e.*

$$\overline{\lim_{n\to\infty}} (\lambda_{m(n),n})^{1/n} \leq \frac{1}{2e^{\alpha}} = \frac{1}{2.298\ldots} < 1 \tag{1.3}$$

where $\alpha = 0.1392\ldots$ *is the real solution of* $2\alpha e^{2\alpha+1} = 1$. *Moreover,*

$$\overline{\lim_{n\to\infty}} (\lambda_{0,n})^{1/n} \geq \frac{1}{6}. \tag{1.4}$$

In [4], the geometric convergence of the Chebyshev constants to zero was extended to a wider class of entire functions. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with $M_f(r) \equiv \sup_{|z|=r} |f(z)|$ its maximum modulus function. Then, f is said to be of *perfectly regular growth* (ρ, B) (cf. Boas [1, p. 8] and Valiron [6, p. 45]) iff there exist two (finite) positive numbers ρ and B such that

$$\lim_{r\to\infty} \frac{\ln M_f(r)}{r^{\rho}} = B. \tag{1.5}$$

In [4], the following was established.

Theorem 2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be any entire function of perfectly regular growth* (ρ, B) *with* $a_k \geq 0$ *for all* $k \geq 0$, *and for any nonnegative integers* m *and* n *with* $0 \leq m \leq n$, *let*

$$\lambda_{m,n} \equiv \inf_{\pi_{m,n}} \left\| \frac{1}{f(x)} - r_{m,n}(x) \right\|_{L_{\infty}[0,\infty]} \tag{1.6}$$

denote the Chebyshev constants for $1/f$ *in* $[0, +\infty)$. *Then, for any sequence* $\{m(n)\}_{n=0}^{\infty}$ *with* $0 \leq m(n) \leq n$ *for each* $n \geq 0$,

$$\overline{\lim_{n\to\infty}} (\lambda_{m(n),n})^{1/n} \leq \frac{1}{2^{1/\rho}} < 1. \tag{1.7}$$

Moreover,

$$\overline{\lim_{n\to\infty}} (\lambda_{0,n})^{1/n} \geq \frac{1}{2^{2+1/\rho}}. \tag{1.8}$$

The results of Theorem 2 give then *sufficient* conditions on $f(z)$ so that its associated Chebyshev constants $\lambda_{m(n),n}$ of (1.6), $0 \leq m(n) \leq n$, converge geometrically to zero as $n \to \infty$. In the spirit of Bernstein's classical inverse-type theorems for polynomial and trigonometric polynomial approximation on finite intervals, the first aim of this paper is to give *necessary* conditions on $f(x)$ for this geometric convergence on the infinite interval $[0, +\infty)$. This will be discussed in §2. In addition, similar necessary conditions for geometric convergence on the infinite interval $(-\infty, +\infty)$ will also be given.

Upon examining the results of Theorem 2, we see that the bounds of (1.7)–(1.8) depend upon ρ, but do not depend upon B. This suggests that the results of Theorem 2, giving sufficient conditions on the entire function $f(z)$ to ensure geometric convergence to zero of the Chebyshev constants, can be generalized to a wider class of entire functions. The second aim of this paper is to consider just such generalizations of Theorem 2. This will be discussed in §3. Finally, in §4 quasi-analytic extensions in the sense of S. N. Bernstein will be given.

2. **A converse theorem.** To establish a converse result to Theorems 1–2 which gives *necessary* conditions of f such that the Chebyshev constants $\lambda_{m,n}$ of $1/f$ converge geometrically to zero, we first introduce some notation. For given $r > 0$ and $s > 1$, let $\mathfrak{S}(r, s)$ denote the unique open ellipse in the complex plane with foci at $x = 0$ and $x = r$ and semimajor and semiminor axes a and b such that $b/a = (s^2 - 1)/(s^2 + 1)$. In more familiar notation,

$$z = x + iy \in \mathfrak{S}(r, s) \quad \text{iff} \quad \frac{(x - r/2)^2}{[(r/4)(s + 1/s)]^2} + \frac{y^2}{[(r/4)(s - 1/s)]^2} < 1. \tag{2.1}$$

If $F(z)$ is any entire function, we denote by $\tilde{M}_F(r, s)$ the maximum modulus of F in $\mathfrak{S}(r, s)$, i.e.

$$\tilde{M}_F(r, s) \equiv \sup\{|F(z)|:\ z \in \mathfrak{S}(r, s)\}. \tag{2.2}$$

We now state one of our main results.

Theorem 3. *Let $f(x)$ be a real continuous function (not $\not\equiv 0$) on $[0, +\infty)$, and assume that there exist a sequence of real polynomials $\{p_n(x)\}_{n=0}^{\infty}$, with $p_n \in \pi_n$ for each $n \geq 0$, and a real number $q > 1$ such that*

$$\overline{\lim_{n \to \infty}} \left\{ \left\| \frac{1}{p_n(x)} - \frac{1}{f(x)} \right\|_{L_\infty[0,\infty]} \right\}^{1/n} \leq \frac{1}{q} < 1. \tag{2.3}$$

Then, there exists an entire function $F(z)$ with $F(x) = f(x)$ for all $x \geq 0$, and F is of finite order ρ, i.e.

$$\overline{\lim_{r \to \infty}} \frac{\ln \ln M_F(r)}{\ln r} \equiv \rho < \infty. \tag{2.4}$$

In addition, for every $s > 1$, there exist constants $K = K(s, q) > 0$, $\theta = \theta(s, q) > 1$, and $r_0 = r_0(s, q) > 0$ such that

$$\tilde{M}_F(r, s) \leq (K \|f\|_{L_\infty[0,r]})^{\theta} \quad \text{for all } r \geq r_0. \tag{2.5}$$

