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# DIFFERENTIABLE STRUCTURES ON FUNCTION SPACES(1)

BY

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ABSTRACT. A $C^s$ differentiable manifold structure is constructed for spaces of maps from a compact $C^r$ manifold $M$ to a $C^{r+s}$ manifold $N$. The method (1) is inspired by Douady; (2) does not require any additional structure on $N$ (such as sprays); (3) includes the case when $N$ is an analytic manifold and concludes that the mapping space is also an analytic manifold; (4) can be used to treat all the classical mapping spaces ($C^r$ functions, $C^r$ functions with Hölder conditions, and Sobolev functions). Several interesting aspects of these manifolds are investigated such as their tangent spaces, their behavior with respect to functions, and realizations of Lie group structures on them. Differentiable structures are also exhibited for spaces of compact maps with noncompact domain.

**Introduction.** Let $\mathcal{F}(M, N)$ be a space of functions from a compact manifold $M$ to a Riemannian manifold $N$. In 1956 J. Eells, Jr. [9] succeeded in showing that for a large variety of such function spaces the Riemannian structure on $N$ determines a differentiable structure on $\mathcal{F}(M, N)$. Since then function space manifolds have come to lie at the heart of modern infinite dimensional nonlinear analysis [8], [10], [11], [12]. The current interest in the topology of infinite dimensional manifolds [13] has not obscured the fact that the only concrete examples of such are still function space manifolds.

The classical method of constructing charts on $\mathcal{F}(M, N)$ involves a utilization of the exponential map induced by the Riemannian structure on $N$. Abraham [1] following Smale has extended this method to the case when $N$ is an infinite dimensional manifold which carries differentiable partitions of unity. Palais [23] and Eliasson [14] have given axiomatic descriptions of the function spaces $\mathcal{F}(M, N)$; more generally, they also include spaces of sections of fibre bundles. The main result, and probably the best result that these classical methods can produce, is one of the following type. *If $M$ is a compact $C^r$ manifold, $N$ is a $C^{r+s+2}$ Banach manifold $(1 \le s \le \infty)$ carrying $C^{r+s+2}$ partitions of unity and*

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$\mathcal{F}(M, N)$ *is a space of functions satisfying certain local properties* (the prototype is always $\mathcal{F} = C^r$), *then* $\mathcal{F}(M, N)$ *can be given the structure of a* $C^s$ *Banach manifold.*

More recently Douady [5] has been interested in the case when $\mathcal{F} = C^r$ $(0 \leq r < \infty)$ and $N$ is an analytic manifold. His construction uses a strong form of the Whitney extension theorem [29]. In this paper we attempt to adapt the method exposed by Douady to a variety of function spaces. By concentrating our initial efforts on the source manifold $M$ we are able to weaken the form of the extension theorem which we need so that a large number of applications will be possible. Our main result (§3C) is the following. *If* $M$ *is a compact* $C^r$ *manifold,* $N$ *is a* $C^{r+s}$ $(0 \leq s \leq \infty, \omega)$ *Banach manifold, and* $\mathcal{F}(M, N)$ *is a space of functions satisfying certain local properties, then* $\mathcal{F}(M, N)$ *can be given the structure of a* $C^s$ *Banach manifold.* The function space $\mathcal{F}$ could be any of the usual examples of such, e.g. differentiable functions $C^r$, differentiable functions satisfying a Hölder condition $C^r_\alpha$, and Sobolev spaces $H^r_p$. The advantages of this result are (1) two degrees of differentiability are not lost, (2) the analytic case is included, and (3) partitions of unity are not required on $N$.

First (§1) we introduce the preliminary notion of a scalloped region on a manifold and develop some properties. Next (§2) we state the axioms for the spaces $\mathcal{F}$, and we obtain some simple consequences. The following section (§3) contains the construction of the differentiable structure on $\mathcal{F}(M, N)$. We go on (§4) to obtain several properties of the manifolds $\mathcal{F}(M, N)$. The results here are fairly standard and follow by methods more or less analogous to known ones, but we include them for completeness. The verification that the classical examples fit into our treatment is in §5. Finally (§6), we show how our methods can be applied to spaces of compact functions. These results are the first of their kind for spaces of functions with noncompact domain.

Penot [26] has presented a treatment similar to ours.

## 1. Scalloped regions on manifolds.

(A) Let $A$ be a subset of $R^m$ (Euclidean $m$-space) and $E$ be a Banach space (real or complex), then we say that a function $f: A \to E$ is of class $C^r$ if $f$ has a $C^r$ differentiable extension to some neighborhood of $A$. Note that if $A$ is the closure of an open set, then the derivatives $(D^p f)_x$ $(1 \leq p \leq r)$ are uniquely determined on $A$.

We define a $C^r$ *scalloped region* $A$ in $R^m$ to be the closure of an open set whose boundary has a finite open cover $\theta_i$ with $C^r$ diffeomorphisms $\psi_i$ such that $\psi_i(\theta_i) = R^m$ and $\psi_i(A \cap \theta_i)$ is one of the following: (a) a *corner of type* $p$ $(1 \leq p \leq m)$ $\{(t_1, \cdots, t_m)$: some $t_i \geq 0$ for $1 \leq i \leq p\}$, or (b) the intersection of a corner of type $p$ with Euclidean half-space $\{(t_1, \cdots, t_m): t_m \geq 0\}$. (Note that Euclidean half-space is a corner of type 1.) Scalloped regions are clearly invariant under $C^r$ diffeomorphisms. Further, we say that a subset $B$ of a $C^r$ manifold $M$ is a $C^r$ *scalloped region on* $M$ if $B$ is contained in some coordinate chart $(U, \psi)$ such that $\psi(B)$ is a $C^r$ scalloped region in $R^m$.

(B) We now want to show that we can cover any compact manifold $M$ (of dimension $m$ and possibly with boundary) by a finite number of arbitrarily small closed disks such that any disk intersected with any number of the remaining disks forms a scalloped region.

We first construct a finite collection of closed convex disks (in some Riemannian metric) $D_1, \cdots, D_n$ centered at points $a_1, \cdots, a_n$ respectively such that (1) any subcollection of the boundaries $\partial M, \partial D_1, \cdots, \partial D_n$ containing $m$ or fewer sets intersect transversally, and (2) any subcollection of the above containing more than $m$ sets has empty intersection. These two conditions will be insured by the following:

(*) Let $v_i(x)$ be the unit velocity vector of the geodesic curve from $x \in \partial D_i$ to $a_i$, and let $u(x)$ be the unit vector at $x \in \partial M$ orthogonal to $T_x(\partial M)$; if $x \in \partial D_{i_1} \cap \cdots \cap \partial D_{i_k}$ (resp. $x \in \partial M \cap \partial D_{i_1} \cap \cdots \cap \partial D_{i_{k-1}}$) then $v_{i_1}(x), \cdots, v_{i_k}(x)$ (resp. $u(x), v_{i_1}(x), \cdots, v_{i_{k-1}}(x)$) are linearly independent.

Since every $C^r$ structure contains a $C^\infty$ structure [22], we will at the outset take differentiable to mean $C^\infty$. As in Penot [26] we adapt a Riemannian metric to a collar of $\partial M$, and we let $\rho(x, y)$ be the associated distance on $M$. There is an $\epsilon > 0$ such that any set of diameter $\leq 2\epsilon$ is contained in some convex chart on $M$; assume $\epsilon = 1$. We begin by covering $M$ by closed convex disks $C_1, \cdots, C_n$ of radius $\frac{1}{2}$.

