VARIETIES OF LINEAR TOPOLOGICAL SPACES

BY

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ABSTRACT. This paper initiates the formal study of those classes of locally convex spaces which are closed under the taking of arbitrary subspaces, separated quotients, cartesian products and isomorphic images. Well-known examples include the class of all nuclear spaces and the class of all Schwartz spaces.

Introduction. Varieties of groups, having roots in the work of G. Birkhoff and B. H. Neumann in the 1930's have come under intensive investigation in the past dozen or so years; an extensive bibliography may be found in Hanna Neumann's monograph [39]. More recently the notion of a variety of topological groups was introduced by the second author of the present paper and in a sequence of papers ([6], [7], [29]–[38]) many basic results have been established indicating this area as a potentially fruitful direction for research.

This paper initiates the study of varieties of locally convex Hausdorff real topological vector spaces (LCS's). Selected results from this paper were announced in [10].

A variety here means a class of LCS's closed under the formation of subspaces (not necessarily closed), separated quotients, arbitrary products and isomorphic images. While such interesting classes of LCS's as the nuclear spaces of Grothendieck [13] constitute a variety, the class of all normed spaces does not (due to the "arbitrary" nature of products). (The varieties "generated by" classes of normed spaces are, of course, of interest.)

The paper proceeds in five sections which we describe briefly below.

§1 forms the bulwark of the paper insofar as it contains the basic results (Theorems 1.1 and 1.4) on the manner in which a variety is generated; examples concerning the relationships of the varieties generated by certain classical Banach spaces are given. A peculiar consequence of the theorems on generating varieties is that a variety generated by a class of Fréchet spaces is closed with respect to completions.

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§2 concerns varieties that are singly generated, that is, generated by a single LCS. Typical of the results characterizing singly generated varieties is the following: A variety $\mho$ is singly generated if and only if $\mho$ is a subvariety of the variety $\mho(l_1(D))$ generated by $l_1(D)$, for some set D. Consequently, a subvariety of a singly generated variety is singly generated. It is also shown that the class of all varieties is "too big" to be a set and that not all varieties are singly generated. This is in marked contrast to the situation in the case of varieties of groups (see [39]). A universal generator of a variety $\mho$ is defined to be a member E of $\mho$ with the property that every LCS in $\mho$ is a subspace of a power of E. It is shown that every singly generated variety has a universal generator. Kōmura and Kōmura [20] proved that the Fréchet space of rapidly decreasing sequences is a universal generator for the variety of nuclear spaces. It is clear from our work that a universal generator exists for the variety of Schwartz spaces (and hence that one also exists for the nuclear variety).

§3 is concerned with the relative size of varieties. We show that there is a smallest, a next to smallest but *no* third smallest (nontrivial) variety. It is also shown that the variety generated by a normed linear space of infinite dimension always contains a maximal proper subvariety. We observe that the nuclear variety is a relatively small one in that it is contained in the variety generated by any infinite-dimensional Banach space.

§4 considers varieties generated by classical Banach spaces. Using the results on generation of varieties, we deduce certain "stability" properties; for example, the only Banach spaces in $\mho(E)$ when E is reflexive, quasi-reflexive, separable, almost-reflexive or Hilbertian are of like type. Comparisons of the varieties generated by the classical spaces are made; for example, Banach showed that, for $1 < p \neq q < \infty$, l_p and l_q are of incomparable linear dimension: we observe that l_p and l_q are strongly incomparable in the sense that $l_p \notin \mho(l_q)$.

§5 presents some miscellaneous results and open questions.

For convenience, we restrict our attention, throughout the entire paper, to vector spaces over the *real* field. Extensions to the complex field are clear.

1. Definitions and basic results.

Definition. A nonempty class $\mho$ of LCS's is said to be a variety if it is closed under the operations of taking subspaces, quotient spaces, (arbitrary) cartesian products and isomorphic images.

The two extreme examples of a variety are the class of all LCS's and the class of all zero-dimensional LCS's. Less obvious examples are

(a) the class of all Schwartz spaces [14, pp. 278, 279],

(b) the class $\mathcal{N}$ of all nuclear spaces [56, p. 103],

(c) the class of all s-nuclear spaces [25],

(d) the class of all LCS's having their weak topologies [14, Chapter 3, §§13 & 14].

A variety is not necessarily closed under the operations of taking strict inductive limits or direct sums; however, it is closed under tne operation of taking projective limits. In fact, a nonempty class of LCS's closed under the operations of taking subspaces, quotient spaces, finite products, projective limits and isomorphic images is a variety.

Definition. Let $\mathcal{C}$ be a class of LCS's and let $\mathcal{V}(\mathcal{C})$ be the intersection of all varieties containing $\mathcal{C}$. Then $\mathcal{V}(\mathcal{C})$ is said to be the variety generated by $\mathcal{C}$. (Clearly this is indeed a variety.) If $\mathcal{C}$ consists of a single LCS E, then $\mathcal{V}(\mathcal{C})$ is written as $\mathcal{V}(E)$ and is said to be singly generated.

Notation. Let $\mathcal{C}$ be any class of LCS's. Then (a) $S(\mathcal{C})$, (b) $Q(\mathcal{C})$, (c) $C(\mathcal{C})$, and (d) $P(\mathcal{C})$ denote, respectively, the class of all LCS's isomorphic to (a) subspaces of LCS's in $\mathcal{C}$, (b) quotient spaces of LCS's in $\mathcal{C}$, (c) cartesian products of families of LCS's in $\mathcal{C}$, and (d) products of finite families of LCS's in $\mathcal{C}$.

We shall see that if $\mathcal{C}$ is any class of LCS's and $E \in \mathcal{V}(\mathcal{C})$, then E can be obtained from $\mathcal{C}$ by a finite number of applications of Q, S and C. This is not obvious, but is a consequence of the following theorem which shows that, indeed, three applications will always suffice.

Theorem 1.1. *Let $\mathcal{C}$ be any class of LCS's. Then $\mathcal{V}(\mathcal{C}) = QSC(\mathcal{C})$.*

Proof. Let $\mathcal{C}$ be any class of LCS's. The following statements are obvious:

(i) $SS(\mathcal{C}) = S(\mathcal{C})$,

(ii) $CC(\mathcal{C}) = C(\mathcal{C})$,

(iii) $QQ(\mathcal{C}) = Q(\mathcal{C})$.

The next three statements are not quite as obvious, but are easily verified:

(iv) $SC(\mathcal{C}) \supseteq CS(\mathcal{C})$,

(v) $QC(\mathcal{C}) \supseteq CQ(\mathcal{C})$,

(vi) $SQ(\mathcal{C}) = QS(\mathcal{C})$.

We now show that $QSC(\mathcal{C})$ is a variety by noting

(a) $Q\{QSC(\mathcal{C})\} = QSC(\mathcal{C})$, by (iii).

(b) $S\{QSC(\mathcal{C})\} = QSSC(\mathcal{C})$, by (vi); $= QSC(\mathcal{C})$, by (i).

(c) $C\{QSC(\mathcal{C})\} \subseteq QCSC(\mathcal{C})$, by (v); $\subseteq QSCC(\mathcal{C})$, by (iv); $= QSC(\mathcal{C})$, by (ii).

The proof is complete.

Corollary 1.2. *If $\mathcal{C}$ is any class of LCS's, then $\mathcal{V}(\mathcal{C}) = SQC(\mathcal{C})$.*

We note that $\mathcal{O}(\mathcal{C})$ need not be equal to $QCS(\mathcal{C})$, $CSQ(\mathcal{C})$ or $CQS(\mathcal{C})$. Whether or not $\mathcal{O}(\mathcal{C}) = SCQ(\mathcal{C})$, always, is unknown to us. However, we do have Theorem 1.4 which, as we shall see in §4, is a powerful tool.

