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PIECEWISE MONOTONE POLYNOMIAL APPROXIMATION

BY

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ABSTRACT. Given a real function f satisfying a Lipschitz condition of order 1 on $[a, b]$, there exists a sequence of approximating polynomials $\{P_n\}$ such that the sequence $E_n = \|P_n - f\|$ (sup norm) has order of magnitude $1/n$ (D. Jackson). We investigate the possibility of selecting polynomials P_n having the same local monotonicity as f without affecting the order of magnitude of the error. In particular, we establish that if f has a finite number of maxima and minima on $[a, b]$ and S is a closed subset of $[a, b]$ not containing any of the extreme points of f, then there is a sequence of polynomials P_n such that E_n has order of magnitude $1/n$ and such that for n sufficiently large P_n and f have the same monotonicity at each point of S. The methods are classical.

Let $C[a, b] = \{f: f \text{ continuous on } [a, b]\}$, with $\|f\| = \max_{x \in [a,b]} |f(x)|$. $\omega(f, h) = \omega(h)$ will denote the modulus of continuity of f. P_n and T_n are, respectively, the spaces of algebraic and trigonometric polynomials of degree less than or equal to n. $C, C_1, C_2, \cdots$ denote absolute positive constants.

$f \in C[a, b]$ will be called *piecewise monotone* if it has a finite number of maxima and minima on $[a, b]$. a, b and the local maxima and minima of f will be called the *peaks* of f. It follows from a result proved independently by W. Wolibner [6] and S. W. Young [7] that if f is a piecewise monotone function on $[a, b]$ and $\epsilon > 0$, then there exists an algebraic polynomial p which increases and decreases simultaneously with f on $[a, b]$ and satisfies $\|f - p\| < \epsilon$. We are concerned here with the accuracy of this type of approximation as a function of the degree of the approximating polynomial. Jackson's classic theorem [2, p. 56] states that for any function $f \in C[a, b]$ there exists $p \in P_n$ such that $\|f - p\| < C_1\, \omega(f, 1/n)$. O. Shisha [5], J. Roulier [4], and G. G. Lorentz and K. L. Zeller [1] have obtained results on the accuracy of the approximation of an increasing function by increasing polynomials of degree less than or equal to n. Lorentz and Zeller have shown that for each increasing function f on $[a, b]$ and for each $n > 0$ there exists an increasing $p \in P_n$ such that $\|f - p\| < C_2\, \omega(f, 1/n)$. They have also shown that for any function f, increasing on $[-\pi, 0]$ and even on $[-\pi, \pi]$, there exists an even polynomial $t \in T_n$, increasing on $[-\pi, 0]$, which satisfies $\|f - t\| < C_3\, \omega(f, 1/n)$. Our main results are Theorem 3 and Theorem 4

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in which we extend, in some sense, the result of Lorentz and Zeller to piecewise monotone functions.

We will let $J_n(x)$ be the Jackson kernel; i.e.,

$$J_n(x) = \frac{1}{\lambda_n}\left(\frac{\sin nx/2}{\sin x/2}\right)^4, \quad \text{where } \lambda_n = \int_{-\pi}^{\pi}\left(\frac{\sin nx/2}{\sin x/2}\right)^4 dx.$$

It is known [2, p. 54] that $n^3 \leq \lambda_n \leq 2\pi n^3$.

Lemma. *$J_n(x)$ has the following properties:*

(i) *If n is even, $0 \leq \delta \leq \pi$, then*

$$\int_0^{\delta} J_n(x)\,dx \geq \int_{\pi-\delta}^{\pi} J_n(x)\,dx. \tag{1}$$

(ii) *If $0 < \delta \leq \pi/2$, then*

$$\int_{\delta}^{\pi} J_n(x)\,dx \leq C_4/n^3\delta^3. \tag{2}$$

Proof. (i) *Case* 1. $0 \leq \delta \leq \pi/2$.

