

HOMOLOGICAL DIMENSIONS OF STABLE HOMOTOPY MODULES AND THEIR GEOMETRIC CHARACTERIZATIONS⁽¹⁾

BY

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ABSTRACT. Projective dimensions of modules over the stable homotopy ring are shown to be either 0, 1 or ∞ ; weak dimensions are shown to be 0 or ∞ . Also geometric characterizations are obtained for projective dimensions 0, 1 and weak dimension 0. The geometric characterizations are interesting; for projective modules they are about the cohomology of geometric realization; while for flat modules they are about homology. This shows that the algebraic duality between "projective" and "flat" is strongly connected with the topological duality between "cohomology" and "homology". Finally, all the homological numerical invariants of the stable homotopy ring—the so-called finitistic dimensions—are completely computed except the one on injective dimension.

1. Introduction and statements of results.

1.1 Introduction. In [7] we investigated the homological dimension of a π_* -module of finitely generated type, where π_* is the stable homotopy ring. In this paper we extend our computations to general π_* -modules. Namely, we find that the projective dimension of a π_* -module is 0, 1 or ∞ ; and its weak dimension is 0 or ∞ . Combining these computations with those of [7], we obtain the finitistic dimensions of π_* , except the injective dimension. Moreover, we obtain geometric characterizations of π_* -modules of projective dimensions 0, 1 and weak dimension 0. These characterizations are interesting: For projective modules they are about the cohomology groups of geometric realization; while for flat modules they are about the homology. Heuristically "projective" and "flat" are dual notions in homological algebra, while "cohomology" and "homology" are dual in topology. These characterizations show that these two heuristic

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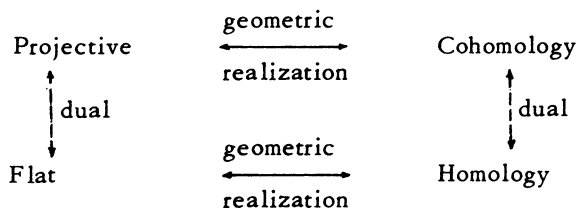
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notions are, surprisingly, well fitted to each other. Schematically



1.2 Main results. Let $\mathfrak{M}$ be, as in [7], the category of π_* -modules bounded from below [7, Definition 2.3]; one can, without loss of generality, regard $\mathfrak{M}$ as the category of nonnegatively graded π_* -modules. As in [7], our spectra will be assumed to be connected, pointed CW-spectra [3]. We will not distinguish between a map and its homotopy class, nor between a spectrum and its homotopy type. A π_* -module which is realizable by some spectrum, say X , is called a stable homotopy module and is denoted by $\pi_*(X)$.

5.7 Theorem (Projective dimension theorem). *Let M be a π_* -module, then*

- (i) *The projective dimension $\text{P d}_{\pi_*} M$ of M is either 0, 1 or ∞ .*
- (ii) *$\text{P d}_{\pi_*} M = 0$ iff there is a wedge R of spheres such that $\pi_*(R) \simeq M$.*
- (iii) *$\text{P d}_{\pi_*} M = 1$ iff there is a spectrum L with trivial homology operations, which is not a wedge of spheres, such that $\pi_*(L) \simeq M$.*

5.9 Theorem (Weak dimension theorem). *Let M be a π_* -module, then*

- (i) *The weak dimension $\text{W d}_{\pi_*} M$ of M is either 0 or ∞ .*
- (ii) *$\text{W d}_{\pi_*} M = 0$ iff there is a spectrum L with trivial homology operations such that $\pi_*(L) \simeq M$.*

5.11 Theorem (Duality theorem). *Let M be a π_* -module, then*

- (i) *M is projective iff there is a spectrum R with trivial cohomology operations such that $\pi_*(R) \simeq M$.*
- (ii) *M is flat iff there is a spectrum L with trivial homology operations such that $\pi_*(L) \simeq M$.*

5.12 Theorem. *If M is a π_* -module, then the following conditions are equivalent.*

- (i) *M is a flat (but not free) π_* -module.*
- (ii) *$\text{P d}_{\pi_*} M = 1$.*
- (iii) *M is π_* -realizable by a wedge $\bigvee L(G_n, n)$ of Moore spectra, where each G_n is torsion free, but at least one G_n is not free.*
- (iv) *M is realizable by a spectrum with trivial homology operations, which is not a wedge of spheres.*

In homological algebra, for a given ring A , there are various dimensions—the

so-called finitistic dimensions—attached to it ([2] or [7]). These dimensions are

$$G D(A) \geq F P D(A) \geq fl P D(A) \geq f P D(A),$$

$$F I D(A) \geq F W D(A) \geq fl W D(A) \geq f W D(A).$$

(Here we assume that A is commutative in the graded sense.) For the stable homotopy ring π_* , we have completely determined all these dimensions, except $F I D(\pi_*)$. Namely,

5.13 Theorem.

$$G D(\pi_*) = \infty > F P D(\pi_*) = 1 > fl P D(\pi_*) = f P D(\pi_*) = 0,$$

$$F I D(\pi_*) \geq 1 > F W D(\pi_*) = fl W D(\pi_*) = f W D(\pi_*) = 0.$$

This paper is a ring theoretical report on the stable homotopy ring of spheres, thus most of the works are algebraic. Theorems 5.7(i), 5.9(i), 5.12(i), (ii) are purely algebraic. They are essentially based only on Milnor-Kervaire's computation of $Im J$, and hold good even for a more general ring (§4). One interesting "feed back" to geometry of these algebraic theorems is that they allow us to realize the flat π_* -modules, even the modules unbounded below (see [10]). Thus it may provide us the possibilities to study the flat modules (including those unbounded below) by geometric means. However one should note that at the present stage, we do not yet have effective means to deal with the nonconnected spectra.⁽²⁾

The paper is organized as follows: §1 is introduction and statement of main results. In §2 we define the higher order homology and cohomology operation in a very general setting. In §3 we characterize the stable spheres in terms of homology operations. In §4 we compute the projective and weak dimensions of general π_* -modules. In §5 we give the proofs to the main theorems stated in this section. We would suggest that the reader, after §1, go directly to §5 and refer to other sections when necessary.