Now suppose we have constructed the closed disks $D_1, \cdots, D_r$ $(0 \leq r < n)$ containing $C_1, \cdots, C_r$ respectively and satisfying (*). For any subset $\{i_1, \cdots, i_k\}$ of $\{1, \cdots, r\}$ with $0 \leq k \leq m$ (resp. $0 \leq k \leq m-1$) the intersection $D_{i_1} \cap \cdots \cap D_{i_k}$ (resp. $\partial M \cap \partial D_{i_1} \cap \cdots \cap \partial D_{i_k}$) is a submanifold of $M$. The image under exp of the set $T$ of all unit vectors orthogonal to the tangent spaces at these intersections has dimension at most $m - 1$. Therefore let $a_{r+1}$ be a point in $C_{r+1}$ and not in $\exp(T)$, and let $D_{r+1}$ be the closed disk at $a_{r+1}$ of radius 1. If $x \in \partial D_{i_1} \cap \cdots \cap \partial D_{i_k} \cap \partial D_{r+1}$ (resp. $x \in \partial M \cap \partial D_{i_1} \cap \cdots \cap \partial D_{i_k} \cap \partial D_{r+1}$), then $a_{r+1} = \exp_x v_{r+1}(x) \notin \exp(T)$; so $v_{r+1}(x)$ is not contained in the space at $x$ orthogonal to the tangent space of the indicated intersection. The vectors $v_{i_1}(x), \cdots, v_{i_k}(x), v_{r+1}(x)$ (resp. $u(x), v_{i_1}(x), \cdots, v_{i_k}(x), v_{r+1}(x)$) are therefore linearly independent. Continuing in this way we obtain a cover $D_1, \cdots, D_n$ of $M$ satisfying (*).

Finally we show that any intersection of the form $D_{i_0} \cap (D_{i_1} \cup \cdots \cup D_{i_k})$ $(0 \leq k \leq n)$ is a scalloped region on $M$. First consider the case when $x \notin \partial M$. Let $\{j_1, \cdots, j_l\}$ be a subset of $\{i_1, \cdots, i_k\}$; let $x$ be contained in $\partial D_{j_1} \cap \cdots \cap \partial D_{j_l}$ and no other $\partial D_{i_1}, \cdots, \partial D_{i_k}$ and let $x \in \operatorname{int} D_{i_0}$ (resp. $x \in \partial D_{i_0}$). Define a map from a neighborhood of $x$ to $R^l$ (resp. $R^{l+1}$) by $y \to (1 - \rho(y, a_{j_1}), \cdots, 1 - \rho(y, a_{j_l}))$ (resp. $y \to (1 - \rho(y, a_{i_0}), 1 - \rho(y, a_{j_1}), \cdots, 1 - \rho(y, a_{j_l})))$. Since this map is a submersion, an application of the inverse function theorem results in a diffeomorphism of a neighborhood $U$ of $x$ onto $R^m$ which takes $U \cap D_{i_0} \cap (D_{i_1} \cup \cdots \cup D_{i_k})$ to a corner of type $l$ or a corner of type $l$ intersected with a half-space depending upon whether $x \in \operatorname{int} D_{i_0}$ or $x \in \partial D_{i_0}$. For the case when $x \in \partial M$, let $g: V \to R$ be a submersion of a neighborhood of $x$ such that $V \cap \partial M = g^{-1}(0)$. Using $g$ as one of the components of the submersion constructed above we obtain corners intersected with half-spaces.

2. **Manifold models.** Given any $C^r$ scalloped region $A$ in $R^m$ and any Banach space $E$ (real or complex), a *manifold model* $\mathfrak{F}(A, E)$ is a Banach space of functions $f: A \to E$ satisfying the following axioms.

(mm1) There are continuous linear inclusions

$$C^r(A, E) \subset \mathfrak{F}(A, E) \subset C^0(A, E).$$

(mm2) If $E$ and $F$ are Banach spaces, $\Theta$ is open in $E$, and $\phi: \Theta \to F$ is $C^r$, then the map $\emptyset_*: \mathfrak{F}(A, \Theta) \to \mathfrak{F}(A, F)$, defined by $\phi_*(f) = \phi \cdot f$, is continuous.

(mm3) If $A$ and $B$ are $C^r$ scalloped regions and $\psi$ is a $C^r$ diffeomorphism of $B$ into $A$, then the linear operator $\psi^*: \mathfrak{F}(A, E) \to \mathfrak{F}(B, E)$, defined by $\psi^*(f) = f \cdot \psi$, is defined, continuous, and a submersion.

(mm4) If $A$ and $B$ are two scalloped regions which intersect in a scalloped region and if a function $f: A \cup B \to E$ has the property that $f|_A \in \mathfrak{F}(A, E)$ and $f|_B \in \mathfrak{F}(B, E)$, then $f \in \mathfrak{F}(A \cup B, E)$.

We now drawseveral conclusions concerning manifold models.

(A) In (mm2) the set $\mathfrak{F}(A, \Theta)$ is defined as $\{f \in \mathfrak{F}(A, E): f(A) \subset \Theta\}$ and by (mm1) is an open subset of $\mathfrak{F}(A, E)$.

(B) Manifold models are additive in the sense that $\mathfrak{F}(A, E \oplus F) = \mathfrak{F}(A, E) \oplus \mathfrak{F}(A, F)$ since the injections $E, F \to E \oplus F$ and projections $E \oplus F \to E, F$ induce maps (mm2) which define the direct sum.

(C) Concerning (mm3) if $\psi^*$ is defined (that is if $f \cdot \psi \in \mathfrak{F}(B, E)$), then $\psi^*$ is automatically a bounded operator. This follows from a verification for $\mathfrak{F} = C^0$ and an application of (mm1) along with the closed graph theorem. Therefore, (mm3) really asks only that each function $g \in \mathfrak{F}(\psi(B), E)$ can be extended to a function $e(g) \in \mathfrak{F}(A, E)$ in a continuous and linear way; $e$ will be a right inverse of $\psi^*$ as desired.

(D) Axioms (mm1), (mm2), (mm3) are almost identical to those which appear in Palais [23] (where one of them is incorrectly stated [27]). The need for (mm4) is an unfortunate technicality; it does not have an analogue in the classical theory, but it does hold for all the classical examples.

(E) We define a continuous linear map $T \to \overline{T}$ of $\mathfrak{F}(A, L^n(E, F))$ into $L^n(\mathfrak{F}(A, E), \mathfrak{F}(A, F))$ by $\overline{T}(g_1, \cdots, g_n)(x) = T(x)(g_1(x), \cdots, g_n(x))$.

**Theorem.** *If in* (mm2) $\phi\colon \Theta \to F$ *is* $C^{r+s}$ $(0 \le s \le \infty, \omega)$, *then* $\phi_*$ *is* $C^s$; *furthermore* $D^p\phi_* = \overline{(D^p\phi)}_*$ *or in other words*

$$(D^p\phi_*)_f(g_1, \cdots, g_n)(x) = (D^p\phi)_{f(x)}(g_1(x), \cdots, g_n(x)).$$

(*For* $0 \le s \le \infty$ *this is just the "$\omega$-lemma" of Smale in our context* [1].)

**Proof.** It is obviously sufficient to prove the theorem for $p = 1$ since the step from $p$ to $p+1$ follows replacing $\phi$ by $D^p\phi$ and $F$ by $L^p(E, F)$ with appropriate identifications.