Since $J_n(x)$ is an even periodic function,

$$\int_{\pi-\delta}^{\pi} J_n(x)\,dx = \int_{\pi}^{\pi+\delta} J_n(x)\,dx.$$

Thus,

$$\int_0^{\delta} J_n(x)\,dx - \int_{\pi}^{\pi+\delta} J_n(x)\,dx = \int_0^{\delta}[J_n(x) - J_n(x+\pi)]\,dx. \tag{3}$$

Now, when n is even,

$$J_n(x) - J_n(x+\pi) = \frac{1}{\lambda_n}\left[\left(\frac{\sin nx/2}{\sin x/2}\right)^4 - \left(\frac{\sin nx/2}{\cos x/2}\right)^4\right].$$

Since $(\tan x/2)^4 \leq 1$ for $0 \leq x \leq \pi/2$, the integrand in (3) is nonnegative, and, hence, (3) is nonnegative for $0 \leq \delta \leq \pi/2$.

Case 2. $\pi/2 \leq \delta \leq \pi$.

Here

$$\int_0^{\delta} J_n(x)\,dx - \int_{\pi-\delta}^{\pi} J_n(x)\,dx = \int_0^{\pi-\delta} J_n(x)\,dx - \int_{\delta}^{\pi} J_n(x)\,dx,$$

and the result follows from Case 1.

(ii) $$\int_{\delta}^{\pi} J_n(x)\,dx \leq \frac{1}{n^3}\int_{\delta}^{\pi}\left(\frac{\sin nx/2}{\sin x/2}\right)^4 dx \leq \frac{1}{n^3}\int_{\delta}^{\pi}\frac{dx}{(\sin x/2)^4}.$$

Now $\sin(x/2) \geq x/\pi$ for $0 \leq x \leq \pi$. Therefore

$$\int_{\delta}^{\pi} J_n(x)\,dx \le \frac{\pi^4}{n^3}\int_{\delta}^{\pi}\frac{dx}{x^4} = \frac{\pi^4}{3n^3}\left[\frac{1}{\delta^3}-\frac{1}{\pi^3}\right] \le \frac{C_4}{n^3\delta^3},$$

and the lemma is established.

Theorem 1. *Let $r(x) = |x|$, $-\pi \le x \le \pi$, and be defined periodically by $r(x + 2k\pi) = r(x)$ for $k = \pm 1, \pm 2, \cdots$. Then there exists $t \in T_n$ such that*

(4) (i) $$\max_{-\pi\le x\le\pi} |t(x) - r(x)| < C_5/n.$$

(ii) *$t(x)$ decreases on $[-\pi, 0]$ and increases on $[0, \pi]$.*

(iii) *$t'(x)$ increases on $[-\pi/2, \pi/2]$.*

(iv) *If $0 < \delta \le \pi/2$, then for all $x \in [\delta, \pi - \delta]$*

(5) $$1 - C_6/n^3\delta^3 \le t'(x) \le 1,$$

and for all $x \in (-\pi + \delta, -\delta)$, $-1 \le t'(x) \le -1 + C_6/n^3\delta^3$.

(v) $\int_{-\pi}^{\pi}|d[t(x) - r(x)]| \le 4C_5/n$.

Proof. For n even let

(6) $$t(x) = \int_{-\pi}^{\pi} J_n(u)r(x-u)\,du.$$

By Jackson's Theorem, $t(x)$ satisfies (i). By periodicity,

$$t(x) = \int_{x-\pi}^{x+\pi} J_n(u)r(x-u)\,du = \int_{x-\pi}^{x} J_n(u)(x-u)\,du - \int_{x}^{x+\pi} J_n(u)(x-u)\,du.$$

Hence

(7) $$\begin{aligned} t'(x) &= \int_{x-\pi}^{x} J_n(u)\,du - \pi J_n(x-\pi) - \int_{x}^{x+\pi} J_n(u)\,du + \pi J_n(x+\pi) \\ &= \int_{x-\pi}^{x} J_n(u)\,du - \int_{x}^{x+\pi} J_n(u)\,du. \end{aligned}$$

Let $x \in [0, \pi]$. By the periodicity and evenness of $J_n(u)$ we then get

$$t'(x) = 2\int_0^x J_n(u)\,du - 2\int_{\pi-x}^{\pi} J_n(u)\,du.$$

Applying (1), $t'(x) \ge 0$. By a similar argument $t'(x) \le 0$ for $-\pi \le x < 0$, proving (ii) for n even. For n odd choose

(8) $$t(x) = \int_{-\pi}^{\pi} J_{n-1}(u)r(x-u)\,du,$$

and the proof follows as above. (i) and (ii) are established.