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2. Higher order homology and cohomology operations. In this section we will define the higher order stable homology and cohomology operations in a very general setting. Roughly speaking, a primary homology (cohomology) operation is a natural transformation of homology functors, and a secondary operation is a natural transformation defined on the kernel of a primary operation with values in the cokernel of primary operations; by proceeding in this way, one can define the

⁽²⁾ Boardman, Eilenberg and Moore are working on "spectral sequences of whole plane" which probably will be a good machinery.

operation of any higher order. We will formulate these ideas in terms of spectral sequences, namely, the first differentials will be the primary operations and the second differentials will be the secondary operations, etc. In order to formulate these ideas more precisely we need some notations for spectral sequences.

The notion of spectral sequences that we need is slightly more general than that in standard texts. Let us recall that a bigraded abelian group E is an indexed collection of abelian groups $E_{s,t}$ for every pair of integers s and t . A differential $d: E \rightarrow E$ of bidegree $(-r, r-1)$ is a collection of homomorphisms $d: E_{s,t} \rightarrow E_{s-r, t+r-1}$, for all s and t , such that $d^2 = 0$. The homology group $H(E)$ is the bigraded group defined by

$$H_{s,t}(E) = [\text{Ker}(d: E_{s,t} \rightarrow E_{s-r, t+r-1})] / d(E_{s+r, t-r+1}).$$

2.1 Definition. A homology spectral sequence is a sequence $\{E^{r,i}, d^{r,i}\}$ such that (a) r_i is an increasing sequence of integers. (Here is the point where we deviate from standard treatment.) (b) $E^{r,i}$ are bigraded abelian groups and $d^{r,i}$ is a differential of degree $(-r_i, r_i-1)$ on $E^{r,i}$. (c) There is a given isomorphism $H(E^{r_i,i}) \simeq E^{r_{i+1},i+1}$.

Note that the differential in the homology spectral sequence is lowering the degree of the first coordinate. There are spectral sequences which raise the degree of the first coordinate. They are called cohomology spectral sequences and are defined similarly. We left the details to the reader.

A homomorphism of spectral sequences is defined as usual and their compositions are obviously still homomorphisms. Thus the collection of all spectral sequences is a category. The category of homology spectral sequences will be denoted by $\mathfrak{E}$.

With these notations we can now define the homology operations. Let Σ be the category of CW-complexes or more general CW-spectra. For convenience we will use Boardman's CW-spectra [3]; and by his formulation one can regard CW-complexes as some connected spectra.

2.2 Definition. Let $\mathfrak{E}$ be the category of homology spectral sequences. Then a covariant functor $T: \Sigma \rightarrow \mathfrak{E}$ is called a Total Homology Operation if the first term $E_{**}^{r_1}$ of the spectral sequence $T(X) = \{E^{r,i}, d^{r,i}\}$ is the ordinary homology group, that is,

$$E_{p,q}^{r_1} = H_p(X, G_q)$$

where each G_q is an abelian group. The differential d^{r_1} is called a Primary Operation and $d^{r,i}$ is called a Homology Operation of the i th order.

Similarly, if we take $\bar{\mathfrak{E}}$ to be the category of cohomology spectral sequence and T to be a contravariant functor, then we get Total Cohomology Operation and cohomology operations of various orders.

Example 1. Let $0 \rightarrow Z \rightarrow Z \rightarrow Z_2 \rightarrow 0$ be an exact sequence. Then there is an exact couple

$$\begin{array}{ccc} H_*(X, Z) & \longrightarrow & H_*(X, Z) \\ & \searrow & \swarrow \\ & H_*(X, Z_2) & \end{array}$$

which gives rise to the Bockstein spectral sequence $B(X)$. Then the functor $T: \Sigma \rightarrow \mathfrak{E}$ defined by $T(X) = B(X)$ is a total homology operation and its differentials are just the usual Bockstein primary and higher order operations.

Example 2. Most common operations arise from a Postnikov system. Let

$$(1) \quad \begin{array}{ccccccc} E_0 & \longleftarrow & E_1 & \longleftarrow & \cdots & \longleftarrow & E_n & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ K_0 & & K_1 & & & & E_n & & \end{array}$$

be a Postnikov system, where K_i is an Eilenberg-Mac Lane spectrum. Then the functor T , which assigns to each $X \in \Sigma$ the spectral sequence of the following exact couple

$$\begin{array}{ccc} \sum_n \pi_*(X \wedge E_n) & \longrightarrow & \sum_n \pi_*(X \wedge E_{n+1}) \\ & \searrow & \swarrow \\ & \sum_n \pi_*(X \wedge K_n) = \sum_n H_*(X) & \end{array}$$

is a total homology operation, and its various differentials are various higher order homology operations.

2.3 Definition. Let $d^{f,i}$ be the i th order homology operation of the total homology operation T . Then the i th order homology operation $d^{f,i}$ is said to be trivial on spectrum X if the i th differential of the spectral sequence $T(X)$ is zero. The total homology operation T is trivial on spectrum X if all the differentials of $T(X)$ are zero. An i th order homology operation $d^{f,i}$ is of degree $\geq n$ if $d^{f,i}$ sends the homology class of dimension s to that of dimension $\leq s - n$. The total homology operation T is of degree $\geq n$ if each $d^{f,i}$ is of degree $\geq n$. Similarly we can define the degree for cohomology operations.

A spectrum X is called a homology (cohomology) operational trivial spectrum abbreviated as HOTS (COTS), if all the possible total homology (cohomology) operations of degree ≥ 1 are trivial on the spectrum.

If the Postnikov system of Example 2 arises from a Postnikov decomposition of a spectrum, then the total homology operations of Example 2 are trivial on the 0-sphere S^0 (by dimension reason). In fact S^0 is a HOTS (COTS).