If, for each $s > 1$, $\tilde{\theta}(s)$ is defined by

$$\overline{\lim_{r \to \infty}} \left\{ \frac{\ln \tilde{M}_F(r, s)}{\ln \|f\|_{L_\infty[0,r]}} \right\} = \tilde{\theta}(s) \tag{2.6}$$

when $\|f\|_{L_\infty[0,r]}$ *is unbounded as* $r \to \infty$, *and* $\tilde{\theta}(s) \equiv 1$ *otherwise, then the order* ρ *of* F *satisfies*

$$\rho \leq \inf_{s>1} \left\{ \frac{\ln \tilde{\theta}(s)}{\ln[\frac{1}{2} + \frac{1}{4}(s + 1/s)]} \right\}, \tag{2.7}$$

and this upper bound for the order ρ *is in general best possible.*

Proof. For any q_1 with $q > q_1 > 1$, it follows from (2.3) that there exists a positive integer $n_1(q_1)$ such that

$$\|1/p_n - 1/f\|_{L_\infty[0,\infty]} \leq 1/q_1^n \quad \text{for all } n \geq n_1(q_1). \tag{2.8}$$

Next, define

$$m_f(r) = \|f\|_{L_\infty[0,r]} \quad \text{for each } 0 \leq r < \infty. \tag{2.9}$$

Fixing $r > 0$, the fact that q_1 exceeds unity implies that there exists a least integer $n_2(r)$ such that

$$q_1^n > q_1^n - m_f(r) \geq q_1^n/2 \quad \text{for all } n \geq n_2(r). \tag{2.10}$$

With $n_3 \equiv \max\{n_1(q_1), n_2(r)\}$, a simple manipulation of the inequality of (2.8), coupled with the second inequality of (2.10), gives us that

$$|p_n(x) - f(x)| \leq m_f^2(r)/(q_1^n - m_f(r)) \leq 2m_f^2(r)/q_1^n$$

for any $n \geq n_3$ and all x with $0 \leq x \leq r$ and $f(x) \neq 0$. But from the continuity of f, this implies that

$$\|p_n - f\|_{L_\infty[0,r]} \leq 2m_f^2(r)/q_1^n \quad \text{for all } n \geq n_3. \tag{2.11}$$

In order to apply results of S. N. Bernstein, we now make the change of variables $r(1 + t)/2 = x$, $0 \leq x \leq r$, $-1 \leq t \leq 1$, and define

$$h(t;\, r) \equiv f\{r(1 + t)/2\}. \tag{2.12}$$

Because f is real and continuous on $[0, +\infty)$, $h(t;\, r)$ is a real continuous function on $[-1, +1]$ for each parameter $r \geq 0$. If

$$E_n\{h(\cdot;\, r)\} \equiv \inf_{\sigma_n \in \pi_n} \|\sigma_n - h(\cdot;\, r)\|_{L_\infty[-1,+1]}$$

denotes the error in the best Chebyshev polynomial approximation in π_n to $h(\cdot\,;\, r)$ on $[-1, +1]$, then the inequality of (2.11) immediately implies that

$$E_n\{h(\cdot;\, r)\} \leq 2m_f^2(r)/q_1^n \quad \text{for all } n \geq n_3.$$

But, as $r > 0$ is fixed and q_1 is an arbitrary number with $q > q_1 > 1$, this implies that

$$\overline{\lim_{n\to\infty}}\,[E_n\{h(\cdot;\,r)\}]^{1/n} \le 1/q \quad \text{for each } r > 0.$$

Hence, using a result of S. N. Bernstein (cf. [3, p. 86]), this means that $h(t;\,r)$ can, for each $r > 0$, be extended to an analytic function in the open ellipse $\mathfrak{S}_q$ with foci ± 1 and semimajor and semiminor axes a and b such that $a + b = q > 1$. Because

$$z = x + iy \in \mathfrak{S}_q \quad \text{iff} \; \frac{x^2}{[\frac{1}{2}(q + 1/q)]^2} + \frac{y^2}{[\frac{1}{2}(q - 1/q)]^2} < 1,$$

this means, using (2.1) and (2.12), that f can be extended to a function $F(z)$ which is analytic in the ellipse $\mathfrak{S}(r,\, q)$. But as r is an *arbitrary* positive real number, and any complex number w is, as is readily verified, in $\mathfrak{S}(r,\, q)$ for all r sufficiently large, then $F(z)$ is evidently analytic in the whole complex plane, i.e. F is an entire function, which proves the first part of the theorem.

Continuing, let $\sigma_n(t;\, r) \equiv p_n\{r(1 + t)/2\}$. Then, the inequality of (2.11) can be rewritten as

$$\|\sigma_n(\cdot;\, r) - h(\cdot;\, r)\|_{L_\infty[-1,+1]} \le 2m_f^2(r)/q_1^n \quad \text{for all } n \ge n_3.$$

With the triangle inequality, and the definitions of (2.9) and (2.12), we deduce from the above inequality that

$$\|\sigma_n(\cdot;\, r)\|_{L_\infty[-1,+1]} \le m_f(r) + 2m_f^2(r)/q_1^n \quad \text{for all } n \ge n_3,$$

and

$$\|\sigma_{n+1}(\cdot;\, r) - \sigma_n(\cdot;\, r)\|_{L_\infty[-1,+1]} \le 2(1 + q_1)\, m_f^2(r)/q_1^{n+1} \quad \text{for all } n \ge n_3.$$

Making use of another result of S. N. Bernstein (cf. [3, p. 92]), the above inequalities can, for any s with $1 < s < q_1$, be extended to any $z \in \mathfrak{S}_s$ by

$$|\sigma_n(z;\, r)| \le (m_f(r) + 2m_f^2(r)/q_1^n)\, s^n \quad \text{for all } n \ge n_3,\; z \in \mathfrak{S}_s, \tag{2.13}$$

and

$$|\sigma_{n+1}(z;\, r) - \sigma_n(z;\, r)| \le 2(1 + q_1)\, m_f^2(r)\,(s/q_1)^{n+1} \quad \text{for all } n \ge n_3,\; z \in \mathfrak{S}_s. \tag{2.14}$$