Proof of (iii). Differentiating in (7),

$$t''(x) = J_n(x) - J_n(x-\pi) - J_n(x+\pi) + J_n(x) = 2[J_n(x) - J_n(x+\pi)].$$

This quantity is nonnegative for $0 \le x \le \pi/2$, as shown in the proof of part (i) of the Lemma. Then $t''(x) \geq 0$ for $-\pi/2 \leq x \leq \pi/2$ by evenness and periodicity of $J_n(x)$.

Proof of (iv). Let $x \in [\delta, \pi - \delta]$. We use (7) to obtain the desired estimates. Since $\int_{-\pi}^{\pi} J_n(u)\,du = 1$ and $J_n(u) \geq 0$, $t'(x) \leq 1$. Now

$$t'(x) \geq \int_{-\delta}^{\delta} J_n(u)\,du - \int_{\delta}^{2\pi-\delta} J_n(u)\,du$$

which, by (2), gives

$$t'(x) \geq 1 - 2C_4/n^3\delta^3 - 2C_4/n^3\delta^3 = 1 - C_6/n^3\delta^3.$$

The estimates for $x \in (-\pi + \delta, -\delta)$ are similarly obtained.

Proof of (v). If $h(x) = t(x) - r(x)$, then $h'(x)$ is ≤ 0 on $[0, \pi]$ by (5). Hence the total variation of $h(x)$ on $[0, \pi]$ is bounded by

$$2 \max_{0 \leq x \leq \pi} |t(x) - r(x)| \leq \frac{2C_5}{n}$$

by (4). Similarly the total variation of $h(x)$ on $[-\pi, 0]$ is bounded by $2C_5/n$. Since $\int_{-\pi}^{\pi} |dh(x)|$ is equal to the total variation of $h(x)$ on $[-\pi, \pi]$, (v) is proved.

Definition. Let f be piecewise monotone on $[a, b]$ with peaks at $a = x_1 < x_2 < \cdots < x_m = b$. A sequence of polynomials $\{p_n\}$ is said to be *comonotone* with f on $[a, b]$ if, for n sufficiently large, p_n increases and decreases simultaneously with f on $[a, b]$. $\{p_n\}$ is said to be *nearly comonotone* with f on $[a, b]$ if, for every ϵ satisfying $0 < \epsilon < \frac{1}{2} \min_i (x_{i+1} - x_i)$, p_n is comonotone with f on $[x_i + \epsilon, x_{i+1} - \epsilon]$, $i = 1, 2, \cdots, m-1$.

Using this terminology, the sequence of polynomials defined by (6) and (8) is comonotone with $r(x)$ on $[-\pi, \pi]$.

Definition. Let $a = y_0 < y_1 < \cdots < y_k = b$ and let $L \in C[a, b]$ be linear on $[y_j, y_{j+1}]$, $j = 0, 1, \cdots, k-1$. Then L will be called *piecewise linear* on $[a, b]$ and $y_0, y_1, \cdots, y_k$ will be called the *nodes* of L. Let S_j be the slope of L on $[y_j, y_{j+1}]$; we let $M(L) = \max_j |S_j|$ and we let $m(L) = \min_j |S_j|$. If $m(L) > 0$ then L will be called a *proper piecewise linear* function.

Theorem 2. *Let L be a proper piecewise linear function on $[-\pi/2, \pi/2]$. Then there exists a nearly comonotone sequence $\{t_n\}$, $t_n \in T_n$, such that*

$$\|L - t_n\| \leq C_7 M(L)/n. \tag{9}$$

Proof. Let $-\pi/2 = x_1 < \cdots < x_m = \pi/2$ be the peaks of L. We will construct a sequence $\{t_n\}$, $t_n \in T_n$, satisfying (9) and such that if $0 < \epsilon < \frac{1}{2}\min(x_{i+1} - x_i)$ then t_n will have the same monotonicity as L on $[x_i + \epsilon, x_{i+1} - \epsilon]$, $i = 1, 2, \cdots, m-1$ for all $n \geq \epsilon^{-1} [C_8 M(L)/m(L)]^{1/3}$.