Next we would like to investigate the question of when a "subspectrum" or "quotient spectrum" of a HOTS is itself a HOTS. We have the following proposition.

2.4 Proposition. *Let X and Y be spectra and suppose there are maps between X and Y such that, for every G , either*

$$0 \rightarrow H_*(X; G) \rightarrow H_*(Y; G)$$

or

$$H_*(Y; G) \rightarrow H_*(X; G) \rightarrow 0$$

is exact. Then if Y is a HOTS, X is also a HOTS.

Proof. Let T be a total homology operation; then one of the lowest order nonzero differentials on $T(X)$ (which is defined on all $H_*(X)$) certainly implies that the corresponding differentials are nonzero on $T(Y)$; a contradiction. Q.E.D.

Next we will investigate the duality between higher order homology and cohomology operations.

Let $\bar{Q}$ be the rational number modulo one. Let $\bar{G} = \text{Hom}_Z(G, \bar{Q})$ be all the Z -homomorphisms of G to $\bar{Q}$, where Z is, as usual, the integers. Let $E = \{E_{p,q}^r, d_{p,q}^r\}$ be a homology spectral sequence. Then

$$\bar{E} = \text{Hom}_Z(E, \bar{Q}) \equiv \{\bar{E}_{r_i}^{p,q}, \bar{d}_{r_i}^{p,q}\}$$

defined by

$$\bar{E}_{r_i}^{p,q} = \text{Hom}_Z(E_{p,q}^r, \bar{Q}) \quad \text{and} \quad \bar{d}_{r_i}^{p,q} = \text{Hom}_Z(d_{p,q}^r, \bar{Q})$$

is a cohomology spectral sequence, because $\text{Hom}_Z(-, \bar{Q})$ is an exact functor.

Let Σ' be the category of connected spectra with finite skeleton. Then we have

2.5 Theorem. *Let $T: \Sigma' \rightarrow \mathfrak{E}$ be a total homology (resp. cohomology) operation. Then there is a total cohomology (resp. homology) operation $\bar{T}: \Sigma' \rightarrow \bar{\mathfrak{E}}$ satisfying the following condition: The i th order cohomology operation $\bar{d}_{r_i}^i$ (= the i th differential of $\bar{T}$) is trivial iff i th order homology operation $d_{r_i}^i$ (= the i th differential of T) is trivial.*

Proof. We will only give the proof for the homology operation T , since the case for cohomology can be obtained by verbatim change. Let $T(X) = \{E_{p,q}^r, d^r\}$, and define $\bar{T}$ as above. Then as remarked above $\bar{T}(X)$ is a cohomology spectral sequence. Thus $\bar{T}$ is a functor $\Sigma' \rightarrow \bar{\mathfrak{E}}$. Moreover, note that since $E_{*,*}^{r_1} = H_*(X, G_*)$, we have

$$E_{r_1}^{*,*} = \text{Hom}_Z(H_*(X; G_*), \bar{Q}) = H^*(X; \bar{G}_*)$$

where the last equality is obtained from [4, p. 120, Proposition 5.1 when T is a homology operation, or Proposition 5.3 when T is a cohomology operation].

Thus $\bar{T}$ is a cohomology operation. Also note that it follows immediately from $\bar{d}_{r_i} = \text{Hom}_Z(d^{r_i}, \bar{Q})$ that $\bar{d}_{r_i} = 0$ iff $d^{r_i} = 0$. This completes the proof.

3. Characterization of stable spheres. In this section we will characterize the stable spheres in terms of homology or cohomology operations. From now on we will assume all the operations are of degree ≥ 1 , for example we will not regard the coefficient homomorphism as a cohomology operation.

The following theorem is more or less a folk theorem (e.g. see [8]).

3.1 Theorem. *Let X be a connected spectrum. Then X is a wedge of spheres iff all the cohomology operations of degree ≥ 1 on X are trivial.*

The dual version is slightly different, due to the incompleteness of the duality, namely,

3.2 Theorem. *Let X be a connected spectrum. Then X is a wedge of spheres iff (i) $H_*(X; Z)$ is a free abelian group, and (ii) all the homology operations of degree ≥ 1 on X are zero.*

Note that (i) is necessary; for example, the Moore spectrum of rational numbers does not admit any nonzero homology operations.

Since the proof is fairly long, perhaps it is desirable to give an outline. First we set up a lemma which establishes the fact that a map between wedges of spheres is detectable by homology operation (in general this may not be true, see [9]). Then we use the co-Postnikov system (or the homology decomposition) to show that the spectrum is obtained by attaching cells successively in a certain "normal" way. Then apply the lemma to show that all these maps are null-homotopic.

3.3 Lemma. *Let R and Y be wedges of spheres and $f: R \rightarrow Y$ be a map which induces a zero map $f_*: H_*(R; G) \rightarrow H_*(Y; G)$ for every G . Then f is null-homotopic iff the mapping cone C_f is a HOTS.*

Proof. The main idea of the proof is first, reduce the discussion of f to its restriction on a finite wedge of spheres. Then, by duality, the problem is reduced to the well-known situation—detecting map by cohomology operations.