The case $p = 1$ will follow from the equality

$$(*) \qquad \phi_*(f+g) - \phi_*(f) - \overline{(D\phi)}_{*f}\, g = \int_0^1 [\overline{(D\phi)}_{*(f+tg)} - \overline{(D\phi)}_{*f}]\,dt\, g.$$

To see this, given any $\epsilon > 0$ choose a $\delta > 0$ utilizing the continuity of $\overline{(D\phi)}_*$, so that if $\|g\| < \delta$ then $\|\overline{(D\phi)}_{*(f+tg)} - \overline{(D\phi)}_{*f}\| < \epsilon$ for $0 \le t \le 1$. It follows from $(*)$ that $\|\phi_*(f+g) - \phi_*(f) - \overline{(D\phi)}_{*f}\, g\| < \epsilon\, \|g\|$, and therefore, $\overline{(D\phi)}_*$ is the derivative of $\phi_*$.

To prove $(*)$ just evaluate both sides at an arbitrary point $x \in A$. By (mm1) evaluation is a continuous linear map and hence passes under the integral sign. We get

$$\phi(f(x) + g(x)) - \phi(f(x)) - (D\phi)_{f(x)}\, g(x) = \int_0^1 [(D\phi)_{(f(x)+tg(x))} - (D\phi)_{f(x)}]\,dt\, g(x)$$

which is just another version of the mean value theorem.

The case when $\phi$ is analytic must be handled separately. If $E$ and $F$ are complex Banach spaces, then the complex analyticity of $\phi_*$ follows from $p = 1$ above [18]. Otherwise form the complexifications $E \otimes_R C$ and $F \otimes_C R$ and the natural injection $i\colon E \to E \otimes_R C$; then $\phi$ extends to an analytic map $\phi^c\colon \Theta^c \to F \otimes_R C$ where $\Theta^c$ is some open set in $E \otimes_R C$ such that $i(E) \cap \Theta^c = i(\Theta)$ [4, p. 209]. The map $\phi^c_*\colon \mathfrak{F}(A, \Theta^c) \to \mathfrak{F}(A, F \otimes_R C)$ is therefore analytic as above, and its restriction to $\mathfrak{F}(A, \Theta^c)$ which is equal to $\phi_*$ is also analytic.

## 3. The manifold structure on $\mathfrak{F}(M, N)$.

(A) Let $M$ be a compact $C^r$ manifold possibly with boundary and $N$ a $C^{r+s}$ manifold without boundary modeled on a Banach space $E$. Given a manifold model $\mathfrak{F}$ we denote by $\mathfrak{F}(M, N)$ the set of all functions $f\colon M \to N$ which are locally in $\mathfrak{F}$; i.e. for every $m \in M$ there is a scalloped region $A$, a chart $(U, \Psi)$ in $M$ with $m$ an interior point of $A \subset U$, and a chart $(V, \phi)$ in $N$ such that $f(U) \subset V$ and $\alpha(f) \equiv (\phi \cdot f \cdot \psi^{-1}) \in \mathfrak{F}(\psi(A), \phi(v))$. Give $\mathfrak{F}(M, N)$ the weak topology generated by the family of functions $\{\alpha\}$ running over scalloped regions and charts.

Our goal is to give $\mathcal{F}(M, N)$ a differentiable structure of class $C^s$.

(B) Let $D_1, \cdots, D_n$ be a covering of $M$ by closed disks as constructed in (§1B), and let $(V_1, \phi_1), \cdots, (V_n, \phi_n)$ be any collection of charts on $N$. We define $\mathcal{F}(D_1, \cdots, D_n; V_1, \cdots, V_n) = \{f \in \mathcal{F}(M, N): f(D_i) \subset V_i, 1 \leq i \leq n\}$; such sets clearly form an open cover of $\mathcal{F}(M, N)$. We give them a differentiable structure of class $C^s$ as follows.

**Lemma.** *The set* $S(D_1, \cdots, D_k; V_1, \cdots, V_k) = \{(f_1, \cdots, f_k): f_i \in \mathcal{F}(D_i, V_i)$ *and* $f_i = f_j$ *on* $D_i \cap D_j$, $1 \leq i, j \leq k\}$ *is a submanifold of* $\Pi_{i=1}^k \mathcal{F}(D_i, V_i) = \mathcal{F}(D_1, V_1) \times \cdots \times \mathcal{F}(D_k, V_k)$.

**Proof.** First we give $\mathcal{F}(D, V)$ a manifold structure of class $C^s$ thus establishing the case $k = 1$. If $(U, \Psi)$ and $(V, \phi)$ are charts with $D \subset U$, then define $\alpha$ as above to be the chart map of $\mathcal{F}(D, V)$ into the open subset $\mathcal{F}(\psi(D), \phi(V))$ of the Banach space $\mathcal{F}(\psi(D), E)$. If $(U_0, \psi_0)$ and $(V_0, \phi_0)$ are another pair of charts with $D \subset U_0$ and $f(D) \subset U_0$, we have $\alpha \cdot \alpha_0^{-1} = (\phi \cdot \phi_0^{-1})_*(\psi_0 \cdot \psi^{-1})^*$ which is $C^s$ by (§2E).

Now assume the lemma for $k$. For simplicity of notation set $D_{k+1} = D$, $V_{k+1} = V$, $B = (D_1 \cup \cdots \cup D_k) \cap D$, and

$$W = \{(f_1, \cdots, f_k) \in S(D_1, \cdots, D_k; V_1, \cdots, V_k): f_i(D_i \cap D) \subset V_i \cap V\}.$$

Consider the diagram

$$\begin{array}{ccc} & & \mathcal{F}(D, V) \\ & & \downarrow J \\ W & \xrightarrow{H} & \mathcal{F}(B, V) \end{array}$$

where $H(f) = f|_B$ and $J(f_1, \cdots, f_i)(x) = f_i(x)$ if $x \in D_i \cap D$; $H$ is well defined by (mm4).

First we assert that $H$ is $C^s$. The map $\Pi\mathcal{F}(D_i, V_i) \to \Pi\mathcal{F}(D_i \cap D, V_i)$ is $C^s$ since it is just a product of restrictions. This map takes the open subset $\Pi\mathcal{F}(D_i, D_i \cap D; V_i, V_i \cap V)$ to the open subset $\Pi\mathcal{F}(D_i \cap D, V_i \cap V)$, and therefore, takes the submanifold $W$ to the subset $S(D_1 \cap D, \cdots, D_k \cap D; V_1 \cap V, \cdots V_k \cap V)$. This last set is an open subset of $S(D_1 \cap D, \cdots, D_k \cap D; V, \cdots, V)$ which is itself a submanifold of $\Pi\mathcal{F}(D_i \cap D, V)$. Since $S(D_1 \cap D, \cdots, D_k \cap D; V, \cdots, V) = \mathcal{F}(B, V)$ (mm4), we are done except for the technicality of showing that the manifold structure on $\mathcal{F}(B, V)$ defined in case $k = 1$ above agrees with the structure on $S(D_1 \cap D, \cdots, D_k \cap D; V, \cdots, V)$ as a submanifold of $\Pi\mathcal{F}(D_i \cap D, V)$. But this is easily seen by the map $\mathcal{F}(D, V) \to S(D_1 \cap D, \cdots, D_k \cap D; V_1, \cdots, V) \subset \Pi\mathcal{F}(D_i \cap D, V)$ where $f \to (f|_{D_1 \cap D}, \cdots, f|_{D_k \cap D})$; the representation of this map as $\mathcal{F}(\psi(C), \phi(V)) \to \Pi\mathcal{F}(\psi(D_i \cap D), \phi(V))$ is just the restriction of a one-one continuous linear operator to an open set and is therefore a $C^s$ diffeomorphism.