Let $-\pi/2 = y_0 < y_1 < \cdots < y_k = \pi/2$ be the nodes of L. Let

$$a_0 = \tfrac{1}{2}(S_0 + S_{k-1}), \qquad a_j = \tfrac{1}{2}(S_j - S_{j-1}), \qquad j = 1, 2, \cdots, k-1.$$

Then

$$L(x) = A + \sum_{j=0}^{k-1} a_j|x - y_j|, \tag{10}$$

where A is a constant. Let t be defined as in Theorem 1 [(6) and (8)] and let

$$t_n(x) = A + \sum_{j=0}^{k-1} a_j t(x - y_j). \tag{11}$$

Then $t_n \in T_n$. It follows [3, p. 147] that

$$|L(x) - t_n(x)| \leq \frac{C_7}{n} \max_i \left|\sum_{j=0}^{i} a_j\right| \leq \frac{C_7}{n} \max_i |S_i| = \frac{C_7 M(L)}{n},$$

thus establishing (9).

Now let $x \in [x_i + \epsilon, x_{i+1} - \epsilon]$. We assume, without loss of generality, that L is increasing on (x_i, x_{i+1}). Let $y_q, y_{q+1}, \cdots, y_{q'}$ be those consecutive nodes (if any) of $L(x)$ which are within ϵ of x; i.e., such that

$$|x - y_q| < \epsilon, \quad |x - y_{q+1}| < \epsilon, \quad \cdots, \quad |x - y_{q'}| < \epsilon. \tag{12}$$

Now, differentiating in (11), we get

$$t_n'(x) = \sum_{0}^{k-1} a_j r_j = \sum_{0}^{q-1} a_j r_j + \sum_{q}^{q'} a_j r_j + \sum_{q'+1}^{k-1} a_j r_j, \tag{13}$$

where r_j denotes $t'(x - y_j)$. For $j = q, \cdots, q'$ we have $|x - y_j| < \epsilon$, hence, from Theorem 1 (iii), r_j is decreasing as j goes from q to q'. Also, from Theorem 1 (ii) and Theorem 1 (iv), $0 \leq r_j \leq 1$ for $x - y_j \geq 0$ and $-1 \leq r_j \leq 0$ for $x - y_j \leq 0$. Applying the estimates from Theorem 1 (iv) for t' to the first and third sums in (13), we get

$$t_n'(x) = \sum_{0}^{q-1} a_j - \sum_{0}^{q-1} a_j E_j + \sum_{q}^{q'} a_j r_j - \sum_{q'+1}^{k-1} a_j + \sum_{q'+1}^{k-1} a_j E_j, \tag{14}$$

where, from (5), we have $0 \leq E_j \leq C_6/n^3\epsilon^3$. Now, from the definition of $\{a_j\}$, rewriting the summands, we obtain

$$\begin{aligned} &\sum_{0}^{q-1} a_j - \sum_{q'}^{k-1} a_j + \sum_{q}^{q'} a_j r_j \\ &= \tfrac{1}{2}\left[S_{q-1}(1 - r_q) + \sum_{q}^{q'-1} S_j(r_j - r_{j+1}) + S_{q'}(1 + r_{q'})\right] \geq m(L) \end{aligned} \tag{15}$$

by positivity of each term. Since $0 \le E_j \le C_6/n^3\epsilon^3$, by partial summation we obtain

$$\left| \sum_{0}^{q-1} a_j E_j - \sum_{q'+1}^{k-1} a_j E_j \right| \le \frac{C_8 M(L)}{n^3 \epsilon^3}. \tag{16}$$

Combining (15) and (16) in (14), we get $t'_n(x) \ge m(L) - C_8 M(L)/n^3\epsilon^3$. This quantity is ≥ 0 for $n \ge \epsilon^{-1}[C_8 M(L)/m(L)]^{1/3}$. Q.E.D.

Remark. In the proof of Theorem 2 we showed that t_n will have the same monotonicity as L on $[x_i + \epsilon, x_{i+1} - \epsilon]$, $i = 1, 2, \cdots, m-1$, for $n \ge \epsilon^{-1}[C_8 M(L)/m(L)]^{1/3}$. It is not difficult to see that t_n and L will have the same monotonicity at a point $x \in [x_i + \epsilon, x_{i+1} - \epsilon]$ for $n \ge \epsilon^{-1}[C_8 M(L)/m(x, \epsilon, L)]^{1/3}$, where $m(x, \epsilon, L)$ denotes the minimum slope of L in an ϵ-neighborhood of x. Indeed, in view of the choice of $q, \cdots, q'$, (12), $S_q, \cdots, S_{q'} \ge m(x, \epsilon, L)$; hence, when making a local estimate in (15), $m(L)$ can be replaced by $m(x, \epsilon, L)$. In particular, if ϵ is less than the distance from x to the nearest node, then n depends only on the slope at x.