The necessity is obvious. To prove the sufficiency, let us assume, to the contrary, that f is not null-homotopic. Since R is a wedge of spheres, there is at least one sphere, say S_0 , in R such that the restriction $\phi = f|_{S_0}$ is not null-homotopic. Then ϕ determines a nonzero element, denoted by ϕ again, in $\pi_*(Y)$. Since Y is a wedge of spheres (and hence $\pi_*(Y)$ is a free π_* -module), there is a finite wedge $Y' \subset Y$ such that ϕ lies in $\pi_*(Y')$, which is a free direct summand of $\pi_*(Y)$. In other words, ϕ can be factored through the finite wedge Y' , that is, there is $\phi': S_0 \rightarrow Y'$ such that the following diagram is commutative

$$\begin{array}{ccc}
 S_0 & \xrightarrow{\phi'} & Y' \\
 & \searrow \phi & \downarrow \\
 & & Y = Y' \vee Y''
 \end{array}$$

where Y'' is the complement of Y' in Y . Obviously the mapping cone $C_{\phi'} = Y' \cup_{\phi'} CS_0$ is a direct summand of the mapping cone $C_\phi = Y \cup_\phi CS_0$, namely,

$$(1) \quad C_\phi = C_{\phi'} \vee Y''.$$

Next, let R'' be the complement of S_0 in R , i.e. $R = S_0 \vee R''$. Then from the geometry we have a mapping cone sequence

$$(2) \quad R'' \xrightarrow{\psi} C_\phi \xrightarrow{j} C_f = Y \cup_f CR$$

where ψ is the composition $R'' \subset R \xrightarrow{f} Y \subset C_\phi$. Note that ψ induces a zero map $\psi_*: H_*(R''; G) \rightarrow H_*(C_\phi; G)$ for every G , since $f_*: H_*(R; G) \rightarrow H_*(Y; G)$ is, by assumption, a zero map for every G . Thus, from (2), we get a monomorphism

$$(3) \quad 0 \longrightarrow H_*(C_\phi; G) \xrightarrow{\phi_*} H_*(C_f; G)$$

for every G .

Now, note that from the assumption that C_f is a HOTS, from (3) above and Proposition 2.4, we conclude that C_ϕ is a HOTS. Also note that since $C_{\phi'}$ is a direct summand of C_ϕ (see (1) above), we can also deduce that $C_{\phi'}$ is a HOTS. Then, by Theorem 2.5, $C_{\phi'}$ is also a COTS (note that $C_{\phi'}$ is finite). Thus, by Peterson's Theorem [12], ϕ' is null-homotopic which is absurd. (Note that $\phi'_*: H_*(S_0; G) \rightarrow H_*(Y; G)$ is zero for every G because ϕ'_* is the restriction of f_* factoring through a direct summand $H_*(Y'; G)$ of $H_*(Y; G)$; and all the cohomology operations are trivial on $C_{\phi'}$.) Thus we conclude that f is null-homotopic.

Proof of Theorem 3.2. Since X is a connected spectrum, there is a co-Postnikov system (or homology decomposition) $\{Y_n\}$ (see [6]):

$$\begin{array}{ccccccc}
 Y_m & \subset & Y_{m+1} & \subset & \cdots & \subset & Y_n \subset Y_{n+1} \subset \cdots; & \bigcup Y_n = Y \simeq X \\
 \uparrow k'_m & & \uparrow k'_{m+1} & & & & \uparrow k'_n & \uparrow k'_{n+1} \\
 R_m & & R_{m+1} & & & & R_n & R_{n+1}
 \end{array}$$

such that

$$(i) \quad H_q(Y_n; Z) = H_q(X; Z), \quad q \leq n;$$

- (ii) $H_q(Y_n; Z) = 0$, $q > n$;
- (iii) $Y_{n+1} = C_{k'_n} = Y_n \cup_{k'_n} CR_n$;
- (iv) R_n is the Moore spectrum of type $(H_{n+1}(X), n)$;
- (v) $k'_{n*}: H_*(R_n; Z) \rightarrow H_*(Y_n; Z)$ is a zero map.

Note that, by assumption, $H_*(X; Z) = H_*(Y; Z)$ is a free abelian group and hence each Moore spectrum R_n is, in fact, a wedge of spheres. Thus in order to show that Y is a wedge of spheres, it is sufficient to show that each k'_n is null-homotopic.

We will show, by induction, that each k'_n is null-homotopic. In order to do this let us assume that it has been shown that each k'_l , $l < n$, is null-homotopic. Then we will show that k'_n is null-homotopic. Note that since k'_l , $l < n$, are assumed to be null-homotopic, Y_n is a wedge of spheres. Thus $k'_n: R_n \rightarrow Y_n$ is a mapping between wedges of spheres. From (v) above $k'_{n*}: H_*(R_n; Z) \rightarrow H_*(Y_n; Z)$ is a zero map. Moreover, by the universal coefficient theorem, $k'_{n*}: H_*(R_n; G) \rightarrow H_*(Y_n; G)$ is zero for every group G (note that both $H_*(R_n; Z)$ and $H_*(Y_n; Z)$ are free). Thus k'_n satisfies the conditions of Lemma 3.3. Therefore it is sufficient to show that Y_{n+1} is a HOTS. From the assumption that $H_*(X; Z)$ is free, we can, by the universal coefficient theorem, improve (i) and (ii) above to the following statements: For every G ,

$$(i') \quad H_q(Y_n; G) = H_q(X; G), \quad q \leq n;$$

$$(ii') \quad H_q(Y_n; G) = 0, \quad q > n.$$

Thus we have a monomorphism

$$0 \rightarrow H_*(Y_n; G) \rightarrow H_*(Y; G)$$

for every G . Then, by Proposition 2.4, we conclude that Y_n is a HOTS. This proves that k'_n is null-homotopic. Thus, by induction, we have shown that all k'_n are null-homotopic and therefore Y is a wedge of spheres. Q.E.D.

4. Dimension of π_* -modules. In this section we will compute all the possible dimensions for general π_* -modules.

First let us note that if M is a π_* -module, then obviously we have $M/M\pi_+ \simeq A \otimes Z$ where $A \otimes Z$ means the tensor product over $\pi_*(^3)$ and $\pi_+ = \{\pi_n, n > 0\}$ is the Jacobson radical of π_* .

4.1 Proposition. *The rational number Q is a flat π_* -module via augmentation.*

Proof. The Z -flatness of Q yields the following exact sequence for any π_* -module M .

$$0 \rightarrow M\pi_+ \otimes_Z Q \rightarrow M \otimes_Z Q \rightarrow M/M\pi_+ \otimes Q \rightarrow 0.$$

(³) Throughout the whole paper $\otimes$ will mean tensor product over π_* .