Let $\{E_i : i \in I\}$ be any family of LCS's and $\mathcal{F}$ be the collection of all nonempty finite subsets of I ordered by inclusion. For each $\sigma \in \mathcal{F}$, we let π_σ denote simultaneously the natural projection mapping of the product space $E = \prod_{i \in I} E_i$ onto the finite product $E_\sigma = \prod_{i \in \sigma} E_i$ and also the natural projection mapping of E_τ onto E_σ for each $\tau \geq \sigma$ in $\mathcal{F}$. Using this notation we have the following

Lemma 1.3. *Let M be a closed subspace of E and M_σ be the closure of $\pi_\sigma(M)$ in E_σ, for each $\sigma \in \mathcal{F}$. For $\tau \geq \sigma$, the formula $f_{\sigma\tau}(x + M_\tau) = \pi_\sigma(x) + M_\sigma$, for $x \in E_\tau$, defines a mapping of E_τ/M_τ into E_σ/M_σ such that $(E_\sigma/M_\sigma, f_{\sigma\tau})$ is a projective system whose projective limit F contains a dense subspace F_0 isomorphic to E/M.*

Proof. Since $\pi_\sigma(M_\tau) \subseteq M_\sigma$ for $\tau \geq \sigma$, $f_{\sigma\tau}$ is a well-defined mapping. It is easily observed that $(E_\sigma/M_\sigma, f_{\sigma\tau})$ is a projective system. The projective limit F is the subspace of $\prod_{\sigma \in \mathcal{F}} E_\sigma/M_\sigma$ formed by those vectors $(y_\sigma)_{\sigma \in \mathcal{F}}$ which satisfy $f_{\sigma\tau}(y_\tau) = y_\sigma$, for $\sigma \leq \tau$. Therefore F contains the subspace F_0 composed of all those vectors of the form $(\pi_\sigma(x) + M_\sigma)_{\sigma \in \mathcal{F}}$, where x ranges in E.

We now show that F_0 is dense in F. Let $y = (y_\sigma)_{\sigma \in \mathcal{F}} \in F$ be given together with n elements $\sigma_1, \cdots, \sigma_n$ of $\mathcal{F}$. Let $\tau = \bigcup_{k=1}^n \sigma_k$ and x be an element of E such that $\pi_\tau(x) + M_\tau = y_\tau$. Then $\pi_{\sigma_k}(x) + M_{\sigma_k} = f_{\sigma_k \tau}(y_\tau) = y_{\sigma_k}$, for $k = 1, \cdots, n$. Thus we have found an element $u = (\pi_\sigma(x) + M_\sigma)_{\sigma \in \mathcal{F}}$ in F_0 such that $y - u$ vanishes on the given coordinates $\sigma_1, \cdots, \sigma_n$. This implies F_0 is dense in F.

Now we show that E/M is isomorphic to F_0. The formula $h(x + M) = (\pi_\sigma(x) + M_\sigma)_{\sigma \in \mathcal{F}}$ for $x \in E$, clearly defines a linear mapping h of E/M onto F_0. For each $\sigma \in \mathcal{F}$, let $h_\sigma : E/M \to E_\sigma/M_\sigma$ be the map given by $h_\sigma(x + M) = \pi_\sigma(x) + M_\sigma$. Clearly, h_σ is continuous. Hence in view of the Embedding Lemma [Kelley, *General topology*, p. 116], it is sufficient to prove that the family $\{h_\sigma : \sigma \in \mathcal{F}\}$ distinguishes points and the closed subsets. Let $\dot{x} = x + M$ be a point not belonging to a closed subset $\dot{C}$ of E/M. We may assume that $\dot{C} = \{y + M : y \in C\}$, where C is a closed subset of E such that $C + M = C$. Then $x \notin C$. Hence there is σ in $\mathcal{F}$ and an open neighborhood U of $\pi_\sigma(x)$ in E_σ such that $\pi_\sigma^{-1}[U] \cap C = \emptyset$. Then for each $\dot{y} = y + M \in \dot{C}$, $U \cap (\pi_\sigma(y) + \pi_\sigma[M]) = \emptyset$ since $y + M \subset C$. Hence $U \cap (\pi_\sigma(y) + M_\sigma) = \emptyset$ or $h_\sigma(\dot{y}) \notin q[U]$, where $q : E_\sigma \to E_\sigma/M_\sigma$ is the quotient map. It follows that $h_\sigma[\dot{C}]$ misses a neighborhood $q[U]$ of $h_\sigma(\dot{x})$.

Theorem 1.4. *If $\mathcal{C}$ is any class of LCS's, then $\mathcal{O}(\mathcal{C}) = SCQP(\mathcal{C})$.*

Proof. Lemma 1.3 shows that $QC(\mathcal{C}) \subseteq SSCQP(\mathcal{C})$. This implies, by Corollary 1.2, that $\mathcal{O}(\mathcal{C}) \subseteq SSSCQP(\mathcal{C}) = SCQP(\mathcal{C})$ and hence that $\mathcal{O}(\mathcal{C}) = SCQP(\mathcal{C})$.

We point out that while Theorem 1.1 has analogues in the theories of varieties of groups [39, p. 15] and varieties of topological groups [6], the same is not true of Theorem 1.4. For varieties of groups the statement corresponding to Theorem 1.4 is false and for varieties of topological groups the situation is not clear. (See [6].)

While a variety is not necessarily closed under the operation of taking completions we do have the following

Theorem 1.5. *If $\mathcal{C}$ is any class of Fréchet spaces and E is in $\mathcal{O}(\mathcal{C})$, then the completion of E is in $\mathcal{O}(\mathcal{C})$.*

Proof. Since finite products and quotients of Fréchet spaces are again Fréchet, we see, from Theorem 1.4, that E is a subspace of a product F of Fréchet spaces. Noting that F is complete, we have that the closure of E in F is complete and hence the desired result.

In §4 we will investigate the varieties generated by some of the well-known Banach spaces.([1]) Of course some information can be immediately obtained from the literature. In particular, noting that

(i) l_p is isomorphic to a subspace of L_p for $1 < p < \infty$ [2],

(ii) every separable Banach space is isomorphic to (a) a subspace of $C[0, 1]$, (b) a quotient of l_1, and (c) a quotient of L_1 [2],

(iii) l_1, $C[0, 1]$ and L_p, for $1 \le p < \infty$, are separable Banach spaces,

(iv) if K is any uncountable compact metric space, then $C(K)$ is isomorphic to $C[0, 1]$ ([28], [45]),

(v) l_1 is isomorphic to a subspace of l_∞ [2],

(vi) l_∞ is isomorphic to L_∞ [42],

we have

Theorem 1.6. *If K is any uncountable compact metric space and $1 < p < \infty$, then $\mathcal{O}(l_p) \subseteq \mathcal{O}(L_p) \subseteq \mathcal{O}(l_1) = \mathcal{O}(C(K)) = \mathcal{O}(L_1) \subseteq \mathcal{O}(l_\infty) = \mathcal{O}(L_\infty)$.*

However, to see which of the above inclusions are proper requires a detailed analysis. This is done in §4 by using Theorem 1.4 as a basic tool in an investigation of the varieties generated by classes of (a) Hilbert spaces, (b) reflexive Banach spaces, and (c) separable Banach spaces. (Of course, the variety generated by the class of *all* Banach spaces is simply the class of all LCS's.

In the light of Banach's results [2, Chapitre XII] and the later work of Paley [40], Kadec [15] and Lindenstrauss and Pełcyzński [22] on compara-

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([1]) Unless specifically indicated otherwise, we take the interval [0, 1] as the underlying space for L_p, $1 \le p \le \infty$.

bility of linear dimension for the spaces l_p and L_p, we will ask: for what p and q is (a) l_p in $\mho(l_q)$ and (b) L_p in $\mho(L_q)$? Again, this will be answered in §4 using Theorem 1.4.

2. Singly generated varieties. In this section we give characterizations of those varieties which are singly generated, and develop the concept of a universal generator.

Definition. Let E be an LCS and m any infinite cardinal. If every neighborhood of zero in E contains a subspace of E of codimension strictly less than m, then E is a $T(m)$-space.[2]

Remarks 2.1. Let E be an LCS of Hamel dimension n, for any given cardinal n. Then

(i) E is a $T(m)$-space for all $m > n$.

(ii) If a continuous norm can be defined on E (in particular, if E is normable) then E is a $T(m)$-space if and only if $m > n$.

(iii) E is a $T(\aleph_0)$-space if and only if E has its weak topology (see [52, Theorem 1.4]).

Noting that products, subspaces and quotients (indeed, continuous linear images) of $T(m)$-spaces are again $T(m)$-spaces, we have

Theorem 2.2. *If $\mathcal{C}$ is a class of $T(m)$-spaces for any fixed infinite cardinal m, then $\mho(\mathcal{C})$ contains only $T(m)$-spaces.*

Corollary 2.3. *If E_1 is an infinite-dimensional normed space and E_2 is an LCS of smaller Hamel dimension, then $E_1 \notin \mho(E_2)$.*

Proof. Let E_1 have Hamel dimension m. From Remarks 2.1 we see that E_2 is a $T(m)$-space but E_1 is not. Hence $E_1 \notin \mho(E_2)$.