From these inequalities, the series

$$\sigma_{n_3}(z;\, r) + \sum_{n=n_3}^{\infty} \{\sigma_{n+1}(z;\, r) - \sigma_n(z;\, r)\}$$

necessarily converges uniformly on $\mathfrak{S}_s$ to an analytic function which, as is easily seen, is $h(z;\, r)$. With the bounds of (2.13) and (2.14) applied to the above sum, it follows that

$$|h(z;\, r)| \le \left(m_f(r) + \frac{2m_f^2(r)}{q_1^{n_3}}\right)s^{n_3} + 2(1+q_1)\, m_f^2(r)\left(\frac{q_1}{q_1 - s}\right)\left(\frac{s}{q_1}\right)^{n_3+1}$$

for all $z \in \mathfrak{S}_s$. Noting from (2.10) that $2m_f(r)/q_1^{n_3} \le 1$, the above inequality becomes

$$|h(z;\, r)| \le m_f(r)\{2 + (1+q_1)\, s/(q_1 - s)\}\, s^{n_3} \quad \text{for all } z \in \mathfrak{S}_s.$$

However, from the definitions of (2.1) and (2.12), this means that

$$|F(z)| \le m_f(r)\{2 + (1+q_1)\, s/(q_1 - s)\}\, s^{n_3} \quad \text{for all } z \in \mathfrak{S}(r, s),$$

which in turn, from (2.2), implies that

$$\tilde{M}_F(r, s) \le m_f(r)\{2 + (1+q_1)\, s/(q_1 - s)\}\, s^{n_3} \quad \text{for all } r > 0, \text{ all } 1 < s < q_1. \tag{2.15}$$

Consider now the nondecreasing function of r, $m_f(r)$. We first assume that $m_f(r)$ is unbounded as $r \to \infty$. Then, for any r_0 sufficiently large with $m_f(r_0) > 1$, define

$$\tau = \tau(r_0) \equiv \left(\frac{\ln s}{\ln q_1}\right)\left\{\frac{\ln 2}{\ln m_f(r_0)} + 1\right\}. \tag{2.16}$$

It can be verified, using the nondecreasing nature of $m_f(r)$ and the fact from (2.10) that $n_3 = n_2(r) \le ((\ln(2q_1) + \ln m_f(r))/\ln q_1)$, that $s^{n_3} \le (m_f(r))^\tau$ for all $r \ge r_0$. In other words, from (2.15),

$$\tilde{M}_F(r, s) \le \{2 + (1+q_1)\, s/(q_1 - s)\}[m_f(r)]^{1+\tau} \quad \text{for all } r \ge r_0, \text{ all } 1 < s < q_1.$$

Because $m_f(r)$, by assumption, is nondecreasing and unbounded as $r \to \infty$, it is clear from (2.16) that, for any s chosen with $1 < s < q_1$, there is an $r_0 > 0$ such that $\tau(r_0) \le 1$, and hence with $K = K(q_1, s) \equiv \{2 + (1+q_1)s/(q_1 - s)\}^{1/2}$, we have that

$$\tilde{M}_F(r, s) \le \{K m_f(r)\}^2 \quad \text{for all } r \ge r_0, \text{ all } 1 < s < q_1. \tag{2.17}$$

For the remaining case when $m_f(r)$ is bounded, i.e. $m_f(r) \le \sigma < \infty$ for all $r \ge 0$, we see from (2.10) that $n_2(r)$, and hence n_3, is independent of r. In this case, (2.15) becomes

$$\tilde{M}_F(r, s) \le \sigma K^2 s^{n_3} \quad \text{for all } r > 0, \text{ all } 1 < s < q_1. \tag{2.18}$$

As is readily verified, if for $s > 1$, $\mu = \mu(s)$ is defined by

$$\mu \equiv \tfrac{1}{2}\{1 + \tfrac{1}{2}(s + 1/s)\} > 1, \tag{2.19}$$

then the open disk $\{z\colon |z| < (\mu - 1)r\}$ is contained in $\mathfrak{S}(r, s)$. Consequently, by the maximum modulus principle,

$$M_F((\mu - 1)\, r) \le \tilde{M}_F(r, s) \quad \text{for all } r > 0. \tag{2.20}$$

Combining the inequalities of (2.18) and (2.20), we see that $M_F(r)$ is bounded for all $r > 0$, so that $F(z) = c$ for all z. Because $F(x)$ agrees with $f(x)$ on $[0, +\infty)$, the assumption that $f(x)$ is not identically zero on $[0, +\infty)$ gives us that $F(z) = c \neq 0$ for all z, and in this case, the inequality of (2.17) is trivially satisfied.

We now show that the entire function F is of finite order ρ, where ρ is defined by (2.4). It is geometrically evident from the definition of μ in (2.19) that the ellipse $\mathcal{E}(r/\mu, s)$, for $1 < s < q_1$, contains all points of the interval $(0, r)$ for any $r > 0$. Hence, by the maximum modulus principle again, $m_f(r) \le \tilde{M}_F(r/\mu, s)$, and thus from (2.17), $\tilde{M}_F(r, s) \le (K\tilde{M}_F(r/\mu, s))^2$ for all $r \ge \mu r_0$. From induction on the above inequality and the inequality of (2.20), it follows that

$$M_F((\mu - 1)\, r) \le \tilde{M}_F(r, s) \le \{K^2 \tilde{M}_F(r/\mu^m, s)\}^{2m} \quad \text{for all } r \ge \mu^m r_0. \tag{2.21}$$

In a similar way, we have that

$$m_f(r) \le \{K^2 m_f(r/\mu^m)\}^{2m} \quad \text{for all } r \ge \mu^m r_0. \tag{2.22}$$