Since $M\pi_+$ is Z -torsion, $M\pi_+ \otimes_Z Q = 0$. Therefore

$$M \otimes_Z Q \simeq M/M\pi_+ \otimes_Z Q \simeq (M \otimes Z) \otimes_Z Q = M \otimes Q.$$

Thus $- \otimes_Z Q$ and $- \otimes Q$ are the same functors; and $- \otimes Q$ is an exact functor. Therefore Q is a flat π_* -module.

4.2 Proposition. *Let M be a flat π_* -module such that $M/M\pi_+$ is a free abelian group. Then M is a free π_* -module.*

Proof. Let $\{e_i\}$ be elements in M such that their images $\{\bar{e}_i\}$ under the natural map constitute a basis of the free abelian group $M/M\pi_+$. Let F be a free π_* -module with basis $\{x_i\}$ such that $\deg x_i = \deg e_i$. Let $f: F \rightarrow M$ be the homomorphism sending x_i to e_i . Tensoring with Z , we have an isomorphism

$$(1) \quad f \otimes Z: F \otimes Z \rightarrow M \otimes Z = M/M\pi_+.$$

From [11, p. 215, Corollary 1.5], we see immediately that f is onto. Thus we obtain a short exact sequence

$$0 \rightarrow \ker f \rightarrow F \xrightarrow{f} M \rightarrow 0.$$

By applying the functor $\text{Tor}_{*,*}^{\pi_*}(-, Z)$ to this short exact sequence, we have

$$\rightarrow \text{Tor}_1^{\pi_*}(M, Z) \rightarrow (\ker f) \otimes Z \rightarrow F \otimes Z \rightarrow M \otimes Z \rightarrow 0$$

where the ring of integers Z is a π_* -module via augmentation. Since M is π_* -flat, $\text{Tor}_1^{\pi_*}(M, Z) = 0$. Combining this with the isomorphism of (1), we conclude that $(\ker f) \otimes Z = 0$. Then from [11, p. 215, Proposition 1.4] we have $\ker f = 0$. In other words, $M \simeq F$ is a free π_* -module.

4.3 Theorem. *A π_* -module M is flat if and only if its projective dimension is equal to or less than one.*

Proof. Let M be a flat π_* -module, and let $0 \rightarrow K \xrightarrow{j} F \rightarrow M \rightarrow 0$ be an exact sequence with F being a free π_* -module. Then from standard homological algebra arguments (e.g. see [7, Proposition 2.4]), we see that K is π_* -flat.

Consider the following commutative diagram

$$(2) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ K \otimes Z & \xrightarrow{j \otimes Z} & F \otimes Z \\ \downarrow & & \downarrow \\ 0 \rightarrow K \otimes Q & \longrightarrow & F \otimes Q \end{array}$$

where both vertical and horizontal lines are exact. The vertical lines are exact

because K and F are flat; and the bottom horizontal line is exact because Q is flat (by Proposition 4.1 above). From commutativity we see easily that $j \otimes Z : K \otimes Z \rightarrow F \otimes Z$ is a monomorphism. Since $F \otimes Z$ is a free abelian group, therefore $K \otimes Z = K/K\pi_+$ is a free abelian group too. From Proposition 4.2, we conclude that K is a free π_* -module. Thus the projective dimension $\text{P d}_{\pi_*} M$ of M is less than or equal to 1.

Conversely, assume that $\text{P d}_{\pi_*} M \leq 1$, then we will show that M is π_* -flat. Obviously, it is sufficient to show this for the case $\text{P d}_{\pi_*} M = 1$. Let $0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ be a free resolution of M [7, Theorem 215]. Let $\{F_1^\alpha, \alpha \in J\}$ be a directed set of finitely generated free submodules of F_1 , namely,

$$F_1^\alpha \subset F_1^\beta, \quad \text{if } \alpha < \beta$$

and

$$\bigcup_{\alpha \in J} F_1^\alpha = F_1 \quad \left(= \varinjlim_{\alpha} F_1^\alpha \right)$$

where $\varinjlim_{\alpha}$ is the direct limit taken over $\{F_1^\alpha, \alpha \in J\}$. Obviously

$$\frac{F_0}{F_1} = \varinjlim_{\alpha} \frac{F_0}{F_1^\alpha}.$$

Moreover since F_1^α is finitely generated, there is a finitely generated free direct summand of F_0 , say F_0' , such that $F_1^\alpha \subset F_0'$ (and $F_0 = F_1' \oplus F_0''$). Since $\text{f W D}(\pi_*) = 0$ [7, Theorem 4.3] and $\text{W d}_{\pi_*}(F_0'/F_1^\alpha) \leq 1$, the module F_0'/F_1^α is flat. Hence $F_0/F_1^\alpha = (F_0'/F_1^\alpha) \oplus F_0''$ is flat. Because the direct limit of flat modules is flat [4], we conclude that $F_0/F_1 = \varinjlim F_0/F_1^\alpha$ is flat. This completes the proof.

4.4 Theorem. *The projective dimension of π_* -modules is either 0, 1, or ∞ .*

Proof. Assume, to the contrary, that there is a π_* -module M of projective dimension $n > 1$. Then, by a standard homological algebra argument (e.g. see [7, Corollary 2.5] for similar arguments) we can assume that $\text{P d}_{\pi_*} M = 2$. Then from [7, Theorem 2.15], we have a free resolution of M ,

$$0 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Or equivalently, we have a short exact sequence

$$0 \rightarrow F_1/F_2 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Then, by Theorem 4.3, F_1/F_2 is a flat π_* -module which is embedded in a free module. By an argument similar to that in Theorem 4.3, F_1/F_0 is a free π_* -module.

Thus $\text{P d}_{\pi_*} M = 1$, a contradiction. This proves the theorem.

Remark. This theorem is true for a more general ring; for example, let A be a ring satisfying (i) $A_0 \cong \mathbb{Z}$ and (ii) for every prime p and integer j there is $a \in A_+$ such that the order of a is precisely p^j .