Since there exist normed spaces of arbitrarily large dimension, we have

Corollary 2.4. *The class of all varieties is not a set.*

Another consequence of Corollary 2.3 is

Corollary 2.5. *The variety of all LCS's is not singly generated.*

Note that the analogues for groups of Corollaries 2.4 and 2.5 are false [39].

Notation. For each infinite cardinal m, let $\mho_m$ denote the class of all $T(m)$-spaces. (In view of Theorem 2.2, $\mho_m$ is a variety, and by Remarks 2.1, $\mho_{\aleph_0}$ is the variety of all LCS's having their weak topology.)

Lemma 2.6. *Let m be any infinite cardinal and E any LCS in $\mho_m$. Then*

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[2] In this definition we may replace "subspace" by "closed subspace" since each neighborhood of zero contains one that is closed.

(a) $E \in SC(\mathfrak{A})$, *where* $\mathfrak{A}$ *is the class of all LCS's in* $\mathcal{O}(E)$ *of Hamel dimension strictly less than m.*

(b) $E \in SC(\mathcal{C})$, *where* $\mathcal{C}$ *is the class of all normed linear spaces in* $\mathcal{O}_m$.

Proof. Let $\{U_i : i \in I\}$ be a base of closed balanced convex neighborhoods of zero in E. Since E is a $T(m)$-space, each U_i contains a closed subspace E_i of E of codimension $< m$. Put $F_i = E/E_i$, for each i. Using the fact that E is Hausdorff, one may verify in a routine manner that the natural mapping of E into $\Pi_{i \in I} F_i$ is an isomorphism of E onto its image. Thus (a) of our statement is valid, since each F_i is in $\mathcal{O}(E)$ and has Hamel dimension less than m.

To prove (b), put G_i equal to the normed linear space E_{U_i} and proceed similarly.[3]

It is *not* true that every variety is generated by its normed spaces (e.g., consider the variety $\mathfrak{N}$ of nuclear spaces). However, Lemma 2.6 shows that every $\mathcal{O}_m$ is.

Theorem 2.7. *A variety* $\mathcal{O}$ *is singly generated if and only if* $\mathcal{O} \subseteq \mathcal{O}_m$ *for some infinite m.*

Proof. If $\mathcal{O}$ is singly generated then $\mathcal{O} = \mathcal{O}(E)$ for some LCS E of Hamel dimension n, say. If m is any infinite cardinal strictly greater than n, then E is a $T(m)$-space, and by Theorem 2.2, $\mathcal{O} \subseteq \mathcal{O}_m$.

Conversely, suppose $\mathcal{O} \subseteq \mathcal{O}_m$ for some m. Denote by $\mathfrak{A}$ the class of all LCS's in $\mathcal{O}$ of Hamel dimension $< m$. Let F be a fixed m-dimensional real vector space (without topology). Let $\mathfrak{B}$ be the set of all LCS's G such that the vector space G, considered without topology, is a subset of F. (Note that $\mathfrak{B}$ is indeed a set.) Let E be that LCS which is the product of the members of the set $\mathfrak{A} \cap \mathfrak{B}$. Thus $E \in \mathcal{O}$ and $S(\{E\}) \supseteq \mathfrak{A}$, since $S(\{E\})$ clearly contains all members of $\mathfrak{A} \cap \mathfrak{B}$ and isomorphic images thereof. Also, by Lemma 2.6 (a), $\mathcal{O} = SC(\mathfrak{A})$. Therefore,

$$\mathcal{O} = SC(\mathfrak{A}) \subseteq SCS(\{E\}) \subseteq SSC(\{E\}) = SC(\{E\}) \subseteq \mathcal{O}(E) \subseteq \mathcal{O},$$

which implies that $\mathcal{O} = SC(\{E\}) = \mathcal{O}(E)$ is singly generated.

Corollary 2.8. *A subvariety of a singly generated variety is singly generated.*

Theorem 2.9. *Let* m *be any infinite cardinal such that* $m^{\aleph_0} = m$, *e.g.,* m *may be any infinite cardinal of the form* $r^{\aleph_0}$. *Let* D *be a set of cardinality* m, *and let* n *be the smallest cardinal greater than* m. *Then* $\mathcal{O}_n = \mathcal{O}(l_1(D))$.

Proof. First note that the Banach space $l_1(D)$ has Hamel dimension m and hence is in $\mathcal{O}_n$; that is, $\mathcal{O}(l_1(D)) \subseteq \mathcal{O}_n$. From Lemma 2.6 (b), we have that

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[3] For the definition of E_{U_i} see [56, p. 97].

$\mathcal{O}_n = \mathcal{O}(\mathcal{C})$, where $\mathcal{C}$ is the class of all normed spaces of Hamel dimension $\leq m$. Now if E is a member of $\mathcal{C}$, then the density character[4] of E is also $\leq m$, and similarly for the completion $\hat{E}$ of E, since $m^{\aleph_0} = m$. Thus it is well known that $\hat{E}$ is isomorphic to a quotient of $l_1(D)$, so that $\hat{E}$, and therefore E, is a member of $\mathcal{O}(l_1(D))$, and $\mathcal{O}(l_1(D)) \supseteq \mathcal{O}(\mathcal{C}) = \mathcal{O}_n$.

The larger the cardinal m, the larger the variety $\mathcal{O}_m$, and thus

Theorem 2.10. *A variety is singly generated if and only if it is a subvariety of* $\mathcal{O}(l_1(D))$, *for some set D.*

Let D_1 be an arbitrary infinite set and let D_2 be a set of the same cardinality as the power set of D_1. Pełczyński [44] showed that the Banach space $l_\infty(D_1)$ contains a subspace isomorphic to $l_1(D_2)$. Hence $\mathcal{O}(l_1(D_2)) = \mathcal{O}(l_\infty(D_1)) = \mathcal{O}_n$, where n is the smallest cardinal greater than the cardinal of D_2. In particular, since every infinite-dimensional Banach space has dimension $\geq c$, $\mathcal{O}(l_\infty)$ is the smallest one of the $\mathcal{O}_m$ varieties which contains an infinite-dimensional Banach space.

Definition. If a variety $\mathcal{O}$ contains an LCS E with the property that every LCS in $\mathcal{O}$ is isomorphic to a subspace of a product of copies of E, then E is said to be a *universal generator* for $\mathcal{O}$.

From Theorem 2.7 and the last line of its proof, we clearly have

Theorem 2.11. *Every singly generated variety has a universal generator.*

Since every separable LCS is isomorphic to a subspace of a product of separable Banach spaces, and since $QP(l_1) = Q(l_1) = \mathcal{C}$, where $\mathcal{C}$ here denotes the class of all separable Banach spaces, it follows that $\mathcal{O}(l_1)$ contains all separable LCS's. Moreover, every separable Banach space is isomorphic to a subspace of $C([0, 1])$, itself separable. Therefore we have

$$SC(C([0, 1])) = SSC(C([0, 1]))$$
$$\supseteq SCS(C([0, 1])) \supseteq SC(\mathcal{C}) = SCQP(l_1) = \mathcal{O}(l_1)$$

which shows that $C([0, 1])$ is a universal generator for $\mathcal{O}(l_1)$ (cf. [23]).

Statement 9, §2 of [59] implies that every Schwartz space is isomorphic to a subspace of a product of separable Banach spaces. Thus we have the theorem of A. Todd (Ph.D. dissertation, University of Florida; in preparation):

Theorem (Todd). $\mathcal{O}(l_1)$ *contains the variety of Schwartz spaces.*

Using Corollary 2.8 and Theorem 2.11 we obtain

Theorem 2.12. *The variety of Schwartz spaces, and hence also the variety of nuclear spaces, has a universal generator.*

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[4] The density character of E is the smallest cardinal of the dense subsets of E.

A deep result of Kōmura and Kōmura [20] is that (s) is a universal generator for the variety of nuclear spaces, where (s) is the Fréchet space of rapidly decreasing sequences. To our knowledge, no one has found a correspondingly serviceable and concrete universal generator for Schwartz spaces.