We conclude the proof as follows. Since $J$ is a $C^s$ submersion (mm3), the pair $(H, J)$ is transversal [3]. Therefore, the fibre product which is just $S(D_1, \cdots, D_{k+1}; V_1, \cdots, V_{k+1})$ is a $C^s$ submanifold of $S(D_1, \cdots, D_k; V_1, \cdots, V_k) \times \mathfrak{F}(D_{k+1}, V_{k+1})$ which by the induction hypothesis is itself a $C^s$ submanifold of $\Pi_{i=1}^{k+1}\mathfrak{F}(D_i, V_i)$.

Applying the lemma to the cover $D_1, \cdots, D_n$ we obtain the fact that $S(D_1, \cdots, D_n; V_1, \cdots, V_n)$ or $\mathfrak{F}(D_1, \cdots, D_n; V_1, \cdots, V_n)$ is a submanifold of $\Pi\mathfrak{F}(D_i, V_i)$ of class $C^s$ (note that (mm4) is not used here).

(C) We have covered $\mathfrak{F}(M, N)$ by a collection of open sets each of which admits a differentiable structure of class $C^s$. It remains to show that these structures agree. Let $\mathfrak{A} = \mathfrak{F}(A_1, \cdots, A_n; V_1, \cdots, V_n)$ and $\mathfrak{B} = \mathfrak{F}(B_1, \cdots, B_m; U_1, \cdots, U_m)$, and consider the identity map $\mathfrak{A} \cap \mathfrak{B} \to \mathfrak{B}$ where $\mathfrak{A} \cap \mathfrak{B}$ is taken to be an open submanifold of $\mathfrak{A}$. This map is of class $C^s$ iff the compositions $\mathfrak{A} \cap \mathfrak{B} \to \mathfrak{B} \subset \Pi\mathfrak{F}(B_i, U_i) \to \mathfrak{F}(B_i, U_i)$ are of class $C^s$ $(1 \le i \le m)$. We are immediately reduced to the problem of showing that if $B$ is any disk in $M$, then the restriction $\mathfrak{F}(A_1, \cdots, A_n, B; V_1, \cdots, V_n, U) \to \mathfrak{F}(B, U)$ is of class $C^s$.

First we modify the method of (§1B) to construct an admissible cover $D_1, \cdots, D_p$ of $M$ subordinate to the open cover $\operatorname{int}(A_1), \cdots, \operatorname{int}(A_n)$ and such that any union of the $D_i$ intersects $B$ in a scalloped region. Next, for each $j$ choose some $i_j$ such that $D_j \subset \operatorname{int}(A_{i_j})$; then we have $\mathfrak{F}(A_1, \cdots, A_n; V_1, \cdots, V_n) \subset \mathfrak{F}(D_1, \cdots, D_p; V_{i_1}, \cdots, V_{i_p})$, and so we are further reduced to showing that $\mathfrak{F}(D_1, \cdots, D_p, B; V_1, \cdots, V_p, U) \to \mathfrak{F}(B, U)$ is of class $C^s$. But this is the same kind of restriction as the map $H$ of (§3B) which was of class $C^s$.

We have therefore proved

**Theorem.** *If $M$ is a compact manifold of class at least $C^r$ $(0 < r < \infty)$ and $N$ is a Banach manifold without boundary of class $C^{r+s}$ $(0 \le s \le \infty, \omega)$, then $\mathfrak{F}(M, N)$ is a Banach manifold of class $C^s$; moreover $\mathfrak{F}(D_1, \cdots, D_n; V_1, \cdots, V_n)$ is a submanifold of $\Pi\mathfrak{F}(D_i, V_i)$ of class $C^s$.*

Penot [26] has a similar treatment which depends upon a very complicated construction of which that in (§1B) forms only the first stage. He is apparently able to replace scalloped regions by disks and drop (mm4) altogether.

## 4. Properties of the structure.

(A) *The vector bundle $\pi_*: \mathfrak{F}(M, TN) \to \mathfrak{F}(M, N)$, where $\pi$ is the projection $TN \to N$, can be taken as the tangent bundle of $\mathfrak{F}(M, N)$.*

It is sufficient to show bundle equivalence over open sets of the form $\mathfrak{A} = \mathfrak{F}(D_1, \cdots, D_n; V_1, \cdots, V_n)$. First we observe that the embedding $\mathfrak{A} \to \Pi\mathfrak{F}(\psi_i(D_i), \phi_i(V_i))$ induces the embedding

$$T\mathfrak{A} \to \prod[\mathfrak{F}(\psi_i(D_i), \phi_i(V_i)) \times \mathfrak{F}(\psi_i(D_i), E)] = \prod \mathfrak{F}(\psi_i(D_i), \phi_i(V_i) \times E),$$

and in fact, the image of $T\mathfrak{A}$ is the set $\{(f_1, v_1), \cdots, (f_n, v_n): f_i(\psi_i(D_i \cap D_j)) \subset \phi_i(V_i \cap V_j), (\phi_j \cdot \phi_i^{-1})_*(\psi_i \cdot \psi_j^{-1})^* f_i = f_j, [D(\phi_j \cdot \phi_i^{-1})_{(f_i \cdot \psi_i \cdot \psi_j^{-1})}]_* \cdot (\psi_i \cdot \psi_j^{-1})^* v_i = v_j, 1 \leq i, j \leq n\}$. The proof of this fact is by induction; assume it is true for $W$ of (§3B). Look at the map $H \times J$: $[\Pi_{i=1}^k \mathfrak{F}(\psi_i(D_i), \phi_i(V_i))] \times \mathfrak{F}(\psi(D), \phi(V)) \to \mathfrak{F}(\psi(B), \phi(V))^2$. Then $H \times J$ takes the point $(f_1, \cdots, f_k, f)$ of the submanifold $W \times \mathfrak{F}(\psi(D), \phi(V))$ to the point $(g, f)$ where $g = (\phi \cdot \phi_i^{-1})_*(\psi_i \cdot \psi^{-1})^* f$, $1 \leq i \leq k$. The result follows by induction and the fact that the tangent bundle to the fibre product of $H$ and $J$ is the pullback $(TH \times TJ)^{-1}(T\Delta)$ where $\Delta$ is the diagonal in $\mathfrak{F}(\psi(B), \phi(V))^2$.

The set $\pi_*^{-1}(\mathfrak{A})$ equals $\mathfrak{F}(D_1, \cdots, D_n; \pi^{-1}(V_1), \cdots, \pi^{-1}(V_n))$ in $\mathfrak{F}(M, TN)$ and can be embedded in $\Pi\mathfrak{F}(\psi_i(D_i), \phi_i(V_i) \times E)$. The image of this embedding is easily seen to be the same as the image of $T\mathfrak{A}$ above. Since both embeddings are linear on the fibres, the bundle equivalence is established.