For each $r \ge r_0$, choose the nonnegative integer m such that $\mu^{m+1} > r/r_0 \ge \mu^m$. A short calculation based on the inequalities of (2.21) gives us that

$$\rho \equiv \overline{\lim_{r \to \infty}} \frac{\ln \ln M_F(r)}{\ln r} \le \frac{\ln 2}{\ln \mu}, \tag{2.23}$$

proving that $F(z)$ is an entire function of finite order $\rho \le (\ln 2)/(\ln \mu)$. But as this inequality is valid for all s with $1 < s < q_1 < q$, it holds also for $s = q$, i.e. from (2.19),

$$\rho \le \frac{\ln 2}{\ln \{\tfrac{1}{2}[1 + \tfrac{1}{2}(q + 1/q)]\}}. \tag{2.24}$$

Continuing the proof of Theorem 3, we now establish the inequality of (2.5). We already know from (2.17) that for any s with $1 < s < q_1$, $\tilde{M}_F(r, s) \le \{Km_f(r)\}^2$ for all $r \ge r_0$. Next, it is easy to verify geometrically that $\mathcal{E}(r, s) \subset \mathcal{E}(r', s')$ for $r' \ge r$, where s' and s are connected by (cf. (2.19))

$$(\mu(s') - 1)\, r' = (\mu(s) - 1)\, r. \tag{2.25}$$

Given any $s > 1$, choose $\alpha = r'/r \ge 1$ such that s', determined from (2.25), satisfies $1 < s' < q_1$; this is always possible for α sufficiently large. Then, fixing α and s', it follows from $\mathcal{E}(r, s) \subset \mathcal{E}(r', s') = \mathcal{E}(\alpha r, s')$ that $\tilde{M}_F(r, s) \le \tilde{M}_F(\alpha r, s')$. But as $1 < s' < q_1$, we can apply (2.17), i.e. $m_f(r) \le \tilde{M}_F(r, s) \le \tilde{M}_F(\alpha r, s') \le (Km_f(\alpha r))^2$ for all $r \ge r_0/\alpha$, the first inequality following from the fact that $[0, r] \subset \mathcal{E}(r, s)$. Making use of the inequality of (2.22), we deduce from the above inequality that

$$\widetilde{M}_F(r, s) \leq \{K^2 m_f(\alpha r/\mu^m(s'))\}^{2m+1} \quad \text{for all } \alpha r \geq \mu^m(s')\, r_0.$$

Now, choose m sufficiently large so that $\alpha \leq \mu^m(s')$. For this choice of m, we have $\widetilde{M}_F(r, s) \leq \{K^2 m_f(r)\}^{2m+1}$ for all $r \geq r_0$, the desired result of (2.5).

To establish the inequality of (2.7), it is sufficient to assume that $\|f\|_{L_\infty[0,r]}$ is unbounded as $r \to \infty$. In this case, if follows from (2.5) that

$$\tilde{\theta}(s) \equiv \varlimsup_{r\to\infty} \left\{ \frac{\ln \widetilde{M}_F(r, s)}{\ln \|f\|_{L_\infty[0,r]}} \right\} \geq 1 \tag{2.26}$$

is finite for each $s > 1$. For $s > 1$ fixed, if follows from (2.26) that for any $\epsilon > 0$, there is an $r_0(\epsilon, s) > 0$ such that

$$\widetilde{M}_F(r, s) \leq \{\|f\|_{L_\infty[0,r]}\}^{(\tilde{\theta}(s)+\epsilon)} \quad \text{for all } r \geq r_0(\epsilon, s),$$

from which it follows, as in (2.21), that

$$\widetilde{M}_F(r, s) \leq \{\widetilde{M}_F(r/\mu^m(s), s)\}^{(\tilde{\theta}(s)+\epsilon)m} \quad \text{for all } r \geq \mu^m(s) \cdot r_0(\epsilon, s).$$

The above inequality, as in the proof of (2.23), gives us that

$$\rho = \varlimsup_{r\to\infty} \frac{\ln \ln M_F(r)}{\ln r} \leq \frac{\ln \tilde{\theta}(s)}{\ln \mu(s)} \quad \text{for each } s > 1,$$

from which we conclude that

$$\rho \leq \inf_{s>1} \left\{ \frac{\ln \tilde{\theta}(s)}{\ln \mu(s)} \right\},$$

the desired result of (2.7).

Finally, to show that the inequality of (2.7) is in general best possible, consider any entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ of perfectly regular growth (ρ, B) (cf. (1.5)) with $a_0 > 0$, $a_k \geq 0$ for all $k \geq 0$. By Theorem 2, the assumptions of Theorem 3 are fulfilled. Because f is of perfectly regular growth (ρ, B), it follows from (1.5) that for any $\delta > 0$,

$$\lim_{r\to\infty} \frac{\ln M_f(\delta r)}{\ln M_f(r)} = \delta^\rho. \tag{2.27}$$

But, because the coefficients a_k of f are all nonnegative, then $\widetilde{M}_f(r, s) = M_f(\mu(s) \cdot r)$ and $\|f\|_{L_\infty[0,r]} = M_f(r)$ for all $r > 0$, all $s > 1$. Thus, from (2.27) and the definition of $\tilde{\theta}(s)$ in (2.6), it follows that $\tilde{\theta}(s) = (\mu(s))^\rho$, so that

$$\rho = \left\{ \frac{\ln \tilde{\theta}(s)}{\ln \mu(s)} \right\} \quad \text{for each } s > 1,$$

i.e. equality holds in (2.7). Q.E.D.

The following are consequences of Theorem 3 and its proof. The first is merely a sharpened restatement of the inequality of (2.24).

Corollary 1. *If $f(x)$ satisfies the hypotheses of Theorem 3, and $\rho > 0$ is the order of the entire function $F(z)$, then*

$$q \leq 2^{1/\rho+1} - 1 + ((2^{1/\rho+1} - 1)^2 - 1)^{1/2}, \tag{2.28}$$

with strict inequality holding if F is of type $B = +\infty$ (cf. (3.8)).