4.5 Theorem. *The weak dimension of a π_* -module is either 0 or ∞ .*

Proof. Assume, to the contrary, that there is a π_* -module M which has finite weak dimension $n \neq 0$, i.e. $\text{W d}_{\pi_*} M = n$, where $0 < n < \infty$. Then from standard homological algebra (e.g. see [7, Corollary 2.5] for similar arguments), we can assume, without loss of generality, that $\text{W d}_{\pi_*} M = 1$. Let us consider the following short exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with F a free π_* -module. Then K is a flat π_* -module embedded in a free module (e.g. see [7, Corollary 2.4]). By arguments similar to those in Theorem 4.3, we conclude that K is a free π_* -module. This proves that $\text{P d}_{\pi_*} M = 1$. By Theorem 4.3, M is flat, a contradiction. This completes the proof.

5. Main theorems. The present section is the central one in this paper, for here we prove the main theorems stated in §1.

5.1 Theorem. *Let X be a connected spectrum, and $b: \pi_*(X) \rightarrow H_*(X; \mathbb{Z})$ be the Hurewicz homomorphism. Then b is an epimorphism if all the homology operations (of degree ≥ 1) are trivial on X . Conversely if $H_*(X; \mathbb{Z})$ is torsion free and b is an epimorphism, then all the homology operations on X are trivial.*

Proof. 1°. Let us assume first that X is a HOTS. Then we will show that the Hurewicz homomorphism $b: \pi_*(X) \rightarrow H_*(X)$ is onto. To show this, first note that there is a spectral sequence [5] $H_*(X; \pi_*) \Rightarrow \pi_*(X)$ with edge homomorphism

$$\pi_n(X) \rightarrow E_{n,0}^\infty \subset E_{n,0}^2 = H_n(X; \pi_0) = H_n(X),$$

which is just the Hurewicz map. Since X is a HOTS and all the differentials are homology operations of degree ≥ 1 , so $E^2 = E^\infty$ whence the Hurewicz map is onto. This proves the first half of the theorem.

2°. Next we will show the second half of the theorem: Let $\{g^\alpha: S^{n_\alpha} \rightarrow X\}$ be a set of generators of $\pi_*(X)$ as a π_* -module, and let $M_0 = \bigvee_\alpha S^{n_\alpha}$ be the wedge of spheres S^{n_α} . Let $f: M_0 \rightarrow X$ be a map such that the following diagram is commutative

$$(1) \quad \begin{array}{ccc} M_0 = \bigvee_\alpha S^{n_\alpha} & \xrightarrow{f} & X \\ \uparrow j & \nearrow g_\alpha & \\ S^{n_\alpha} & & \end{array}$$

where j is the natural inclusion.

Let ζ_α be the fundamental class of $H_{n_\alpha}(S^{n_\alpha})$. Then by the definition of the Hurewicz homomorphism $b(g_\alpha) = g_{\alpha*}(\zeta_\alpha)$ where $g_{\alpha*}: H_*(S^{n_\alpha}) \rightarrow H_*(X)$ is induced by g_α . Thus the onto-ness of b implies that the set $\{g_{\alpha*}(\zeta_\alpha) \mid \text{for all } \alpha\}$ generates the whole group $H_*(X)$. Note that $H_*(M_0) = \sum_\alpha H_*(S^{n_\alpha})$, and hence $\{j_{\alpha*}(\zeta_\alpha) \mid \text{for all } \alpha\}$ generates $H_*(M_0)$. Moreover by the commutativity of the "triangle" (1) above, we have $f_*j_{\alpha*}(\zeta_\alpha) = g_{\alpha*}(\zeta_\alpha)$. Thus $f_*j_{\alpha*}(\zeta_\alpha)$ generates the whole group $H_*(X)$. In other words, $\text{Im } f_* = H_*(X)$. That is, $f_*: H_*(M_0) \rightarrow H_*(X)$ is an epimorphism.

Note that, by assumption, $H_*(X; Z)$ is torsion free; and hence, by the universal coefficient theorem, we have an epimorphism

$$f_*: H_*(M_0; G) = H_*(M_0) \otimes_Z G \rightarrow H_*(X) \otimes_Z G = H_*(X; G)$$

for every G . Now, since M_0 , as a wedge of spheres, is a HOTS, we conclude from Proposition 2.4 that X is a HOTS.

Thus we have completed the proof of the theorem.

5.2 Proposition. *A π_* -module M with finite projective or weak dimension is realizable as a stable homotopy module $\pi_*(X)$ for some spectrum X , i.e. $M \simeq \pi_*(X)$. In particular every flat π_* -module is realizable.*

Proof. From Theorem 4.4, the finiteness of $\mathbb{W}d_{\pi_*} M$ or $\mathbb{P}d_{\pi_*} M$ implies $\mathbb{P}d_{\pi_*} M \leq 1$. Let

$$0 \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$$

be a free resolution of M [7, Theorem 2.15]. Then by [7, Theorem 2.18] F_i and d_1 can be realized by wedges of spheres R_i , $i = 0, 1$, and a map $f: R_1 \rightarrow R_0$. That is, $\pi_*(R_i) \simeq F_i$ and $f_* = d_1$. Note that the monomorphism of $d_1 = f_*$ reduces the Puppe sequence of the mapping cone sequence $R_1 \rightarrow R_0 \rightarrow C_f$ into the following short exact sequence

$$0 \rightarrow \pi_*(R_1) \rightarrow \pi_*(R_0) \rightarrow \pi_*(C_f) \rightarrow 0$$

where C_f is the mapping cone of f . Comparing the two short exact sequences above, we obtained the isomorphism $M \simeq \pi_*(C_f)$. This proves the proposition.