(B) *If $g: N \to N'$ is a $C^{r+s}$ map of $C^{r+s}$ Banach manifolds $N$ and $N'$, then $g_*$: $\mathfrak{F}(M, N) \to \mathfrak{F}(M, N')$ is differentiable of class $C^s$ and $Tg_* = (Tg)_*$.*

Given any $f \in \mathfrak{F}(M, N)$ and $x \in M$, we obtain charts $V$ containing $f(x)$ in $N$ and $V'$ containing $g(f(x))$ in $N'$ such that $g(V) \subset V'$. Taking a finite subcover of $\{V\}$ and corresponding $\{V'\}$ and applying the construction of (§1B) we get open sets $\mathfrak{A} = \mathfrak{F}(D_1, \cdots, D_n; V_1, \cdots, V_n) \subset \mathfrak{F}(M, N)$ and $\mathfrak{B} = \mathfrak{F}(D_1, \cdots, D_n; V_1', \cdots, V_n') \subset \mathfrak{F}(M, N')$ where $f \in \mathfrak{A}$ and $g_*(\mathfrak{A}) \subset \mathfrak{B}$. Embedding $\mathfrak{A}$ and $\mathfrak{B}$ in $\Pi\mathfrak{F}(\psi_i(D_i), \phi_i(V_i))$ and $\Pi\mathfrak{F}(\psi_i(D_i), \phi_i'(V_i'))$ respectively we have that the map $g_*$ is just a restriction of the map $\Pi(\phi_i' \cdot g \cdot \phi_i^{-1})_*$ which is of class $C^s$. To compute the derivative of $g_*$ just look at the derivative of $\Pi(\phi_i' \cdot g \cdot \phi_i^{-1})_*$ and use (A) above.

(C) *If $g: N \to N'$ is* (i) *an immersion,* (ii) *an embedding,* (iii) *a submersion,* (iv) *a subimmersion, then $g_*: \mathfrak{F}(M, N) \to \mathfrak{F}(M, N')$ is of the same type respectively.*

Let $N$ and $N'$ be modeled on $E$ and $E'$ respectively. If $g$ is an immersion, we have for every $x \in N$ charts $(V, \phi)$ in $N$ and $(V', \phi')$ in $N'$ and a continuous linear map $l: E' \to E$ such that $\phi' \cdot g|_V \cdot \phi^{-1}$ is the restriction of a continuous linear map $\gamma: E \to E'$ where $l \cdot \gamma = 1_E$. Therefore for any $f \in \mathfrak{F}(M, N)$ we can find open sets $\mathfrak{A} = \mathfrak{F}(D_1, \cdots, D_n, V_1, \cdots, V_n) \subset \mathfrak{F}(M, N)$ and $\mathfrak{B} = \mathfrak{F}(D_1, \cdots, D_n; V_1', \cdots, V_n') \subset \mathfrak{F}(M, N')$ where $(V_i, \phi_i)$ and $(V_i', \phi_i')$ are related as above. Let $\tilde{E} = \Pi\mathfrak{F}(D_i, E)$, $\tilde{E}' = \Pi\mathfrak{F}(D_i, E')$, $G = \Pi\gamma_i$, and $L = \Pi l_i$. Then we also have that the embeddings $\Phi: \mathfrak{A} \to \Pi\mathfrak{F}(D_i, \phi_i(V_i)) \subset \tilde{E}$ and $\phi: \mathfrak{B} \to \Pi\mathfrak{F}(D_i, \phi_i'(V_i')) \subset \tilde{E}'$ have the property that $\Phi' \cdot g_*|_{\mathfrak{A}} \cdot \Phi^{-1}: \Phi(\mathfrak{A}) \to \Phi'(\mathfrak{B})$ is a restriction of $G$.

Letting $\tilde{\mathfrak{A}} = \Phi(\mathfrak{A})$ and $\tilde{\mathfrak{B}} = \Phi(\mathfrak{B})$ we have the following situation: Banach spaces $\tilde{E}$, $\tilde{E}'$ with respective submanifolds $\tilde{\mathfrak{A}}$, $\tilde{\mathfrak{B}}$, and continuous linear maps $G: \tilde{E} \to \tilde{E}'$, $L: \tilde{E}' \to \tilde{E}$ such that $G(\tilde{\mathfrak{A}}) \subset \tilde{\mathfrak{B}}$ and $L \cdot G = 1_{\tilde{E}}$. We want to show that $G$ is an immersion of $\tilde{\mathfrak{A}}$ into $\tilde{\mathfrak{B}}$. Let $T$ and $T'$ be the tangent spaces to $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{B}}$ at points $a$ and $G(a)$ respectively. Then regarding $T$ and $T'$ as linear varieties in $\widetilde{E}$ and $\widetilde{E}'$ respectively and using the fact that $(TG)_a = G|_T$ we have that $G(T) \subset T'$ and $(L|_{T'}) \cdot (G|_T) = 1_T$. Therefore $G|_T: T \to T'$ is a continuous linear injection whose image splits in $T'$. This is just another characterization of an immersion [3]. Translating this back to the local representation of $g_*$ we conclude that $g_*$ is also an immersion.

Suppose that $g$ is an embedding. In the above choose the $V_i$ so that $g(V_i)$ is a chart in $g(N)$ and $g(V_i) = g(N) \cap \theta_i$ where $\theta_i$ is open in $N'$. This will insure that $g_*(\mathfrak{A}) = \mathcal{F}(D_1, \cdots, D_n; g(V_1), \cdots, g(V_n))$ is open in $\mathcal{F}(M, g(N))$ and equals $\mathcal{F}(M, g(N)) \cap \mathcal{F}(D_1, \cdots, D_n; \theta_1, \cdots, \theta_n)$. Therefore $g_*$ is an injective immersion (since $g$ is), and $g_*(\mathcal{F}(M, N) = \mathcal{F}(M, g(N))$ has the induced topology of $\mathcal{F}(M, N')$.

Case (iii) is completely analogous to (i), and (iv) follows since every subimmersion is the composition of a submersion and an immersion.

(D) *If $M'$ is a closed submanifold of $M$, then the restric i ı map $\rho: \mathcal{F}(M, N) \to \mathcal{F}(M', N)$ is a submersion of class $C^s$.*

To be precise, $M'$ could be a submanifold of the interior of $M$ or a submanifold of $M$ which meets $\partial M$ orthogonally (i.e. if $x \in M \cap \partial M$, then $T_xM$ is spanned by $T_x\partial M$ and $T_xM'$), or $M'$ could even be the boundary of $M$ itself.

The crux of the problem lies in constructing admissible covers $D_1, \cdots, D_n$ of $M$ such that for every disk $D_i$ the intersection $D_i \cap M'$ is either a disk in $M'$ or empty. In this case $D_1 \cap M', \cdots, D_n \cap M'$ will be an admissible cover of $M'$. To do this we slightly alter the construction of (§1B). First choose a Riemannian metric on $M$ so that $M'$ becomes a totally geodesic submanifold of $M$; this is possible in all the cases here considered [15]. Then follow the construction of (§1B) making sure that the centers of the disks chosen avoid the set $\exp S$ where $S$ is the set of unit vectors tangent to $M$ at $M'$ and orthogonal to $TM$. Since each $D_i$ is isometric to a Euclidean disk and $D_i \cap M'$ is totally geodesic in $D_i$, the result follows.

Now given any $\mathcal{F}(D_1, \cdots, D_n; V_1, \cdots, V_n)$ where $D_1 \cap M', \cdots, D_m \cap M'$ is an admissible cover of $M'$ and $D_i \cap M' = \phi$, $m < i \leq n$, we have that $\rho$: $\mathcal{F}(D_1, \cdots, D_n; V_1, \cdots, V_n) \to \mathcal{F}(D_1 \cap M', \cdots, D_m \cap M'; V_1, \cdots, V_m)$. The local representation of $\rho$ is just a restriction of the continuous linear map $\Pi_{i=1}^n \mathcal{F}(D_i, E) \to \Pi_{i=1}^m \mathcal{F}(D_i \cap M', E)$ which is a submersion by (mm3). The completion of the proof follows from (§4C).