We remark that the bound of (2.28) simultaneously improves the upper bound $q \leq 6$ of (1.4) of Theorem 1 for the case $\rho = 1$, as well as the upper bound $q \leq 2^{2+1/\rho}$ of (1.8) of Theorem 2 for the more general case $\rho > 0$.

Corollary 2. *If $f(x)$ satisfies the hypotheses of Theorem 3 and $\|f\|_{L_\infty[0,\infty]}$ is bounded, then $f(x) \equiv c \neq 0$.*

The point of Corollary 2 is that it is easy to give examples of entire functions which are real on $[0, +\infty)$ which cannot possess the property of (2.3). For example $F(z) = \cos z + 2$ is such a function. A less obvious example which makes use of the inequality of (2.5) is $F(z) = (z+1)(2+\cos z)$, which is an entire function of order $\rho = 1$, for which $r + 1 \leq m_f(r) \leq 3(r+1)$ for all $r \geq 0$. But since $|F(z)|$ grows exponentially along the imaginary axis, it is clear that the necessary condition of (2.5) of Theorem 3 cannot hold for all r sufficiently large.

It is of interest to know that there is an analogue of Theorem 3 which deals with approximations on the infinite interval $(-\infty, +\infty)$. In fact, this extension to $(-\infty, +\infty)$ directly follows the steps of the proof of Theorem 3, with the interval $[0, r]$ being replaced by $[-r, +r]$. If $\hat{\mathfrak{E}}(r, s)$, for $r > 0$ and $s > 1$, denotes the open ellipse in the complex plane with foci at $\pm r$ and semimajor and semiminor axes a and b such that $b/a = (s^2 - 1)/(s^2 + 1)$, then, in analogy with (2.2), define

$$\hat{M}_F(r, s) = \sup\{|F(z)|: z \in \hat{\mathfrak{E}}(r, s)\}. \tag{2.29}$$

We then have

Theorem 4. *Let $f(x)$ be a real continuous function (not $\not\equiv 0$) on $(-\infty, +\infty)$, and assume that there exist a sequence of real polynomials $\{p_n(x)\}_{n=0}^{\infty}$, with $p_n \in \pi_n$ for each $n \geq 0$, and a real number $q > 1$ such that*

$$\overline{\lim_{n\to\infty}} \left\{ \left\| \frac{1}{p_n(x)} - \frac{1}{f(x)} \right\|_{L_\infty[-\infty,+\infty]} \right\}^{1/n} = \frac{1}{q} < 1. \tag{2.30}$$

Then, there exists an entire function $F(z)$ with $F(x) = f(x)$ for all x with $-\infty < x < +\infty$, and F is of finite order ρ (cf. (2.24)). In addition, for every $s > 1$, there exist constants $K = K(s, q) > 0$, $\theta = \theta(s, q) > 1$, and $r_0 = r_0(s, q) > 0$ such that

(2.31) $$\hat{M}_F(r, s) \leq \{K \|f\|_{L_\infty[-r, +r]}\}^\theta \quad \text{for all } r \geq r_0.$$

If, for each $s > 1$, $\hat{\theta}(s)$ *is defined by*

(2.32) $$\overline{\lim_{r \to \infty}} \left\{ \frac{\ln \hat{M}_F(r, s)}{\ln \|f\|_{L_\infty[-r, +r]}} \right\} = \hat{\theta}(s)$$

when $\|f\|_{L_\infty[-r,+r]}$ *is unbounded as* $r \to \infty$, *and* $\hat{\theta}(s) \equiv 1$ *otherwise, then the order* ρ *of* F *satisfies*

(2.33) $$\rho \leq \inf_{s>1} \left\{ \frac{\ln \hat{\theta}(s)}{\ln [s/2 + 1/2s]} \right\},$$

and this upper bound for the order ρ *is in general best possible.*

We conclude this section with the remark that $f(z) = e^{z^2} = \sum_{k=0}^{\infty} a_k z^k$, of perfectly regular growth ($\rho = 2$, $B = 1$) and $a_k \geq 0$ for all $k \geq 0$, necessarily satisfies (by virtue of Theorem 2) the hypotheses of Theorem 4, and gives the case of equality in (2.32).

3. **Some sufficient conditions for geometric convergence.** In this section, we make use of the property of (2.5) of Theorem 3 to establish new sufficient conditions on f for the geometric convergence to zero of its Chebyshev constants, as in (2.3).

Theorem 5. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function with* $a_0 > 0$ *and* $a_k \geq 0$ *for all* $k \geq 1$. *If there exist real numbers* $A > 0$, $s > 1$, $\theta > 0$, *and* $r_0 > 0$ *such that*

(3.1) $$\tilde{M}_f(r, s) \leq A (\|f\|_{L_\infty[0,r]})^\theta \quad \text{for all } r \geq r_0,$$

then there exist a sequence of real polynomials $\{p_n(x)\}_{n=0}^{\infty}$ *with* $p_n \in \pi_n$ *for each* $n \geq 0$, *and a real number* $q \geq s^{1/(1+\theta)} > 1$ *such that*

(3.2) $$\overline{\lim_{n \to \infty}} \left\{ \left\| \frac{1}{p_n(x)} - \frac{1}{f(x)} \right\|_{L_\infty[0,\infty]} \right\}^{1/n} = \frac{1}{q} < 1.$$

Consequently, if $\lambda_{m,n}$ *denote the Chebyshev constants for* $1/f$ *in* $[0, +\infty)$ (*cf.* (1.6)), *then for any sequence of positive integers* $\{m(n)\}_{n=0}^{\infty}$ *with* $0 \leq m(n) \leq n$, *then*

(3.3) $$\overline{\lim_{n \to \infty}} \{\lambda_{m(n),n}\}^{1/n} \leq \frac{1}{q} < 1.$$

Proof. For each $r > 0$, let $q_n(x; r) \in \pi_n$ denote the best Chebyshev approximation to f in $[0, r]$, i.e.