5.3 Theorem. *A π_* -module M is flat iff M can be realized as a stable homotopy module $\pi_*(X)$ by spectrum X with trivial homology operations.*

Proof. It follows from Proposition 5.2 that M is realizable, that is, there is a spectrum X such that $M \simeq \pi_*(X)$. Then, from [7, Corollary 4.10], we see that X is a HOTS. Thus we have shown the necessity.

To prove the sufficiency, note that from Theorem 5.1 we have an onto

Hurewicz homomorphism $b: \pi_*(X) \rightarrow H_*(X; Z) \rightarrow 0$. Let M_0 be the wedges of spheres which are "generators" of $\pi_*(X)$ as π_* -modules (M_0 is constructed as in the proof 2° of Theorem 5.1). Then, arguing as in the proof 2° of Theorem 5.1, the onto-ness of b implies the onto-ness of f_* .

$$H_*(M_0; Z) \xrightarrow{f_*} H_*(X; Z) \rightarrow 0$$

where $f: M_0 \rightarrow X$ is constructed as in the proof 2° of Theorem 5.1.

By tensoring with abelian group G , we get

$$H_*(M_0; Z) \otimes G \rightarrow H_*(X; Z) \otimes G \rightarrow 0.$$

Since both X , by assumption, and M_0 are a wedge of spheres, they are both HOTS. So the homology groups $H_*(X; Z)$ and $H_*(M_0; Z)$ are torsion free [7, Proposition 4.14]. Therefore, by the universal coefficient theorem, the above exact sequence can be rewritten as $H_*(M_0; G) \rightarrow H_*(X; G) \rightarrow 0$. Then, by the Puppe sequence of $M_0 \xrightarrow{f} X \rightarrow C_f$, we get from the above sequence the following exact sequence $0 \rightarrow H_*(C_f; G) \rightarrow H_*(M_0; G)$, for every G . Thus, by Proposition 2.4, we conclude that C_f is a HOTS (since M_0 is). Moreover $H_*(C_f; Z)$, as a subgroup of a free group $H_*(M_0; Z)$, is a free group. Therefore, by Theorem 3.2, we conclude that C_f is a wedge of spheres. In other words, the Puppe sequence

$$0 \rightarrow \pi_*(C_f) \rightarrow \pi_*(M_0) \rightarrow \pi_*(X) \rightarrow 0$$

is a free π_* -resolution of $\pi_*(X)$. Then, by Theorem 4.3, $\pi_*(X)$ is flat. This completes the proof.

5.4 Theorem. *A spectrum with trivial homology operations has the same homotopy type as a wedge of Moore spectra of torsion free groups.*

Proof. By Theorems 5.3 and 4.3, the stable homotopy module of the HOTS X has projective dimension 1, i.e. there is a free resolution

$$(1) \quad 0 \rightarrow F_1 \xrightarrow{d_0} F_0 \rightarrow \pi_*(X) \rightarrow 0.$$

From [7, Theorem 2.18], there is a realization for (1). Namely, there are wedges of spheres R_0, R_1 and a map $f: R_1 \rightarrow R_0$ such that the following sequence

$$(2) \quad \begin{array}{ccccc} 0 & \longrightarrow & \pi_*(R_1) & \xrightarrow{f_*} & \pi_*(R_0) \longrightarrow \pi_*(X) \longrightarrow 0 \\ & & S \parallel & & S \parallel \\ & & F_1 & & F_0 \end{array}$$

is exact and $f_* = d_0$. Note that $\pi_*(X)$ is flat (by Theorem 4.3), hence, by tensoring with Z , we still have an exact sequence

$$(3) \quad 0 \rightarrow F_1 \otimes Z \rightarrow F_0 \otimes Z \rightarrow \pi_*(X) \otimes Z \rightarrow 0.$$

From [7, Corollary 4.9] and [13], we have $\pi_*(X) \otimes Z = \pi_*(X \wedge K(Z)) = H_*(X; Z)$ (where $K(Z)$ is Eilenberg-Mac Lane spectrum of integers). Similarly we get $F_1 \otimes Z = H_*(R_1; Z)$ and $F_0 \otimes Z = H_*(R_0; Z)$. Let L be a wedge of Moore spectra of torsion free groups such that $H_*(L; Z) = H_*(X; Z)$. Then it is easy to see that (3) is merely a free resolution for the group $H_*(L; Z)$ which is realizable by the wedges of spheres R_0, R_1 and the map $f: R_0 \rightarrow R_1$. Thus we have the following homotopy commutative diagram

$$\begin{array}{ccccccc} R_0 & \xrightarrow{f} & R_1 & \longrightarrow & X & \longrightarrow & R_0 \\ \downarrow = & & \downarrow = & & \downarrow g_1 & & \downarrow = \\ R_0 & \longrightarrow & R_1 & \longrightarrow & L & \longrightarrow & R_0. \end{array}$$

Then, by [3, J.12], there is a map g such that the above diagram is homotopy commutative. Thus g induces an isomorphism $H_*(g): H_*(X) \rightarrow H_*(L)$. By Whitehead's theorem, we have $X \simeq L$. Q.E.D.

5.5 Theorem. *A π_* -module P is projective iff P can be realized as a stable homotopy module $\pi_*(R)$ by spectrum R with trivial cohomology operations.*

Proof. The necessity follows immediately from Theorem 3.1. To prove the sufficiency, let us assume that $\pi_*(X)$ is projective. Then by [1, Eilenberg Lemma], there is a large free module F_1 such that $\pi_*(X) \oplus F_1 = F_2$, where F_2 is a free module. Then, by [7, Theorem 2.18], we have wedges of spheres R_1 and R_2 realizing F_1 and F_2 . Thus we get

$$(1) \quad \pi_*(X \vee R_1) \simeq \pi_*(R_2).$$

Moreover there is a map $f: R_2 \rightarrow X \vee R_1$ which induces the above isomorphism (1); see e.g. [7, Theorem 2.18]. By Whitehead's theorem, we get $X \vee R_1 \simeq R_2$. Hence we have

$$H^*(N_2; G) = H^*(X; G) \oplus H^*(R_1; G).$$

Thus $H^*(X; G)$, as a direct summand of $H^*(R_2, G)$, does not admit any nonzero cohomology operations of degree ≥ 1 . Thus, by Theorem 3.1, X is a wedge of spheres; and hence is a HOTS. This completes the proof.