3. **Relative size of varieties.** Using Banach's [2] assertion (proved, e.g. in [3]) that every infinite-dimensional Banach space has an infinite-dimensional closed subspace with a Schauder basis, and the fact [20] that (s) is a universal generator for the variety $\mathfrak{N}$ of nuclear spaces, the third author [53] proved that if B is any infinite-dimensional Banach space then $\mathfrak{N} \subseteq SC(B)$. (For the case $B = l_p$, $1 \le p \le \infty$, this result is due to Grothendieck [13]; cf. [56, p. 101].) Consequently we have

Theorem 3.1. *The variety generated by any infinite-dimensional Banach space contains the variety of nuclear spaces.*

The proof as given in [53] also yields

Theorem 3.2. *The variety generated by any infinite-dimensional sequentially-complete barrelled space with a normalized Schauder basis contains the nuclear variety.*

(A *normalized* Schauder basis is one which is both bounded and bounded away from zero; see [18].)

Example 3.3. Let E be the vector space which is the intersection of the spaces $l_{1+1/n}$, and give E the metric topology induced by the totality of the $l_{1+1/n}$-norms $(n = 1, 2, \cdots)$. Then E is a Fréchet space with a normalized basis, so that $\mathfrak{N} \subseteq \mathfrak{V}(E)$, by Theorem 3.2. Theorem 3.1 is not applicable since, by [51], E contains no infinite-dimensional Banach space.

Further comments on normalized bases are made in §5.

Theorem 3.4. *Any variety generated by an infinite-dimensional normed linear space has a maximal proper subvariety.*

Proof. Let N be an infinite-dimensional normed linear space, and let $\mathcal{C}$ be a *set* of LCS's in $\mathfrak{V}(N)$ such that a subvariety of $\mathfrak{V}(N)$ is proper if and only if it is generated by an element in $\mathcal{C}$ (cf. proof of Lemma 2.6 and Theorem 2.7). Clearly, $\mathcal{C}$ is nonempty. For E_1 and E_2 in $\mathcal{C}$, define $E_1 \le E_2$ if $\mathfrak{V}(E_1) \subseteq \mathfrak{V}(E_2)$.

Let $\mathcal{S}$ be any ascending chain in $\mathcal{C}$. We show that $N \notin \mathfrak{V}(\mathcal{S})$: if $N \in \mathfrak{V}(\mathcal{S})$, then $N \in SQP(\mathcal{S})$ by Theorem 4.1 (proved independently) so that $N \in SQ\{E_1 \times \cdots \times E_n\}$ where $E_1, \cdots , E_n \in \mathcal{S}$, and we may assume that $E_i \le E_n$ for $i = 1, \cdots n$. But then $N \in \mathfrak{V}(E_1 \times \cdots \times E_n) = \mathfrak{V}(E_n)$, which contradicts the fact that $\mathfrak{V}(E_n)$ is a proper subvariety of $\mathfrak{V}(N)$. Therefore, $N \notin \mathfrak{V}(\mathcal{S})$, so that $\mathfrak{V}(\mathcal{S}) = \mathfrak{V}(F)$ for some $F \in \mathcal{C}$. Clearly F is an upper bound for $\mathcal{S}$. By Zorn's lemma there is a maximal element M in $\mathcal{C}$, and hence $\mathfrak{V}(M)$ is a maximal proper subvariety of $\mathfrak{V}(N)$.

Notation. Let ω denote the Fréchet space of all real sequences with the usual product topology. Let φ denote the strong dual of ω: φ is an $\aleph_0$-dimensional vector space with the strongest locally convex topology. We note that ω has its weak topology and is reflexive.

For each infinite cardinal m, let φ_m denote an m-dimensional vector space given the strongest locally convex topology; in particular, $\varphi_{\aleph_0} = \varphi$. We point out that every strict (LF)-space contains a complemented copy of φ [54].

Theorem 3.5. *Let n be any infinite cardinal and let m be the smallest cardinal greater than n. Then*

   (i) *φ_n is a universal generator for $\mho(\varphi_n)$;*

   (ii) *$\mho(\varphi_m)$ has a (unique) maximum proper subvariety, namely $\mho(\varphi_n)$, and*

   (iii) *$\mho(\varphi_m) \cap \mho_m = \mho(\varphi_n)$.*

Proof. By Theorem 1.4, $\mho(\varphi_n) = SCQP(\varphi_n)$. Note that every LCS in $P(\varphi_n)$ is isomorphic to φ_n, so that $\mho(\varphi_n) = SCQ(\varphi_n)$. Also each quotient of φ_n is isomorphic to φ_k for some $k \leq n$ and hence is in $S(\varphi_n)$. Thus $\mho(\varphi_n) = SCQ(\varphi_n) = SCS(\varphi_n) = SC(\varphi_n)$, which shows that φ_n is a universal generator.

Let $\mho$ be a subvariety of $\mho(\varphi_m)$ and let E be any of its members. Since φ_m is a universal generator, E is isomorphic to a subspace of a product $\Pi_{i \in I} F_i$, where each $F_i = \varphi_m$. Let E_i be the projection of E on F_i, for each $i \in I$. Then E_i with the induced topology is isomorphic to φ_{k_i}, for some $k_i \leq m$. Clearly, if each $k_i < m$, then $E \in SC(\varphi_n) = \mho(\varphi_n)$. If $k_i = m$ for some i, then there is a continuous linear mapping of E onto φ_m. Since every linear mapping onto φ_m is open, E has a quotient isomorphic to φ_m and hence $\mho(E) = \mho(\varphi_m)$. Thus there are only two possibilities: $\mho = \mho(\varphi_m)$ or $\mho \subseteq \mho(\varphi_n)$.

Finally, to see that $\mho(\varphi_m) \cap \mho_m = \mho(\varphi_n)$, we only have to observe that $\varphi_m \notin \mho_m$ and $\varphi_n \in \mho(\varphi_m) \cap \mho_m$.

Theorem 3.6. (i) *The variety of all LCS's with their weak topology has R, the reals, as a universal generator and hence is the smallest (nontrivial) variety.*

   (ii) *$\mho(\varphi)$ is the (unique) second smallest variety, in the sense that every variety properly containing $\mho(R)$ contains $\mho(\varphi)$.*

   (iii) *There is no third smallest variety.*

Proof. It is well known that an LCS has its weak topology if and only if it is isomorphic to a subspace of a product of copies of R (see [19], or Theorem 1.4 of [52]), and one easily observes that $SCQP(R) = SC(R)$: (i) is easily verified by noting that if $\mho$ is any nontrivial variety, then it contains an LCS E with a 1-dimensional subspace, necessarily isomorphic to R, so that $\mho(R) \subseteq \mho(E) \subseteq \mho$.

Now suppose that $\mho$ is a variety strictly larger than $\mho(R)$. This means that $\mho$ contains an LCS E which does not have its weak topology. But by Theorem

1.4 of [52], E does not have its weak topology if and only if the product space E^I contains a subspace isomorphic to φ for each indexing set I with cardinality $\geq c$. It therefore follows that $\mho \supseteq \mho(E) \supseteq \mho(\varphi)$.

To prove (iii) we find two varieties properly containing $\mho(\varphi)$ whose intersection is $\mho(\varphi)$. Let $\aleph_1$ be the smallest cardinal greater than $\aleph_0$. Since $\mho(\varphi)$ contains only $T(\aleph_1)$-spaces, $\varphi_{\aleph_1} \notin \mho(\varphi)$, and thus $\mho(\varphi_{\aleph_1})$ properly contains $\mho(\varphi)$. From the proof of Theorem 3.5(ii) it is clear that if E is in $\mho(\varphi)$, then either E has its weak topology or some quotient of E is isomorphic to φ. If we let E be an $\aleph_0$-dimensional normed linear space, then $E \notin \mho(\varphi)$ since E does not have its weak topology and φ is not metrizable; but E is in $\mho_{\aleph_1}$. Thus $\mho_{\aleph_1}$ also properly contains $\mho(\varphi)$. But by Theorem 3.5, $\mho_{\aleph_1} \cap \mho(\varphi_{\aleph_1}) = \mho(\varphi)$, and the proof is complete.

Corollary 3.8. *An LCS E has its weak topology if and only if $\varphi \notin \mho(E)$.*

We note the curiosity that the mutually dual spaces ω and φ are universal generators for the first and second varieties, respectively.