$$(3.4)\qquad \|f - q_n(\cdot;\ r)\|_{L_\infty[0,r]} = \inf_{\sigma_n \in \pi_n} \|f - \sigma_n\|_{L_\infty[0,r]} \equiv \delta_n(r).$$

It is well known that $q_n(x;\ r)$ can be regarded as a polynomial interpolation of $f(x)$ on $[0,\ r]$, i.e. there exist $0 < x_1(r) < x_2(r) < \cdots < x_{n+1}(r) < r$ such that $q_n(x_j(r);\ r) = f(x_j(r)),\ 1 \le j \le n+1$. Express $q_n(x;\ r)$ in Newton interpolation series form, i.e.

$$q_n(x;\ r) = f(x_1(r)) + f[x_1(r),\ x_2(r)](x - x_1(r))$$

$$+ \cdots + f[x_1(r), \cdots, x_{n+1}(r)] \cdot \prod_{j=1}^{n} (x - x_j(r)),$$

where $f[x_1(r), \cdots, x_{j+1}(r)]$ are divided differences of f in the points $x_1(r), \cdots, x_{j+1}(r)$. As is well known, $f[x_1(r), \cdots, x_{j+1}(r)] = f^{(j)}(\xi)/j!$ where $x_1(r) < \xi < x_{j+1}(r)$, and because of the assumption that the Taylor coefficients are all nonnegative, these divided differences are evidently all nonnegative. Thus, $q_n(x;\ r)$ is monotonically increasing as a function of x for all $x \ge r$. Next, let $p_n(x;\ r) \equiv q_n(x;\ r) + \delta_n(r)$ for each $n \ge 0$. From (3.4), it is clear that

$$(3.5)\qquad p_n(x;\ r) \ge f(x) \ge f(0) > 0 \quad \text{for all } x \in [0,\ r].$$

Similarly, from the monotonic nature of $p_n(x;\ r)$ as a function of x for all $x \ge r$, we also have that $p_n(x;\ r) \ge f(r)$ for all $x \ge r$. But as $f(x)$, by hypothesis, is nondecreasing on $[0,\ +\infty)$, we also have that $f(x) \ge f(r) > 0$ for all $x \ge r$. Hence, from the above two inequalities,

$$(3.6)\qquad |1/f(x) - 1/p_n(x;\ r)| \le 2/f(r) \quad \text{for all } x \ge r.$$

For the interval $[0,\ r]$ on the other hand, it follows from (3.5) that

$$\left|\frac{1}{f(x)} - \frac{1}{p_n(x;\ r)}\right| = \frac{|p_n(x;\ r) - f(x)|}{f(x) \cdot p_n(x;\ r)} \le \frac{2\delta_n(r)}{f^2(0)}, \qquad x \in [0,\ r].$$

Next, since f is by hypothesis an entire function, another result of S. N. Bernstein (cf. [3, p. 91]) gives us that

$$\delta_n(r) = \|f - q_n(\cdot;\ r)\|_{L_\infty[0,r]} \le \tilde{M}_f(r;\ s)/(s-1)s^n$$

for any $s > 1$ and any $n \ge 0$. Now choose the $s > 1$ for which (3.1) is valid. Then, using the hypothesis of (3.1) and noting that $\|f\|_{L_\infty[0,r]} = f(r)$, it follows from the last two inequalitites that

$$(3.7)\qquad \|1/f(x) - 1/p_n(x;\ r)\|_{L_\infty[0,r]} \le Bf^{\theta}(r)/s^n \quad \text{for all } n \ge 0, \text{ all } r \ge r_0,$$

where $B \equiv 2A/\{(s-1)f^2(0)\}$. Now, the assumption that $f(r)$ is unbounded as $r \to \infty$ gives us that there exist a positive integer n_4 and an $r(n) \ge r_0$ such that

$f(r(n)) = s^{n/(1+\theta)}$ for all $n \geq n_4$. Consequently, if we set $p_n(x) \equiv p_n(x; r(n))$, we see from (3.6) and (3.7) that $\|1/f(x) - 1/p_n(x)\|_{L_\infty[0,\infty]} \leq C/s^{n/(1+\theta)}$ for all $n \geq n_4$, where $C \equiv \max\{2, B\}$, from which it follows that

$$\overline{\lim_{n\to\infty}} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{L_\infty[0,\infty]} \right\}^{1/n} \leq \frac{1}{s^{1/(1+\theta)}} < 1,$$

the desired result of (3.2). Finally, from (1.6), it is obvious that the Chebyshev constants for $1/f$ satisfy $0 \leq \lambda_{n,n} \leq \lambda_{n-1,n} \leq \cdots \leq \lambda_{0,n}$, and the result of (3.3) follows immediately from (3.2). Q.E.D.

We remark that the assumption in Theorem 4 that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ have only nonnegative Taylor coefficients a_k is probably stronger than is needed in general. For example, for $f(z) = e^z + 2e^{-z}$, which has every other Taylor coefficient negative, the conclusion of (3.2) can be shown to be valid. On the other hand, if we consider $\hat{f}(z) = e^{2z} \sin z$, then $\hat{f}(z)$ does satisfy the growth condition of (3.1), and has Taylor coefficients a_k which are *not* all nonnegative. But as $\hat{f}(\pi k) = 0$ for all nonnegative integers k, it is clear that (3.2) could not hold for *any* sequence $\{p_n(x)\}_{n=0}^{\infty}$ with $p_n \in \pi_n$ for each $n \geq 0$.

To show that the result of Theorem 5 contains the results of Theorems 1 and 2 as special cases, we prove

Theorem 6. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with $a_0 > 0$ and $a_k \geq 0$ for all $k \geq 0$, and assume that there exist finite numbers $\rho > 0$ and $0 < b \leq B$ such that*

$$\overline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{r^\rho} = B, \qquad \underline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{r^\rho} = b. \tag{3.8}$$

Then, there exist real numbers $A > 0$, $s > 1$, and $r_0 > 0$ such that

$$\tilde{M}_f(r, s) \leq A(\|f\|_{L_\infty[0,r]})^\theta \quad \textit{for all } r \geq r_0. \tag{3.9}$$

Thus, the conclusions of Theorem 5 are valid.