In fact we have proved more, namely,

5.6 Corollary. *Every projective π_* -module P is realizable by a wedge of spheres, and hence is free.*

5.7 Theorem. *Let M be a π_* -module. Then*

- (i) *The projective dimension $\text{P d}_{\pi_*} M$ of M is either 0, 1 or ∞ .*
- (ii) *$\text{P d}_{\pi_*} M = 0$ iff there is a wedge R of spheres such that $M \simeq \pi_*(R)$.*
- (iii) *$\text{P d}_{\pi_*} M = 1$ iff there is spectrum L with trivial homology operations, which is not a wedge of spheres, such that $M \simeq \pi_*(L)$.*

Proof. (i) follows from Theorem 4.4; (ii) follows from Corollary 5.6; and (iii) follows from (ii) and Theorems 5.3 and 4.3.

5.8 Corollary. *$\pi_*(X)$ has projective dimension ∞ iff there is a nonzero homology operation on $H_*(X)$.*

Proof. By Theorem 5.7, the projective dimension of $\pi_*(X)$ cannot be 0 or 1 iff X is not a HOTS. So the corollary follows.

5.9 Theorem. *Let M be a π_* -module. Then*

- (i) *The weak dimension $\text{W d}_{\pi_*} M$ of M is either 0 or ∞ .*
- (ii) *$\text{W d}_{\pi_*} M = 0$ iff there is a spectrum L with trivial homology operations such that $M \simeq \pi_*(L)$.*

Proof. (i) follows from Theorem 4.5, and (ii) follows from Theorem 5.3.

5.10 Corollary. *$\text{W d}_{\pi_*}(\pi_*(X)) = \infty$ iff there is a nonzero homology operation on $H_*(X)$.*

5.11 Theorem. *Let M be a π_* -module, then*

- (i) *M is projective iff there is a spectrum R with trivial cohomology operations such that $M \simeq \pi_*(R)$.*
- (ii) *M is flat iff there is a spectrum L with trivial homology operations such that $M \simeq \pi_*(L)$.*

Proof. Follows from Theorems 5.7 and 5.9.

5.12 Theorem. *If M is a π_* -module, then the following conditions are equivalent:*

- (1) *M is a flat (but not free) π_* -module.*
- (2) *$\text{P d}_{\pi_*} M = 1$.*
- (3) *M is realizable by a spectrum L with trivial homology operations, which is not a wedge of spheres.*
- (4) *M is a wedge $\bigvee L(G_n, n)$ of Moore spectra, where each G_n is torsion free, but at least one is not free.*

Proof. (1) $\Leftrightarrow$ (2) follows from Theorem 4.3. (3) $\Leftrightarrow$ (4) by Theorem 5.4. (1) $\Leftrightarrow$ (3) by Theorem 5.9.

5.13 Theorem.

$$G D(\pi_*) = \infty > F P D(\pi_*) = 1 > f l P D(\pi_*) = f P D(\pi_*) = 0,$$

$$F I D(\pi_*) \geq 1 > F W D(\pi_*) = f l W D(\pi_*) = f W D(\pi_*) = 0.$$

Proof. $f l P D(\pi_*) = f P D(\pi_*) = f W D(\pi_*) = 0$ is computed in [7, §1, Theorem 3]. $F P D(\pi_*) = 1$ and $F W D(\pi_*) = 0$ follow immediately from Theorems 4.4 and 4.5 respectively; $G D(\pi_*) = \infty$ follows immediately from the fact that $F P D(\pi_*) \neq f P D(\pi_*)$. Thus the only thing left to be proved is that $F I D(\pi_*) \geq 1$.

To prove this, let us assume, to the contrary, that $F I D(\pi_*) = 0$. Then, by [2, Theorem 7.1 (c), p. 483], every π_* -module, in particular the integer Z as π_* -module, contains a nonzero submodule isomorphic to a principal ideal. On the other hand one notes that every principal ideal in π_* must contain some (additive) torsion element. (Let (x) be a principal ideal in π_* and $x \in \pi_0 = Z$. Let p be a prime which is not a factor of x . Then (x) contains all the elements in the p -primary component and hence contains an (additive) torsion element.) Thus Z cannot contain a submodule isomorphic to any principal ideal of π_* , a contradiction. This proves that $F I D(\pi_*) \geq 1$. Thus we have completed the proof of the theorem.

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Open problems. (1) $F I D(\pi_*) = ?$ It seems likely that $F I D(\pi_*) = 1$.

(2) $I d_{\pi_*}(\pi_*) = ?$ P. Freyd has shown that the self-injective dimension $I d_{\pi_*}(\pi_*) \neq 0$ (see Proc. La Jolla Conf. on Categorical Algebra (La Jolla, Calif., 1965), Springer-Verlag, New York, 1966, pp. 121–172). It seems to us that $I d_{\pi_*}(\pi_*) = \infty$.

(3) $K\text{-dim}(\pi_*) = ?$ It seems that $K\text{-dim}(\pi_*) = 1$. This question is very interesting. The Krull dimension $K\text{-dim}(\pi_*)$ is a "measure" of prime ideals (namely, the maximal length of chains of prime ideals) and which is closely related to the nilpotency problem. For example, the Barratt's conjecture means that there is only one (nonmaximal) prime ideal in π_* (which implies $K\text{-dim}(\pi_*) = 1$). Therefore the answer to this question is not only interesting by itself, it would also give a good approximation to the old question. We would like to point out here that in the theory of Noetherian ring the $K\text{-dim}$ is closely related to the finitistic dimensions. If π_* were a Noetherian ring (but it is *not*), then our computations would imply $K\text{-dim} \pi_* = 1$.

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