4. Varieties generated by classes of Banach spaces. In the present section we concern ourselves with varieties generated by classes of Banach spaces. Two problems are dealt with: first, we consider properties of a class $\mathcal{B}$ of Banach spaces which are inherited to a greater or lesser extent by members of $\mho(\mathcal{B})$; second, we investigate the varieties generated by many of the classical Banach spaces.

The consideration of general inheritance properties is based to a large extent on Theorem 1.4 and one of its consequences, Theorem 4.1. It seems likely that the techniques used here connot be easily extended to varieties generated by classes of Fréchet spaces or more general classes.

Our treatment of the varieties generated by special (usually the classical) Banach spaces is, of course, based upon the extensive literature concerning these spaces. Invariably, the conclusion that one space is *not* in the variety generated by another is a consequence of Thoerem 4.1 and some distinctive relationship between the spaces.

A remark on notation: if $\mathcal{C}$ is a class of LCS's then $P_c(\mathcal{C})$ denotes the class of all LCS's isomorphic to a countable product of members of $\mathcal{C}$.

Theorem 4.1. *Let $\mathcal{C}$ be a class of LCS's and let $E \in \mho(\mathcal{C})$. Then*
  (i) *If E is normable, then $E \in SQP(\mathcal{C}) = QSP(\mathcal{C})$; and*
  (ii) *if E is metrizable, then $E \in SP_cQP(\mathcal{C})$.*

Proof. (i) In light of Theorem 1.4 and the fact that $SPQP(\mathcal{C}) = SQPP(\mathcal{C}) = SQP(\mathcal{C})$, we need only show that if the normed space E is a subspace of the product $F = \Pi_{i \in I} E_i$ of LCS's E_i $(i \in I)$, then E is isomorphic to a subspace of the product $\Pi_{i \in \sigma} E_i$ for some finite subset σ of I.

Let U be the unit ball of the normed space E. Then U contains a set of the form $\mathcal{O} \cap E$ where $\mathcal{O}$ is a basic open set in F, i.e. $\mathcal{O}$ is of the form

$$\mathcal{O} = \prod_{i \in I} \mathcal{O}_i$$

with each $\mathcal{O}_i$ a neighborhood of zero in E_i and $\mathcal{O}_i = E_i$ for all i not in some finite subset σ of I.

We assert that the natural projection π_σ of E onto its image (under π_σ) in $E_\sigma = \Pi_{i \in \sigma} E_i$ is an isomorphism.

In fact, π_σ is clearly linear and continuous. If $x \in E$ but $x \notin U$ then $x \notin \mathcal{O}$ so $\pi_\sigma(x) \neq 0$, so that π_σ is one-one. Finally, if $y' \in (\Pi_{i \in \sigma} \mathcal{O}_i) \cap \pi_\sigma(E)$, then there exists $y \in E$ such that $\pi_\sigma(y) = y'$, from which $y \in \mathcal{O}$ follows with the consequence that $\pi_\sigma(U)$ is a neighborhood of zero in $\pi_\sigma(E)$, and π_σ is open.

(ii) If E is metrizable, then its topology is given by a countable collection of "unit balls" and a countable union of finite subsets of I do the work of σ in the above proof.

Remarks. (1) Many of the classical Banach spaces are isomorphic to their own square. If E is such a space then (a) a Fréchet space F is in $\mathcal{O}(E)$ if and only if F is in $SP_c Q(E)_i$, and (b) a Banach space B is in $\mathcal{O}(E)$ if and only if it is in $SQ(E)$.

(2) It is also worth mentioning that the usual duality of subspaces and quotients (see [14], [56]) is an effective weapon in determining the normed linear spaces in a given variety; this is especially true for most of the classical Banach spaces whose duals are well known.

(3) Finally, we mention that by the Open Mapping Theorem of Banach [2], we have that a Banach space E is a quotient of a Banach space F if and only if E is a continuous linear image of F.

Of course, this remark applies to the case where E and F are both Fréchet spaces or both strict (LF)-spaces; however, we have yet to make any real use of this comment in the latter case.

Theorem 4.2. *Let $\mathcal{B}$ be a class of reflexive Banach spaces. Suppose $E \in \mathcal{O}(\mathcal{B})$. Then the completion of E is semireflexive. Consequently, if E is infrabarrelled, then the completion of E is reflexive.*

In particular, any Fréchet space (more particularly, any Banach space) or strict (LF)-space in $\mathcal{O}(\mathcal{B})$ is reflexive.

Proof. By Theorem 1.5, $\mathcal{O}(\mathcal{B})$ is closed with respect to completions. But (by [14] or [56]), $CQP(\mathcal{B})$ contains only semireflexive LCS's. As any (quasi-) complete subspace of a semireflexive LCS is semireflexive [56, p. 144], we get the proof of the first assertion. The second assertion is an immediate consequence of the

fact ([14] or [56]) that an infrabarrelled, semireflexive LCS is reflexive.

The other assertions follow from the infrabarrelled, complete nature of the classes of spaces cited.

Remark. We have actually proved that any quasi-complete LCS in the variety generated by a class of reflexive Banach spaces is semireflexive.

The corresponding statement in case of a class of reflexive Fréchet spaces is false. In fact, an example of Grothendieck and Köthe yields a Fréchet-Montel space with l_1 as a quotient.

Corollary 4.3. For $1 < p < \infty$, neither c_0 nor L_1 is in $\mho(L_p)$. Consequently, $\mho(L_p) \subsetneqq \mho(L_1) = \mho(l_1) = \mho(C([0, 1]))$.

Theorem 4.4. Let $\mathcal{H}$ be a class of Hilbert spaces and B be a Banach space in $\mho(\mathcal{H})$. Then B is (isomorphic to) a Hilbert space.

Proof. By Theorem 4.1, $B \in SQP(\mathcal{H})$. Since $QP(\mathcal{H})$ contains only spaces isomorphic to Hilbert spaces, and every closed subspace of a Hilbert space is a Hilbert space, we obtain the desired result.

Corollary 4.5. For $1 \leq p \; (\neq 2) \leq \infty$, $l_p \notin \mho(l_2) = \mho(L_2)$. Consequently, $L_p \notin \mho(l_2)$.

Theorem 4.6. For $1 < p \neq q < \infty$, $l_p \notin \mho(l_q)$.

Proof. Since l_p is isomorphic to its own square, any Banach space in $\mho(l_p)$ is a member of $QS(l_p)$. Suppose $p < q$. Then for any (closed) subspace S_q of l_q, and any continuous linear operator $T: S_q \to l_p$, we have by Theorem A2 of [51] that T is compact. Thus T is never an onto operator. It follows that $l_p \notin QS(l_q)$ and, therefore, $l_p \notin \mho(l_q)$.

Suppose $q < p$. Then $p' < q'$, where $(1/p) + (1/p') = 1 = (1/q) + (1/q')$. If $l_p \in \mho(l_q)$, then $l_p \in SQ(l_q)$, so by duality, $l_{p'} \in QS(l_{q'})$ which is—as we have just seen—impossible. The proof is complete.

Remark. Actually, Rosenthal has shown more than we have applied above; in fact, using the deep results of [3] and [43], he has shown that if μ and ν are *any* atomic measures and $1 < p < q < \infty$, then every continuous linear operator from any subspace of $L_q(\mu)$ to $L_p(\nu)$ is compact.

Of course, the varietal consequence of this fact is the following: *For any atomic measures μ and ν and any pair $p, q: 1 < p \neq q < \infty$, $L_p(\nu) \notin \mho(L_q(\mu))$.*

Theorem 4.7. Let $1 < p \neq q < \infty$. Then the following statements are equivalent:
   (i) $L_p \in \mho(L_q)$;
   (ii) $l_p \in \mho(L_q)$; and
   (iii) either $q < p \leq 2$ or $2 \leq p < q$.

Proof. Since for each $p \geq 1$, l_p is isomorphic to a subspace of L_p [2, p. 206], (i) implies (ii).

For $1 < q < p \leq 2$, L_p is isomorphic to a subspace of L_q [22]. Thus by duality, if $2 \leq p < q < \infty$, then L_p is isomorphic to a quotient of L_q. In either case, (i) follows from (iii).

Finally, taking note of the fact that each L_p is isomorphic to its own square, we proceed as in Theorem 4.6 to apply Theorem A2 of [51] to yield non-(iii) implying non-(ii).

Corollary 4.8. *If* $1 < p \ (\neq 2) < \infty$, *then* $\mathcal{O}(l_p) \subsetneqq \mathcal{O}(L_p)$.