Proof. For any $0 < \epsilon < b$, choose any $\theta > 1$ such that $\theta(b-\epsilon)/(B+\epsilon) > 1$. It follows from (3.8) that there exists an $r_0(\epsilon)$ such that

$$e^{(b-\epsilon)r^\rho} < M_f(r) < e^{(B+\epsilon)r^\rho} \quad \text{for all } r \geq r_0(\epsilon). \tag{3.10}$$

Because the Taylor coefficients a_k of f are all nonnegative, we see that for any $s > 1$, and any $r > 0$, $\tilde{M}_f(r, s) = f(\mu r) = M_f(\mu r)$, where $\mu > 1$ is given by (2.19). Thus, to prove the inequality of (3.9) it suffices to show that there is a $\mu > 1$ for which $M_f(\mu r) \leq (M_f(r))^\theta$ for all $r \geq r_0(\epsilon)$. But from the inequalities of (3.10), this is true for $\mu = [\theta(b-\epsilon)/(B+\epsilon)]^{1/\rho} > 1$. Q.E.D.

We remark that as the assumption for perfectly regular growth of f in (1.5), as used in Theorem 2, is obviously stronger than the assumption of (3.8) of Theorem 6, then Theorems 5 and 6 generalize the results of Theorem 2.

The next result, like that of Theorem 6, makes specific use of the sufficient conditions of Theorem 5, and extends Theorem 6 to special entire functions of order $\rho = 0$.

Theorem 7. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of order $\rho = 0$ with $a_0 > 0$ and $a_k \geq 0$ for all $k \geq 0$, such that there exist finite numbers*

$$1 < \rho_l \equiv \overline{\lim_{r\to\infty}} \frac{\ln \ln M_f(r)}{\ln \ln r},$$

and $0 < b_l \leq B_l$ such that

$$\overline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{(\ln r)^{\rho_l}} = B_l, \qquad \underline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{(\ln r)^{\rho_l}} = b_l. \tag{3.11}$$

Then, there exists a sequence of real polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with $p_n \in \pi_n$ for each $n \geq 0$ such that

$$\overline{\lim_{n\to\infty}} \left\{ \left\| \frac{1}{p_n(x)} - \frac{1}{f(x)} \right\|_{L_\infty[0,\infty]} \right\}^{1/n} = 0. \tag{3.12}$$

Thus, the Chebyshev constants for $1/f$ satisfy (3.3) with $q = \infty$.

Proof. For any $0 < \epsilon < b_l$, choose $\theta > 1$ such that $\theta(b_l - \epsilon)/(B_l + \epsilon) > 1$. It follows from (3.11) that there is an $r_0(\epsilon)$ such that

$$e^{(b_l-\epsilon)(\ln r)^{\rho_l}} < M_f(r) < e^{(B_l+\epsilon)(\ln r)^{\rho_l}} \quad \text{for all } r \geq r_0(\epsilon). \tag{3.13}$$

Again, as in the proof of Theorem 6, the nonnegative Taylor coefficients of f give us that $\tilde{M}_f(r, s) = f(\mu r) = M_f(\mu r)$ for any $s > 1$, any $r > 0$, where μ is given by (2.19). But, for *any* choice of $s > 1$, it follows from (3.13) that there is an $r(s, \epsilon) > 0$ such that $\tilde{M}_f(r, s) < (M_f(r))^{\theta}$ for all $r \geq r(s, \epsilon)$. Thus, from Theorem 5, (3.2) is evidently valid for $q = \infty$, which is the desired result of (3.12). Q.E.D.

Finally, consider the entire function $f(z)$ defined by

$$f(z) = \int_1^{\infty} e^{-t} t^{z-1}\, dt. \tag{3.14}$$

As is readily verified, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then its Taylor coefficients, given by

$$a_k = \frac{1}{k!} \int_1^{\infty} e^{-t} (\ln t)^k \frac{dt}{t}, \qquad k \geq 0,$$

are all positive. It can be verified that f is of order $\rho = 1$, but it is *not* of finite type, i.e. $\overline{\lim}_{r\to\infty} \ln(M_f(r))/r = +\infty$ (cf. (3.15)). As such, Theorem 6 cannot be applied. But, we now show that Theorem 5 is applicable. Because $a_k > 0$ for all $k \geq 0$, then as before, $\tilde{M}_f(r, s) = f(\mu r) = M_f(\mu r)$ and $m_f(r) = f(r)$ for all $r > 0$, where $\mu > 1$ is given by (2.19). For $r \geq 1$, it follows from

$$f(r) = \int_0^\infty e^{-t} t^{r-1}\, dt - \int_0^1 e^{-t} t^{r-1}\, dt = \Gamma(r) - \int_0^1 e^{-t} t^{r-1}\, dt$$

that $-1 + e^{-1} + \Gamma(r) < f(r) < \Gamma(r)$ for all $r > 1$. Using Stirling's formula to estimate $\Gamma(r)$, we see that there are two positive constants C_1 and C_2 such that

$$C_1 e^{-r} r^{r-1/2} < f(r) < C_2 e^{-r} r^{r-1/2} \quad \text{for all } r > 1. \tag{3.15}$$

Then, it is readily verified that for any fixed $s > 1$ and any $\theta > \mu(s)$, $\tilde{M}_f(r, s) = M_f(\mu r) \leq (M_f(r))^\theta$ for all r sufficiently large. Hence, Theorem 5 is applicable for any $s > 1$. We similarly remark that the functions

$$f(z) = 1 + \sum_{n=2}^{\infty} \frac{z^n}{(n \ln n)^{n/\rho}}, \qquad g(z) = 1 + \sum_{n=2}^{\infty} z^n \left(\frac{\ln n}{n}\right)^{n/\rho}, \qquad \rho > 0,$$

entire functions of order $\rho > 0$ but of type resp., $B = 0$ and $B = +\infty$, (cf. (3.8)), fail to satisfy the hypotheses of Theorem 6, but as in the previous example, Theorem 5 is, however, applicable.