A piecewise monotone function f will be called *proper piecewise monotone* if it satisfies the following: for any $\epsilon > 0$ and two successive peaks x_i, x_{i+1} of f there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \ge \delta$$

for all x, y in $[x_i + \epsilon, x_{i+1} - \epsilon]$, $x \ne y$.

Theorem 3. *Let f be a proper piecewise monotone function on $[-\pi/2, \pi/2]$ such that $f \in \mathrm{Lip}_M 1$ (i.e., such that f satisfies $\omega(f, h) \le Mh$). Then there is a nearly comonotone sequence $\{t_n\}$, $t_n \in T_n$, such that*

$$\|f - t_n\| \le C_9 M/n. \tag{17}$$

Proof. Let L_n be the proper piecewise linear function on $[-\pi/2, \pi/2]$ which has nodes at the peaks of f and at the points $-\pi/2 + j\pi/n$, $j = 0, 1, \cdots, n$, such that $L_n(x) = f(x)$ at the nodes. Then L_n and f have the same peaks and the same monotonicity for all $x \in [-\pi/2, \pi/2]$. Also,

$$\|f - L_n\| \le M\pi/n \tag{18}$$

and $M(L_n) \le M/n$. Hence, by Theorem 2, there is a polynomial $t_n \in T_n$ such that

$$\|L_n - t_n\| \le C_7 M/n. \tag{19}$$

Now $\|f - t_n\| \le \|f - L_n\| + \|L_n - t_n\| \le C_9 M/n$ from (20) and (21), establishing (19).

Since f is proper piecewise monotone, for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that the slope S_i of L_n in $[y_i, y_{i+1}]$ satisfies $S_i \ge \delta$ whenever $[y_i, y_{i+1}]$ is not within ϵ of a peak. Let $n \ge \epsilon^{-1} [C_8 M/\delta(\epsilon)]^{1/3}$. In view of the remark following Theorem 2, t_n has the same monotonicity as L_n (and, hence, as f) at all points not within ϵ of a peak.

Remark. Theorems 2 and 3 were stated for the interval $[-\pi/2, \pi/2]$, but are easily extended to the interval $[a, a+\pi]$ for any real number a via the translation $x = u + a + \pi/2$.

Theorem 4. *Let f be a proper piecewise monotone function on $[a, b]$ such that $f \in \mathrm{Lip}_M 1$. Then there is a nearly comonotone sequence $\{p_n\}$, $p_n \in P_n$, such that $\|f - p_n\| \le C_{10} M/n$.*

This theorem is proved by use of the standard transformation $x = \cos\theta$, with some modifications.

Note that the class of functions for which Theorem 3 and Theorem 4 are proved includes all f which have a continuous derivative that does not vanish except at the peaks. If we view monotonicity more "locally" and less "globally", we can state our results more precisely, in a sense, than we have in Theorem 3 and Theorem 4. This is done in Theorem 3′ and Theorem 4′, which are actually corollary (indeed, equivalent) to the results already established.

Theorem 3′. *Let $f \in \mathrm{Lip}_M 1$ on $[-\pi/2, \pi/2]$. Then there is a sequence $\{t_n\}$, $t_n \in T_n$, such that*

$$\|f - t_n\| < C_9 M/n.$$

Moreover, f and t_n will have the same monotonicity at x for all

$$n \ge \epsilon^{-1}[C_8 M/\delta(\epsilon)]^{1/3},$$

where ϵ is the distance from x to the nearest peak of f and

$$\delta(\epsilon) = \inf_{0<|h|<\epsilon} \left| \frac{f(x+h) - f(x)}{h} \right|.$$

Theorem 4′. *Let $f \in \mathrm{Lip}_M 1$ on $[a, b]$. Then there is a sequence $\{p_n\}$, $p_n \in P_n$, such that*

$$\|f - p_n\| \le C_{10} M/n.$$

Moreover, f and p_n will have the same monotonicity at x for all $n \ge \epsilon^{-1}[C_8 M/\delta(\epsilon)]^{1/3}$, where ϵ and $\delta(\epsilon)$ are the same as in Theorem 3′.

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