Proof. By Theorem 4.7, $l_2 \in \mathcal{O}(L_p)$ while by Theorem 4.6, $l_2 \notin \mathcal{O}(l_p)$.

Modifying the terminology of [21], we call an LCS E *almost reflexive* whenever every bounded sequnce in E has a weak Cauchy subsequence. Of course, Eberlein's theorem (see [56]) yields the almost reflexivity of all reflexive Fréchet spaces. The spaces c_0 and $C(\Omega)$ for Ω a compact, Hausdorff topological space containing no perfect subsets provide nonreflexive examples of almost reflexive Banach spaces (see [21] or [46] for discussions of almost reflexivity).

Theorem 4.9. *Let* $\mathcal{B}$ *be any class of almost reflexive Banach spaces and suppose* $E \in \mathcal{O}(\mathcal{B})$. *If* E *is a Fréchet space (more particularly, if* E *is a Banach space) or if* E *is a strict (LF)-space, then* E *is almost reflexive.*

Proof. A simple application of the diagonal procedure along with the well-known duality between products and direct sums of LCS's yields that the countable product of almost reflexive LCS's is almost reflexive. As was noted in [21] any quotient of an almost reflexive *Banach space* (again the example cited after Theorem 4.2 shows necessity of Banach spaces for the verity of the statement of the present theorem) is again almost reflexive. Clearly, any subspace of an almost reflexive space is almost reflexive. Thus in the case that E is a Fréchet space (in fact, if E is just a metrizable LCS) in $\mathcal{O}(\mathcal{B})$, then Theorem 4.1 and the above comments yield that E is almost reflexive.

If E is a strict (LF)-space in $\mathcal{O}(\mathcal{B})$, then supposing E is the strict inductive limit of the Fréchet spaces E_n $(n \in N)$, we have that each E_n is a subspace of E, so each $E_n \in \mathcal{O}(\mathcal{B})$. From the preceding paragraph, each E_n is almost reflexive. But a bounded subset B of E is by [56] contained in some E_{n_0} and is bounded therein. The rest of the proof is now clear.

Theorem 4.10. *Let* $\mathcal{B}$ *be a class of almost reflexive Banach spaces. Then any Fréchet space or strict (LF)-space which is weakly sequentially complete and is in* $\mathcal{O}(\mathcal{B})$ *is reflexive.*

Proof. Eberlein's theorem applies to Fréchet spaces and strict (LF)-spaces to yield equivalence in such spaces of the notions of (relative) weak compactness

and weak sequential compactness. But clearly almost reflexivity plus weak sequential completeness of an LCS yields the weak sequential compactness of bounded sets. Thus, by the barrelled nature of Fréchet spaces and strict (LF)-spaces, almost reflexive such spaces are reflexive. Applying Theorem 4.9 we get the assertion of Theorem 4.10.

The best known nonreflexive, weakly sequentially complete Banach spaces are the infinite-dimensional $L_1(\mu)$-spaces. Consequently,

Corollary 4.11. *For any nontrivial measure μ, we have $L_1(\mu) \notin \mathcal{O}(c_0)$; more generally, for any dispersed (containing no perfect subsets) compact Hausdorff topological space Ω, $L_1(\mu) \notin \mathcal{O}(C(\Omega))$.*

Corollary 4.12. *Let Ω be a dispersed compact Hausdorff topological space and let Γ be a nondispersed compact Hausdorff topological space. Then $C(\Gamma) \notin \mathcal{O}(C(\Omega))$.*

Proof. If $C(\Gamma) \in \mathcal{O}(C(\Omega))$, then by the results of [46], $L_1 \in \mathcal{O}(C(\Omega))$ which contradicts Corollary 4.11.

Remark. An important example of $C(\Gamma)$ spaces for Γ a nondispersed, compact Hausdorff topological space is any $L_\infty(\mu)$-space where μ is a nontrivial localizable measure (see [44] or [46]).

Theorem 4.13. $\mathcal{O}(c_0)$ *contains no infinite-dimensional weakly sequentially complete Banach space; in particular, no infinite-dimensional reflexive Banach space is in $\mathcal{O}(c_0)$.*

Proof. If E is a weakly sequentially complete Banach space in $\mathcal{O}(c_0)$, then E is, by Theorem 4.9, reflexive. As c_0 is isomorphic to its own square, $E \in QS(c_0)$. Let T be a continuous linear operator from a subspace S_0 of c_0 onto E. By the reflexivity of E, T is weakly compact. By [12, p. 171], T is compact. Thus E is finite-dimensional.

Thus we see that every infinite-dimensional Banach space in $\mathcal{O}(c_0)$ is almost reflexive but nonreflexive.

Corollary 4.14. *For any $1 \leq p \leq \infty$, $l_p \notin \mathcal{O}(c_0)$.*

Theorem 4.15. *Let $\mathcal{F}$ be a class of Fréchet spaces each of density character $\leq m$ (m some infinite cardinal). Then any Fréchet space or strict (LF)-space in $\mathcal{O}(\mathcal{F})$ also has density character $\leq m$.*

Proof. An easy check of $P(\mathcal{F})$, $QP(\mathcal{F})$, $P_cQP(\mathcal{F})$ and $SP_cQP(\mathcal{F})$ shows that any metrizable space in $\mathcal{O}(\mathcal{F})$ has density character $\leq m$; moreover, any strict (LF)-space defined by a sequence of Fréchet spaces each of density character $\leq m$ is itself of density character $\leq m$, a fact easily verified. The proof is therefore complete.

Corollary 4.16. *Let $\mathcal{F}$ be a class of separable Fréchet spaces and $E \in \mho(\mathcal{F})$. Then, if E is a Fréchet space or a strict (LF)-space, E is separable.*

Corollary 4.17. $\mho(l_1) \subsetneqq \mho(l_\infty)$.

Proof. We have already noted that $\mho(l_1) \subseteq \mho(l_\infty)$; however, l_∞ is not separable (see [14]) while l_1 is—thus Corollary 4.16 applies.

Corollary 4.18. *Let S be any infinite set and $P(S)$ be the power set of S. Then $\mho(l_1(P(S))) = \mho(l_\infty(S))$.*

Proof. As is well known, the density character of $l_\infty(S)$ is 2^m, where m is the cardinality of S. Thus $l_\infty(S)$ is a quotient of $l_1(P(S))$, and $l_\infty(S) \in \mho(l_1(P(S)))$.

On the other hand, $l_1(S)$ has dual $l_\infty(S)$, so [44, Proposition 3.3], $l_\infty(S)$ contains $l_1(P(S))$. Thus, $l_1(P(S)) \in \mho(l_\infty(S))$ obtains.

Theorem 4.19. *Let $\mathcal{B}$ be any class of Banach spaces each of which has a separable dual space; then every Banach space in $\mho(\mathcal{B})$ has a separable dual space.*

Proof. Another—by now—routine application of Theorem 4.1.

In particular, we have from Theorem 4.19 an alternative derivation of

Corollary 4.20. *If $1 < p < \infty$, then none of the spaces l_1, L_1, $C([0, 1])$ is in any of the varieties $\mho(c_0)$, $\mho(l_p)$, $\mho(L_p)$.*

In the same fashion as before we have

Theorem 4.21. *Let $\mathcal{B}$ be a class of quasi-reflexive Banach spaces (see [8] for definition and basic properties). Then every Banach space in $\mho(\mathcal{B})$ is quasi-reflexive.*

Remark. If B is a quasi-reflexive Banach space then for each n, $B^{(2n)}$, the $(2n)$th dual space of B, is in $\mho(\mathcal{B})$. How close does this phenomenon come to characterizing quasi-reflexivity?

5. Miscellaneous results and open questions. The problem of embedding a topological space in an LCS has been considered by several authors notably Markov [24], Kakutani [17], Arens and Eells [1] and Michael [27]. In particular, they showed

(i) Every Tychonoff (completely regular) space can be embedded in an LCS as a closed subset and a Hamel basis.

(ii) Every metric space can be embedded isometrically as a closed linearly independent set in a normed linear space.

Definition. Let X be a topological space and $\mho$ a variety. Then an LCS in $\mho$ is said to be a free locally convex space of $\mho$ on X, denoted by $F(X, \mho)$, if there is a continuous mapping ν of X into $F(X, \mho)$ such that, for every continuous

mapping ϕ of X into any LCS E in $\mho$, there exists a continuous linear transformation Φ of $F(X, \mho)$ into E such that $\Phi\nu = \phi$.