4. **Quasi-analytic extensions.** Following S. N. Bernstein (cf. [5, p. 373]), let $QA[0, +\infty]$ denote the set of all real-valued continuous functions on $[0, +\infty)$ such that if $f_1, f_2 \in QA[0, +\infty]$ with $f_1(x) \equiv f_2(x)$ on some finite interval $[a, b]$ with $0 < a < b < +\infty$, then $f_1(x) = f_2(x)$ for all $x \in [0, +\infty)$. Then, the following result is an easy extension of Theorem 3 to the set of quasi-analytic functions, in the sense of S. N. Bernstein.

Theorem 8. *Let $f(x)$ be a real continuous function on $[0, +\infty)$ with at most a finite number of points $\{x_i\}_{i=1}^m$ in $[0, +\infty)$ for which $f(x_i) = 0$. Assume that there exist a sequence of real polynomials $\{p_n(x)\}_{n=0}^\infty$ with $p_n \in \pi_n$ for each $n \geq 0$, and a real number $q > 1$ such that*

$$\varlimsup_{n \to \infty} \left\{ \left\| \frac{1}{p_n(x)} - \frac{1}{f(x)} \right\|_{L_\infty[0,\infty]} \right\}^{1/n} = \frac{1}{q} < 1. \tag{4.1}$$

Then, $f \in QA[0, +\infty]$.

Proof. For any q_1 with $q > q_1 > 1$, it follows from (4.1) that there exists a subsequence $\{p_{n_k}(x)\}_{k=0}^\infty$ with $n_0 < n_1 < \cdots$ such that

$$\|1/p_{n_k} - 1/f\|_{L_\infty[0,\infty]} \leq 1/q_1^{n_k} \quad \text{for all } k \geq 0. \tag{4.2}$$

Fixing $r > 0$, let $k_1(r)$ be the least integer such that

$$q_1^{n_k} > q_1^{n_k} - m_f(r) \geq q_1^{n_k}/2 \quad \text{for all } k \geq k_1(r). \tag{4.3}$$

Thus, as in the proof of Theorem 3 (cf. (2.11)), we have that

$$\|p_{n_k} - f\|_{L_\infty[0,r]} \leq 2m_f^2(r)/q_1^{n_k} \quad \text{for all } k \geq k_1(r), \tag{4.4}$$

which implies that

$$E_{n_k}(f;\ 0,\ r) \equiv \inf_{\sigma_{n_k} \in \pi_{n_k}} \|\sigma_{n_k} - f\|_{L_\infty[0,r]} \leq \frac{2m_f^2(r)}{q_1^{n_k}}, \qquad k \geq k_1(r). \tag{4.5}$$

But, by a well-known result of S. N. Bernstein (cf. [5, p. 373]), the inequality of (4.5) implies that f is quasi-analytic on $[0, r]$ for each $r > 0$, i.e. if $g_1(x)$ and $g_2(x)$ are quasi-analytic on $[0, r]$ with $g_1(x) \equiv g_2(x)$ on some proper subinterval $[\alpha, \beta] \subset [0, r]$, then $g_1(x) = g_2(x)$ for all $x \in [0, r]$. As this holds for each $r > 0$, then $f \in QA[0, +\infty]$. Q.E.D.

We remark that other known results for quasi-analytic functions on finite intervals can be directly applied to Theorem 3. For example, using the technique of Bernstein (cf. [5, p. 373]), if the sequence $\{n_k\}_{k=0}^{\infty}$ in (4.2) is such that the ratio n_{k+1}/n_k is bounded as $k \to \infty$, then $f(x)$ can be extended to an entire function because of Theorem 3. Similar extensions of Theorem 4 to quasi-analytic functions on $(-\infty, +\infty)$ can also be made.

Added in proof. Recently Reddy [*A contribution to rational Chebyshev approximation to certain entire functions in* $[0, \infty)$ (to appear)] replaced $\overline{\lim}$ by $\underline{\lim}$ in (1.8) and also he proved under the assumptions of Theorem 6 that

$$\overline{\lim_{n\to\infty}}\, (\lambda_0, n)^{1/n} \leq \frac{1}{e^{b/\rho e B'(1+2^{1/\rho})^\rho}}.$$

REFERENCES

1. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954. MR 16, 914.

2. W. J. Cody, G. Meinardus and R. S. Varga, *Chebyshev rational approximations to* e^{-x} *in* $[0, +\infty)$ *and applications to heat-conduction problems*, J. Approximation Theory 2 (1969), 50–65. MR 39 #6536.

3. Günter Meinardus, *Approximation of functions: Theory and numerical methods*, Springer Tracts in Natural Philosophy, vol. 13, Springer-Verlag, New York, 1967. MR 36 #571.

4. Günter Meinardus and Richard S. Varga, *Chebyshev rational approximation to certain entire functions in* $[0, +\infty)$, J. Approximation Theory 3 (1970), 300–309.

5. A. F. Timan, *Theory of approximation of functions of a real variable*, Internat. Series of Monographs in Pure and Appl. Math., vol. 34, Macmillian, New York, 1963. MR 33 #465.

6. G. Valiron, *Lectures on the general theory of integral functions*, Chelsea, New York, 1949.

7. Richard S. Varga, *Matrix iterative analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962. MR 28 #1725.

8. ———, *Some results in approximation theory with applications to numerical analysis*, Numerical Solution of Partial Differential Equations, vol. 2, Academic Press, New York, 1971, pp. 623–649.

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