(E) *If $M'$ is a closed submanifold of $M$ and $N'$ is a submanifold of $N$, then the set $\mathcal{F}(M, M'; N, N') = \{f \in \mathcal{F}(M, N): f(M') \subset N'\}$ is a submanifold of $\mathcal{F}(M, N)$.*

The map $\rho: \mathcal{F}(M, N) \to (M', N)$ is a submersion and is therefore transversal to the submanifold $\mathcal{F}(M', N')$ [3]. Hence $\rho^{-1}(\mathcal{F}(M', N')) = \mathcal{F}(M, M'; N, N')$ is a submanifold of $\mathcal{F}(M, N)$.

(F) *The manifolds $\mathcal{F}(M, N \times N')$ and $\mathcal{F}(M, N) \times \mathcal{F}(M, N')$ are $C^s$ diffeomorphic.*

Let $\pi$, $\pi'$ be the projections $N \times N' \to N, N'$ respectively. Consider a local representation of the homeomorphism $f \to (\pi_*(f), \pi'_*(f))$ of the first manifold onto the second. Apply (§2B).

(G) *If* $G$ *is a (Banach) Lie group of class* $C^{r+s}$, *then* $\mathfrak{F}(M, G)$ *is a (Banach) Lie group of class* $C^s$.

Let $i: G \to G$ and $\mu: G \times G \to G$ be the $C^{r+s}$ group operations of inversion and multiplication. Then $i_*$ and the composition of the diffeomorphism $\mathfrak{F}(M, G) \times \mathfrak{F}(M, G) \to \mathfrak{F}(M, G \times G)$ followed by $\mu_*$ define $C^s$ group operations on $\mathfrak{F}(M, G)$. We denote the multiplication on $\mathfrak{F}(M, G)$ simply by $\mu_*$.

*If* $G$ *is modeled on* $E$ *and has (Banach) Lie algebra* $g$, *then* $\mathfrak{F}(M, G)$ *is modeled on* $\mathfrak{F}(M, E)$ *and has (Banach) Lie algebra* $\mathfrak{F}(M, g)$.

Let $e$ be the constant map of $M$ into the identity $\epsilon$ of $G$. The tangent space at $e$ is by (§4A) $\pi_*^{-1}(e) = \{f \in \mathfrak{F}(M, TG): \pi \cdot f = e\}$ which is clearly isomorphic to $\mathfrak{F}(M, E)$. The norm on $\mathfrak{F}(M, E)$ is induced from the norm on the tangent space at $e$.

The (Banach) Lie algebra $g$ consists of the Banach space $E$ with a Lie algebra structure such that $\|[a, b]\| \le \|a\|\|b\|$ for $a, b \in g$ [10]. We define the bracket in $\mathfrak{F}(M, g)$ as the composition of the isomorphism in (§2B) followed by $[\ ,\ ]_*$; this composition we will denote simply as $[\ ,\ ]_*$. Since $\|[A, B]_*\| \le$ (constant) $\|A\|\|B\|$ for $A, B \in \mathfrak{F}(M, g)$, we can slightly alter the norm on $\mathfrak{F}(M, E)$ to obtain a Banach Lie algebra structure on $\mathfrak{F}(M, g)$. Now let $A \in \mathfrak{F}(M, g)$, then $A$ induces a left invariant vector field $\overline{A}$ on $\mathfrak{F}(M, G)$ where $\overline{A}_f = (T_2\mu_*)_{(f,\ e)}A$. To show that $\mathfrak{F}(M, g)$ is the Lie algebra of $\mathfrak{F}(M, G)$ it is sufficient to compute the bracket $[[\overline{A}, \overline{B}]]$ in $\mathfrak{F}(M, G)$ and show that at $e$ it is just the bracket $[A, B]_*$ in $\mathfrak{F}(M, g)$. Working locally we have $\overline{A}_f$ is $(D_2\mu_*)_{(f,e)}A$ and $[[A, B]]_e$ is $(D_1D_2\mu_*)_{(e,e)}(A, (D_2\mu_*)_{(e,e)}B) - (D_1D_2\mu_*)_{(e,e)}(B, (D_2\mu_*)_{(e,e)}A)$. This last expression evaluated at $x \in M$ is just $(D_1D_2\mu)_{(\epsilon,\epsilon)}(A(x), (D_2\mu)_{(\epsilon,\epsilon)}B(x)) - (D_1D_2\mu)_{(\epsilon,\epsilon)}(B(x), (D_2\mu)_{(\epsilon,\epsilon)}A(x))$ which is exactly the local representation of the bracket $[A, B]_*$ in $\mathfrak{F}(M, g)$.

*If* exp *is the exponential map from* $g$ *to* $G$ *then* $\exp_*$ *is the exponential map from* $\mathfrak{F}(M, g)$ *to* $\mathfrak{F}(M, G)$.

The 1-parameter subgroup of $G$ generated by $a \in g$ is the unique solution of $\dot{\phi}(t, a) = (D_2\mu)_{(\phi(t,a),\epsilon)}a$ with initial condition $\phi(0, a) = G$. We then have by definition $\phi(1, a) = \exp a$. Similarly, for any $A \in \mathfrak{F}(M, g)$ the unique solution of $\dot{\Phi}(t, A) = (D_2\mu_*)_{(\Phi(t,\ A),\ e)}A$ with $\Phi(0, A) = e$ defines the exponential map of $\mathfrak{F}(M, g)$ to $\mathfrak{F}(M, G)$ [20]. But if we evaluate this differential equation at $x \in M$ we get $\dot{\Phi}(t, A)(x) = (D_2\mu)_{(\Phi(t,A)(x),\epsilon)}A(x)$ with $\Phi(0, A)(x) = \epsilon$. We conclude by uniqueness of solutions that $\Phi(t, A)(x) = \phi(t, A(x))$, and therefore, $\Phi(1, A)(x) = \phi(1, A(x)) = \exp A(x)$ or $\Phi(1, A) = \exp_* A$.

(H) We call a surjective $C^r$ $(r \ge 0)$ map $p: N \to M$ a *foliation* if for every $y \in N$ there are charts $(V, \phi)$ and $(U, \psi)$ containing $y$ and $p(y)$ respectively such that the following diagram is satisfied

$$\begin{array}{ccc} V & \xrightarrow{\phi} & \psi(U) \times W \\ {\scriptstyle p}\downarrow & & \downarrow \\ U & \xrightarrow[\psi]{} & W \end{array}$$

where $W$ is an open subset of some Banach space and the unlabeled map is projection on the first factor. If $r \geq 1$ then $p$ is a foliation iff it is a submersion; foliations clearly include fibrations.

*Let* $\Gamma(M, N) = \{f \in \mathcal{F}(M, N): p \cdot f = 1_M\}$ *then* $\Gamma(M, N)$ *is a* $C^s$ *submanifold of* $\mathcal{F}(M, N)$.

Given any $f \in \Gamma(M, N)$ we can find an admissible cover $D_0, \cdots, D_n$ of $M$ subordinate to charts $(U_0, \psi_0), \cdots, (U_n, \psi_n)$, charts $(V_0, \phi_0), \cdots, (V_n, \phi_n)$ in $N$, and open sets $W_0, \cdots, W_n$ in some Banach space all satisfying the diagram and such that $f \in \mathcal{F}(D_0, \cdots, D_n; V_0, \cdots, V_n)$. We show that $\Gamma(M, N) \cap \mathcal{F}(D_0, \cdots, D_n; V_0, \cdots, V_n)$, which we write as $\Gamma(D_0, \cdots, D_n; V_0, \cdots, V_n)$, is a submanifold of $\mathcal{F}(D_0, \cdots, D_n; V_0, \cdots, V_n)$ by induction on $n$.