The usual Freyd adjoint functor theorem argument ([17], [29]) shows that: for any topological space X and variety $\mho$, $F(X, \mho)$ exists and is unique.

However, we can say more.

Theorem 5.1. *For any Tychonoff space X and (nontrivial) variety $\mho$,*

(i) *the canonical mapping ν is a homeomorphism of X onto $\nu(X)$ (so we will identify X with $\nu(X)$),*

(ii) *X is a Hamel basis for $F(X, \mho)$, and*

(iii) *X is a closed subset of $F(X, \mho)$.*

Proof. Since X is a Tychonoff space it can be embedded in a product R^I of copies of R. As $R^I \in \mho$, there exists a continuous linear transformation Φ of $F(X, \mho)$ into R^I such that $\Phi\nu$ acts identically on X. This clearly implies that ν is a homeomorphism of X onto $\nu(X)$. (We identify X with $\nu(X)$ in the sequel.)

Let $x_1, \cdots, x_n$ be in X and $\alpha_1, \cdots, \alpha_n$ in R such that the element of $F(X, \mho)$, $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$. Let $\{e_1, \cdots, e_n\}$ be a Hamel basis for R^n. Since X is Tychonoff, there exists a continuous mapping ϕ of X into R^n such that $\phi(x_i) = e_i$, $i = 1, \cdots, n$. As $R^n \in \mho$, there exists a continuous linear transformation Φ of $F(X, \mho)$ into R^n such that $\Phi|X = \phi$. Thus $\Phi(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_1 e_1 + \cdots + \alpha_n e_n$. Since $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$, we have $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$ which implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Hence X is a Hamel basis for $F(X, \mho)$.

Let $\beta(X)$ be the Stone-Čech compactification of X. Then $F(\beta(X), \mho)$ exists and, by the above remarks, has $\beta(X)$ as a basis. Let ϕ be the natural mapping of X into $\beta(X)$. Then there exists a continuous linear transformation Φ of $F(X, \mho)$ into $F(\beta(X), \mho)$ such that $\Phi|X = \phi$. Clearly, $\Phi^{-1}(\beta(X)) = X$ and since $\beta(X)$ is a closed subset of $F(\beta(X), \mho)$ we have that X is closed in $F(X, \mho)$.

Unfortunately we have, as yet, found no "use" for free locally convex spaces. However, we point out that free groups are important in the study of varieties of groups, and free topological groups, at least, provide a source of interesting examples ([36], [24]). We conclude our comments on free locally convex spaces by stating a few facts which can be proved in the same way as those for free topological groups.

1. If X is any LCS is a variety $\mho$, then X is a quotient of $F(X, \mho)$.

2. If X is any nondiscrete Tychonoff space then $F(X, \mho)$ is not metriazble.

3. There exist nonhomeomorphic spaces X and Y such that $F(X, \mho)$ is isomorphic to $F(Y, \mho)$.

4. If φ_m is in a variety $\mho$, then φ_m is $F(X, \mho)$ for a discrete space X.

Noting [39] that every variety of groups is generated by its finitely generated groups, it is natural to ask if every variety of LCS's is generated by its finite-

dimensional members. Of course, this is not true. A more likely analogue is that every variety of LCS's is generated by its compactly generated members. (An LCS is said to be compactly generated if it has a compact spanning set.) However, this too is *false*. To see this, suppose the variety $\mathcal{O}$ of all LCS's is generated by the class $\mathcal{C}$ of all compactly-generated LCS's. (Incidently, $\mathcal{O}(\mathcal{C})$ is not singly-generated.) Let D be a set of cardinality 2^c. Then by Theorem 4.1, $l_1(D) \in SQP(\mathcal{C}) = S(\mathcal{C})$. Noting that any discrete *subgroup* of a compactly-generated LCS is countable while the unit vectors in $l_1(D)$ generate an uncountable discrete subgroup of $l_1(D)$ (cf. [38]), we have a contradiction.

Related to the opening remarks of §3 we comment that

(a) Using the terminology and results of [49] one can show [55] that no infinite-dimensional Schwartz space has a normalized *e*-Schauder basis; in particular, no barrelled Schwartz space has a normalized Schauder basis.

(b) Let E be a Fréchet space with Schauder basis $\{x_i\}$. The basis is said to be *normal* if for some sequence $\{a_i\}$ of positive scalars, $\{a_i x_i\}$ is a normalized basis for E. The basis is *abnormal* if it is not normal [18]. In [55] the following are shown to be equivalent:

(i) some subbasis of $\{x_i\}$ has all its subbases abnormal;

(ii) some subbasis of $\{x_i\}$ spans a nuclear space;

(iii) some subbasis of $\{x_i\}$ spans a Schwartz space;

(iv) some subbasis of $\{x_i\}$ spans a Montel space.

(In (ii)–(iv) we are considering the *closed* linear span of *infinite* subbases.)

(c) In [5] it is shown that every nonnormable infinite-dimensional Fréchet space F contains an infinite-dimensional nuclear space F_0; indeed, one sees that if F does not have its weak topology and does not contain an infinite-dimensional Banach space, then the nuclear subspace F_0 may be chosen so as not to have its weak topology. Thus we have

If F is a Fréchet space, then either it has its weak-topology (and $\mathcal{O}(F) = \mathcal{O}(R)$) or $\mathcal{O}(F) \cap \mathcal{N} \neq \mathcal{O}(R)$. Indeed, in the latter case, $\mathcal{O}(F) \cap \mathcal{N} \neq \mathcal{O}(\varphi)$, since each LCS in $\mathcal{O}(\varphi)$ either has its weak topology or contains φ and φ is nonmetrizable (cf. Theorem 3.1).

This prompts us to ask

Question 1. What are the subvarieties of $\mathcal{N}$?

We have already mentioned $\mathcal{O}(R)$, $\mathcal{O}(\varphi)$, $\mathcal{O}(s)$ and the *s*-nuclear spaces. The works [11] and [48] are relevant. As a corollary to Theorem 3.2 of [18] and our Theorem 3.2, we have

If E is a complete barrelled space with a symmetric Schauder basis, then

(i) $\mathcal{O}(E) = \mathcal{O}(\omega) = \mathcal{O}(R)$,

(ii) $\mathcal{O}(E) = \mathcal{O}(\varphi)$, *or*

(iii) $\mathcal{O}(E) \supseteq \mathcal{N}$.

We have proved the existence of a universal generator for the Schwartz spaces.

But we ask

Question 2. Does there exist a *serviceable* universal generator for the Schwartz spaces? (What is it?)

We have noted that $\mathcal{O}(l_1)$ contains all the Schwartz spaces, so we ask

Question 3. For what Banach spaces B is it true that $\mathcal{O}(B)$ contains all the Schwartz spaces? In particular, does $\mathcal{O}(l_p)$, $1 < p < \infty$, or $\mathcal{O}(c_0)$ contain all the Schwartz spaces?

Related to the above question is

Question 4. Are the only Banach spaces in $\mathcal{O}(l_p) \cap \mathcal{O}(l_q)$, $1 < p \neq q < \infty$, the finite-dimensional ones?

Question 5. If $1 < p \neq q < \infty$, for what r does $l_r \in \mathcal{O}(\{l_p, l_q\})$?

As was noted in §1, if $\mathcal{O}$ is a variety generated by a class of Fréchet spaces, then $\mathcal{O}$ is closed under the formation of completions; notice that $\mathcal{O}(\varphi)$ is likewise closed under completions but is not generated by Fréchet spaces. Therefore we ask

Question 6. Is there an internal characterization of varieties closed under the formation of completions?

In the same spirit is

Question 7. Is there an internal characterization of varieties closed under the formation of direct sums, inductive limits or tensor products (or other operations of interest)?

If $1 \leq p < \infty$ and μ is a nonatomic measure, then $l_2 \in \mathcal{O}(L_p(\mu))$; also as $L_p(\mu, l_2)$, the space of Bochner pth power integrable l_2-valued functions, is isomorphic to $L_p(\mu)$ (see [22]), we have $L_p(\mu, l_2) \in \mathcal{O}(L_p(\mu))$. This motivates

Question 8. Let $1 \leq p < \infty$ and μ be any nonatomic measure. Is it true that if E is a Banach space in $\mathcal{O}(L_p(\mu))$ then $L_p(\mu, E) \in \mathcal{O}(L_p(\mu))$?