The result is easy for $n = 0$. For the general case consider the diagram of (§3B). The sets $\Gamma(D, V)$, $\Gamma(B, V)$, and $\Gamma(D_1, \cdots, D_n; V_1, \cdots, V_n) \cap W$ are all submanifolds of $\mathcal{F}(D, V)$, $\mathcal{F}(B, V)$, and $W$ respectively, and $J$ restricted to $\Gamma(D, V)$ is still a submersion. Therefore, the fibre product of $\Gamma(D, V)$ and $\Gamma(D_1, \cdots, D_n; V_1, \cdots, V_n)$, which is just $\Gamma(D_0, \cdots, D_n; V_0, \cdots, V_n)$, is a submanifold of the fibre product of $\mathcal{F}(D, V)$ and $\mathcal{F}(D_1, \cdots, D_n; V_1, \cdots, V_n)$ which is just $\mathcal{F}(D_0, \cdots, D_n; V_0, \cdots, V_n)$.

5. **Applications.** In this section we consider some concrete examples of manifold models for which the preceding theory holds.

(A) Let $A$ be a $C^r$ scalloped region in $R^m$ and $E$ be a Banach space; we define $C^r(A, E)$ to be the space of all $C^r$ maps $f: A \to E$ with norm $\|f\|_{C^r} = \max_{0 \leq q \leq r}(\sup_{x \in A}\|(D^q f)_x\|)$. To show completeness of the norm let $f_n$ be a Cauchy sequence in $C^r(A, E)$, then $f_n, Df_n, \cdots, D^r f_n$ all converge to functions $f, f^1, \cdots, f^r$ on $A$ with the property that $D^q f = f^q$ $(0 \leq q \leq r)$ on int(A). We now recall a special form of the Whitney extension theorem ([30] and Theorem 1 of [31]) which says that such a function $f$ will have a $C^r$ extension to $R^m$ as long as $A$ satisfies a condition which prevents the occurrence of interior cusps on its boundary. Scalloped regions will easily satisfy this condition. In fact, due to the simplicity of scalloped regions, we need only appeal to the particular version of the Whitney theorem due to Hestenes [17]. According to Hestenes the extension of $f$ can be made locally by a generalized reflection across corners of the type described in (§1A). Furthermore, these reflections are clearly linear and continuous with respect to $f$. We therefore obtain $f_n \to f \in C^r(A, E)$. It should be mentioned that the Whitney and Hestenes theorems are stated only for $E = R^1$, but the proofs are valid without change for arbitrary Banach spaces [2].

The verification of (mm2) follows from a generalized form of the chain rule [2] which tells us that

$$D^q(\phi f) = \sum_{i_1+\cdots+i_p=q;\ 1\le p\le q} a_p (D^p\phi)_f (D^{i_1}f, \cdots, D^{i_p}f)$$

where $i_1, \cdots, i_p \ge 1$ and the $a_p$ are integers. We obtain

$$\sup_{x\in A} \|D^q(\phi f)_x\| \le \sum |a_p| \sup_{x\in A} \|(D^p\phi) f(x)\| \max_{0\le j\le p}\left(\sup_{x\in A} \|(D^{i_j}f)_x\|\right) \le (\text{constant}) \|f\|_{C^r},$$

and, therefore, $\|\phi_*(f)\|_{C^r} \le (\text{constant}) \|f\|_{C^r}$.

The axioms (mm3) and (mm4) are again direct applications of the Hestenes form of the Whitney theorem.

We therefore obtain the main result Theorem (3C) along with all the results of (§4) for the manifold model $\mathfrak{F} = C^r$.

Furthermore, we have the following additional result. *If $M$ and $N$ are smooth compact manifolds and $P$ is a smooth Banach manifold, then composition of functions induces a map $C^r(M, N) \times C^{r+s}(N, P) \to C^r(M, P)$ which is of class $C^s$. Similarly, taking $M$ to be a point, we find that the evaluation map $N \times C^t(N, P) \to P$ is $C^t$.* (The smoothness condition on the manifolds could be relaxed, but we take it for simplicity.)

Reducing the problem to a local one as usual we find that it is sufficient to show that the map $\kappa: C^r(D, V) \times C^{r+s}(V, E) \to C^r(D, E)$ is of class $C^s$ where $D$ and $V$ are closed and open disks in Euclidean spaces respectively, $P$ is modeled on $E$, and $\kappa(f, g) = g \cdot f$. But this is immediate from the observation that the map $\kappa(f,\ ): C^{r+s}(V, E) \to C^r(D, E)$ is a bounded operator (§2C) and the map $\kappa(\ , g): C^r(D, V) \to C^r(D, E)$ is just $g_*$ which is $C^s$. The result for the evaluation map is verified in a similar fashion. More generally we obtain that $N \times \mathfrak{F}(N, P) \to P$ is $C^t$ if $\mathfrak{F}(N, P)$ is a subspace of $C^t(N, P)$.

(B) Let $C^r_\alpha(A, E)$ $(r \ge 0,\ 0 < \alpha \le 1)$ be the space of all functions $f: A \to E$ of class $C^r$ such that $(D^r f)_x$ satisfies a Hölder condition of order $\alpha$ on $A$, and let

$$\|f\|_{C^r_\alpha} = \|f\|_{C^r} + \max_{0\le q\le r} \sup_{x,y\in A;\, x\ne y} \frac{\|(D^q f)_x - (D^q f)_y\|}{\|x - y\|}.$$

The proof that $C^r_\alpha(A, E)$ is a Banach space is similar to that for $C^r(A, E)$. It is only necessary to note that $D^r f$ converges to a function $f^r$ which also satisfies a Hölder condition of order $\alpha$ on $A$ and that the Hestenes reflection of $f$ will have the property that $D^r f$ satisfies a Hölder condition of order $\alpha$ on $R^n$. (The analog of the Whitney theorem for $C^r_\alpha(A, E)$ appears in [28] stated there for $E = R^1$.)

It is clear that $C^{r+1} \subset C^r_\alpha \subset C^0$. For axiom (mm2) to hold we must assume that $\phi$ is at least $C^{r+1}$, but this causes no problems in Theorem (2C) or Theorem (3C). By the chain rule we have

$$\begin{aligned}
&\|D^q(\phi\cdot f)_x - D^q(\phi\cdot f)_y\| \\
&\quad = \Big\| \sum a_p (D^p\phi)_{f(x)}((D^{i_1}f)_x, \cdots, (D^{i_p}f)_x) \\
&\qquad\quad - \sum a_p (D^p\phi)_{f(y)}((D^{i_1}f)_y, \cdots, (D^{i_p}f)_y)\Big\| \\
&\quad \le \sum |a_p| \, \|(D^p\phi)_{f(x)} - (D^p\phi)_{f(y)}\| \max(\|(D^{i_1}f)_x\|, \cdots, \|(D^{i_p}f)_x\|) \\
&\qquad + \sum |a_p| \, \|(D^p\phi)_{f(y)}\| \max(\|(D^{i_1}f)_x - (D^{i_1}f)_y\|, \|(D^{i_2}f)_x\|, \cdots, \|(D^{i_p}f)_x\|) \\
&\qquad + \cdots + \sum |a_p| \, \|(D^p\phi)_{f(y)}\| \max(\|(D^{i_1}f)_y\|, \cdots, \|(D^{i_p}f)_x - (D^{i_p}f)_y\|),
\end{aligned}$$

and, therefore, $\|D^r(\phi\cdot f)_x - D^r(\phi\cdot f)_y\| \le (\text{constant})\|f\|_{C^r_\alpha}$.

The verifications of (mm2) and (mm3) for $C^r_\alpha$ are also similar to those for $C^r$ again using the fact that the Hestenes reflection satisfies a Hölder condition on its $r$th derivative.

Therefore Theorem (3C) and all the results of (§4) hold for $\mathcal{F} = C^r_\alpha$ with one alteration. The necessity of assuming that $\phi$ was at least $C^{r+1}$ in (mm2) forces us to assume that $M$ is a compact manifold of class $C^{r+1}$ in all those results, otherwise the $C^r$ structure could not be defined locally as in (§3B).

(C) Let $r$ be an integer $(0 \le r < \infty)$, $p$ be a real number $(1 \le p < \infty)$, $A$ be a scalloped region in $R^m$, and $E$ be a Banach space. We define the Sobolev space $H^r_p(A, E)$ as the completion of the space $C^r(A, E)$ with respect to the norm $\|f\|_{H^r_p} = \Sigma_{\alpha \le r} \sup_{x\in A} \|(D^\alpha f)_x\|_{L_p}$.

If $r > m/p$, we then have continuous inclusions $C^r(A, E) \dot{\subset} H^r_p(A, E) \subset C^0(A, E)$. The second inclusion follows from the Sobolev embedding theorem [16], [21]. Axiom (mm2) is Theorem 9.10 of [23]. Axiom (mm3) is a simple restatement of the Calderón extension theorem [21], [28], and (mm4) is clear from the equivalent characterization of $H^r_p(A, E)$ as the space of functions $f \in L_p(A, E)$ whose derivatives $Df, \cdots, D^r f$ exist in the weak sense and are in $L^p(A, E)$ [16]. The application of the Sobolev and Calderón theorems is possible since scalloped regions clearly satisfy the hypotheses of every formulation quoted of those theorems; that is $A$ satisfies the cone property [16], is strongly Lipschitz [21], and minimally smooth [28]. It should again be admitted that the above results are stated only for $E = R^1$, but as always the proofs hold for the general case without change.

We therefore have Theorem (3C) and the results of (§4) for $H^r_p(M, N)$ where $r > m/p$ and $m$ is the dimension of $M$.

## 6. Spaces of compact functions.

(A) We have been purposely vague concerning the case $\mathcal{F} = C^0$ in the preceding discussion. It is easy to see, however, that not only is $C^0$ an appropriate manifold model but also that for this case some of the proofs can be enormously simplified, e.g. scalloped regions can be eliminated completely. This will be clear from the following more general considerations.

Let $X$ be a metric space and $N$ be a $C^s$ Banach manifold $(0 \leq s \leq \infty, \omega)$. We define the set $K^0(X, N)$ of *compact maps* as those continuous functions $f: X \to N$ for which $f(X)$ is relatively compact in $N$. We give $K^0(X, N)$ the Hausdorff topology generated by the subbasis $(A, V) = \{f \in K^0(X, N): \overline{f(A)} \subset V\}$ where $A$ is any subset of $X$ and $V$ is open in $N$. If $N$ happens to have a metric $d$ (a necessary and sufficient condition for the existence of one is that $N$ be paracompact—Stone and Smirnov), then this topology coincides with the uniform topology induced by $\rho(f, g) = \sup_{x \in X} d(f(x), g(x))$. Furthermore, the space $K^0(X, E)$ is a Banach space with norm $\|f\| = \sup_{x \in X} \|f(x)\|$. To verify completeness let $f_n$ be a Cauchy sequence, then $f_n$ converges to a continuous function $f$ on $X$. The compactness of $f$ follows easily since an $\epsilon/3$-net for $\overline{f_n(X)}$ where $\|f_n - f\| < \epsilon/3$ will serve as an $\epsilon$-net for $\overline{f(X)}$.

In attempting to apply the manifold model axioms to $K^0(X, E)$ we see that all the consequences of (mm1) which we need, that is (2A), (2C), and the continuity of the evaluation map hold for $K^0(X, E)$. Axiom (mm2) is verified as follows. Since $\overline{f(X)}$ is compact in $\Theta$ a simple argument shows that for every $\epsilon > 0$ there is a $\delta > 0$ such that for $\theta_1$ and $\theta_2$ in $\Theta$ with $\theta_2 \in \overline{f(X)}$ and $|\theta_1 - \theta_2| < \delta$ we have $\|\phi(\theta_1) - \phi(\theta_2)\| < \epsilon$. Let $f_n \to f$, then $\|f_n(X) - f(X)\| < \delta$ for large $n$ and all $x \in X$ which gives $\|\phi(f_n(x)) - \phi(f(x))\| < \epsilon$ or $\|\phi_*(f_n) - \phi_*(f)\| < \epsilon$. Axiom (mm3) can be simplified to require only that for a set $A$ closed in $X$ the restriction $K^0(X, E) \to K^0(A, E)$ is a submersion. This is a simple consequence of (1) the Dugundji extension theorem [7], which supplies us with a linear form preserving extension operator $e$ such that $e(f)(X) \subset \text{co}(f(A)) =$ the convex hull of $f(A)$, and (2) the theorem of Mazur [6], which insures us that the closure $\overline{\text{co}(f(A))}$ is compact. Finally (mm4) holds trivially.

Let $A_1, \cdots, A_n$ be a closed cover of $X$, and let $V_1, \cdots, V_n$ be charts on $N$. Then the manifold structure on sets of the form $K^0(A_1, \cdots, A_n; V_1, \cdots, V_n) = \{f \in K^0(X, N): \overline{f(A_i)} \subset V_i, 1 \leq i \leq n\}$ is constructed as in (§3). Theorem (3C) and the results of (§4) are proved for $K^0(X, N)$ without alteration. Furthermore, it can be seen that these results will hold for any function space model which is a subspace of $K^0(X, E)$ and which satisfies (mm2)–(mm4).

(B) We describe here as an afterthought an interesting method of reducing the case of $K^0$ to that of $C^0$. First we recall the following as a straightforward consequence of the Stone-Čech theorem [7]. *If $X$ is a Tychonoff space and if $Y$ is Hausdorff, then $K^0(X, Y)$ is homeomorphic to $C^0(\check{X}, Y)$ where $\check{X}$ is the Stone-Čech compactification of $X$ and $C^0(\check{X}, Y)$ is equipped with the compact-open* *topology*. Granting this we can apply the results of Eells [9], Palais [23], Abraham-Smale [1], and Eliasson [14] to $C^0(\check{X}, N)$ and obtain the following. *If* $X$ *is a Tychonoff space and* $N$ *is a* $C^{s+2}$ *Banach manifold without boundary but with* $C^{s+2}$ *partitions of unity, then* $K^0(S, N)$ *is a* $C^s$ *Banach manifold where* $1 \leq s \leq \infty$. The advantage of this approach is that it does not require $X$ to be metrizable; but again it suffers the usual disadvantages of the classical theory in that $N$ must carry differentiable partitions of unity, two degrees of differentiability are lost, and the analytic case is not included.

(C) One could replace compactness of the maps by an asymptotic condition. In practice, say, look at all $H^s$ maps which are $H^s$ close to a given one at infinity. This case is also interesting and is now being worked out by M. Cantor, thesis, University of California at Berkeley.

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