The Lindenstrauss-Pełczyński [22] paper introduces the $\mathcal{L}_p$-spaces which characterize geometrically the complemented subspaces of $L_p(\mu)$ for $1 < p < \infty$. Of course the uncomplemented subspaces are harder to handle.

Question 9. What geometric properties characterize the Banach spaces in $\mathcal{O}(L_p(\mu))$, $1 < p < \infty$?

Rosenthal's proof [50] that every operator from a subspace of l_p to l_q, $1 \leq q < p < \infty$, is compact is dependent upon the nature of l_p spaces; it does not (immediately) generalize to Orlicz sequence spaces. It seems possible, however, that the result (Theorem 4.6) still holds for reflexive Orlicz spaces with the proper relationship between them.

Question 10. What can be said of the varieties generated by Orlicz spaces (with generating functions satisfying the usual growth conditions)?

Question 11. Let l^Φ, l^Ψ be distinct Orlicz sequence spaces. Are they (necessarily) of incomparable linear dimension?

Of course a classical result of Banach [2] states that the different l_p spaces $(1 \leq p < \infty)$ are of incomparable linear dimension.

Question 12. Let $\mathcal{F}$ denote the class of all separable reflexive Fréchet spaces and $\mathcal{B}$ the class of all Banach spaces in $\mathcal{F}$. Does there exist

(a) $E \in \mathcal{F}$ such that $\mathcal{F} \subseteq \mathcal{O}(E)$;

(b) $E \in \mathcal{B}$ such that $\mathcal{F} \subseteq \mathcal{O}(E)$;

(c) $E \in \mathcal{F}$ such that $\mathcal{B} \subseteq \mathcal{O}(E)$;

(d) $E \in \mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{O}(E)$?

Szlenk [58] showed that there does not exist $E \in \mathcal{B}$ such that every $B \in \mathcal{B}$ is isomorphic to a subspace (or a quotient) of E. A negative answer to (d) would result if one could show the nonexistence of $E \in \mathcal{B}$ such that every $B \in \mathcal{B}$ is the continuous image of some closed subspace of E.

Using the universality of $C[0, 1]$ for the class of separable Banach spaces and the incomparability of l_p for $1 < p < \infty$, it follows that there exists precisely a continuum of nonisomorphic members of $\mathcal{B}$, say $\{X_i : i \in R\}$. Then keeping the notation of [9], $P_{l_2(R)}(X_i)$ is a reflexive Banach space which has density character c, and clearly contains every member of $\mathcal{B}$ isomorphically. Thus by passing to a "slightly higher" density character we can get a reflexive Banach space which by Szlenk's result and the continuum hypothesis is of minimal density character to be universal for the separable reflexive Banach spaces.

In another direction, $C(-\infty, \infty)$ has been shown by Mazur and Orlicz [26] to be universal for the class of all separable Fréchet spaces; here $C(-\infty, \infty)$ denotes the linear space of all continuous real-valued functions of a real variable with topology that of uniform convergence on compact sets. Consequently, there exists precisely a continuum of nonisomorphic members of $\mathcal{F}$, denoted by $\{Y_i : i \in R\}$. Consider $Y = \Pi_{i \in R} Y_i$. Y is a separable [47] reflexive [14] LCS which clearly contains every member of $\mathcal{B}$. (The result of [47] applies as a consequence of the Anderson-Kadec theorem [4].)

We know that every LCS of density character $\leq \aleph_0$ is in $\mathcal{O}(C[0, 1])$.

Question 13. Given an infinite cardinal m, does there exist a compact Hausdorff topological space K_m such that the density character of $C(K_m)$ is m and every Banach space (or LCS) of density character $\leq m$ is in $\mathcal{O}(C(K_m))$?

Question 14. Let K be a dispersed compact Hausdorff space. Does every (separable) subspace of $C(K)$ possess the strict Dunford-Pettis property of [12]?

If so, then $\mathcal{O}(C(K))$ contains no infinite-dimensional reflexive Banach spaces. (Apply the results of [12] and [46].)

We digress momentarily to varieties of topological groups. In [6] it was shown that the variety of all topological groups is not generated by the class $\mathcal{C}$ of locally compact groups. This was done by showing that certain linear topological spaces, regarded as topological groups, are not in the variety of topological groups

$V(\mathcal{C})$ generated by $\mathcal{C}$. We point out here that $V(\mathcal{C})$ contains no infinite-dimensional Banach spaces (regarded as topological groups). This is easily seen using Theorem 4 of [6] and the following generalization of Theorem 4.1:

If the Banach space B (regarded as a topological group) is a subgroup of a product $\prod_{i \in I} A_i$ of topological groups A_i, then B is isomorphic to a subgroup of a product $\prod_{i \in \sigma} A_i$, where σ is a *finite* subset of I.

Further comments on this are made in [38].

Returning to topological vector spaces, we note that up to now we have only considered varieties of *locally convex* spaces. A recent result of Peck [41] indicates that varieties of general topological vector spaces have a rich structure and often contain many locally convex subvarieties. This prompts the question: What can be said about varieties of (Hausdorff) topological vector spaces?

Of course many of our results on locally convex varieties carry over to the non-locally convex case, in particular, those relating to generation of varieties and singly generated varieties do (see Theorems 1.1, 1.2, 1.4, 2.8 and 2.11).

We will need the following result which extends Theorem 4.1:

Let $\mathcal{C}$ be a class of (Hausdorff) topological vector spaces. If E is a locally bounded topological vector space in $\mathcal{O}(\mathcal{C})$, then $E \in QSP(\mathcal{C})$. (Recall that a topological vector space is called locally bounded if it has a bounded neighborhood of zero.)

Of particular interest in our present discussion is consideration of some of the classical nonlocally convex topological vector spaces: l_p and L_p for $0 < p < 1$. In analogy with the case $p = 1, 2, \infty$ but in contradistinction to other $p \geq 1$, we have

Theorem 5.2. *For* $0 < p < 1$, $\mathcal{O}(l_p) = \mathcal{O}(L_p)$.

Proof. As in the case $p \geq 1$, l_p can be isomorphically embedded in L_p, so $\mathcal{O}(l_p) \supseteq \mathcal{O}(L_p)$. On the other hand, as shown in [57], L_p is a quotient of l_p, so $\mathcal{O}(L_p) \subseteq \mathcal{O}(l_p)$.

Theorem 5.3. *If* $0 < p < q < 1$, *then* $\mathcal{O}(l_q) \subsetneqq \mathcal{O}(l_p)$.

Proof. That $\mathcal{O}(l_q) \subseteq \mathcal{O}(l_p)$ is clear from [57]. Recall that if E is a locally bounded topological vector space and U is a bounded circled neighborhood of 0 in E, then $2U \subseteq U + U \subseteq kU$ for some $k \geq 2$. The greatest lower bound of the k's for which the above is true is denoted by k_U and called the modulus of concavity of U. The modulus of concavity of E is denoted by $\mathcal{K}(E)$ and is defined to be $\inf\{k_U\}$ as U ranges over all bounded circled neighborhoods of E. ($\mathcal{K}(E) \geq 2$.)

Observe that if $F \in S(E)$ or $Q(E)$, then $\mathcal{K}(F) \leq \mathcal{K}(E)$.

Consider the possibility $L_p \in \mathcal{O}(L_q)$ where $0 < p < q < 1$. Then, as L_q is isomorphic to its own square and is locally bounded, $L_p \in SQ(L_q)$. This would imply

that $K(L_p) \leq K(L_q)$ which is false since $K(L_p) = 2^{1/p} > 2^{1/q} = K(L_q)$ (using Rolewicz [50]). Thus using Theorem 5.1, $\mathcal{O}(l_q) \not\subseteq \mathcal{O}(l_p)$.

Another important example of a nonlocally convex topological vector space is the space S of all real-valued Lebesgue measurable functions on the interval [0, 1] with topology (a complete, metric linear topology) that of convergence in measure. As is noted in [57], S has L_p as a subspace for $1 < p \leq 2$; of course S has no locally convex quotient spaces. We ask

Question 15. For what p is L_p in $\mathcal{O}(S)$?

Addendum. Much progress has been made on answering the questions posed in §5. For an account of this progress, we refer the reader to the paper by the first two authors entitled *Some remarks on varieties of locally convex linear topological spaces*, which should appear in the near future.

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