ON THE NULL-SPACES OF ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS IN R^n

BY

HOMER F. WALKER

ABSTRACT. The objective of this paper is to generalize the results of Lax and Phillips [4] and Walker [6] to include elliptic partial differential operators of all orders whose coefficients approach constant values at infinity with a certain swiftness. An example is given of an elliptic operator having an infinite-dimensional null-space whose coefficients slowly approach constant limiting values.

1. **Introduction.** Let $L_2(R^n; C^k)$ denote the usual Hilbert space of equivalence classes of C^k-valued functions on R^n whose absolute values are Lebesgue-square-integrable over R^n. Given a positive integer m, let $H_m(R^n; C^k)$ denote the Hilbert space consisting of those elements of $L_2(R^n; C^k)$ which have (strong) partial derivatives of order m in $L_2(R^n; C^k)$. Denote the usual norm on $L_2(R^n; C^k)$ by $\| \|$, and take

$$\|u\|_m = \left\{ \sum_{|\alpha| \le m} \left\| \frac{\partial^\alpha}{\partial x^\alpha} u \right\|^2 \right\}^{1/2}$$

to be the norm on $H_m(R^n; C^k)$, the notation being standard multi-index notation. In the following, each linear partial differential operator

$$Au(x) = \sum_{|\alpha| \le m} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

of order m is assumed to have domain $H_m(R^n; C^k)$ in $L_2(R^n; C^k)$ and to have coefficients continuous in x on R^n. Such an operator is said to be *elliptic* if

$$\det \left| \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \right| \neq 0$$

for all x in R^n and all nonzero ξ in R^n.

Consider a linear elliptic partial differential operator

$$A_\infty u(x) = \sum_{|\alpha| = m} a_\alpha^\infty \frac{\partial^\alpha}{\partial x^\alpha} u(x)$$

of order m which has constant coefficients and no terms of order less than m. Suppose that there is given a second elliptic operator

$$A_0 u(x) = \sum_{|\alpha| \leq m} a_{\alpha}^0(x) \frac{\partial^{\alpha}}{\partial x^{\alpha}} u(x)$$

of order m whose coefficients converge at infinity swiftly and uniformly to those of A_{∞} as follows: There exists a nonnegative real-valued continuous function ϕ on $\mathbf{R}^n$ and a number ϵ, $0 < \epsilon < \frac{1}{2}$, satisfying

(i)
$$|(A_0 - A_{\infty}) u(x)| \leq \phi(x) \left(\sum_{|\alpha| \leq m} \left| \frac{\partial^{\alpha}}{\partial x^{\alpha}} u(x) \right|^2 \right)^{1/2}$$

for all x in $\mathbf{R}^n$ and all u in $H_m(\mathbf{R}^n; \mathbf{C}^k)$.

(ii)
$$|x|^{m+2\epsilon} \phi(x) \text{ is bounded in } \mathbf{R}^n.$$

(Note for later reference that it follows in particular from (ii) that $(1 + |x|)^{m-n/2+\epsilon} \phi(x)$ is in $L_2(\mathbf{R}^n; \mathbf{C}^1)$ and that $\lim_{|x| \to \infty} (1 + |x|)^{m+\epsilon} \phi(x) = 0$.)

Then, given a positive R, denote by $E(A_0, R)$ the set of all linear elliptic partial differential operators

$$Au(x) = \sum_{|\alpha| \leq m} a_{\alpha}(x) \frac{\partial^{\alpha}}{\partial x^{\alpha}} u(x)$$

of order m whose coefficients are equal to those of A_0 outside the ball B_R^n of radius R about the origin in $\mathbf{R}^n$. Note that if A is an operator in $E(A_0, R)$, then there exist constants C_1 and C_2 depending on A such that the standard elliptic estimate $\|u\|_m \leq C_1 \|u\| + C_2 \|Au\|$ holds for all u in $H_m(\mathbf{R}^n; \mathbf{C}^k)$. (See [1], [5], and others for the derivation of such estimates.) It follows from this estimate that an operator in $E(A_0, R)$ with domain $H_m(\mathbf{R}^n; \mathbf{C}^k)$ is a closed operator.

The objective of this paper is to generalize the results of [4] and [6], which concern first-order elliptic operators whose coefficients become constant outside a bounded subset of $\mathbf{R}^n$, to include the operators in $E(A_0, R)$ described here. Specifically, it is shown in the sequel that the dimension of the null-space $N(A)$ of an operator A in $E(A_0, R)$ is finite and depends upper-semi-continuously on the operator in a certain sense. The line of reasoning followed here parallels exactly that followed in [6]. In particular, the proofs of Lemma 3, Theorem 1, and Theorem 2 may be transcribed almost verbatim from their counterparts in [6] and will not be given here. In the concluding section, an example is given of an elliptic operator with an infinite-dimensional null-space whose coefficients approach their limiting values at infinity more slowly than do the coefficients of the above operator A_0.

2. **Preparatory lemmas.** Given a positive R and linear elliptic partial differential operators A_∞ and A_0 of order m as described above, consider the set

$$M(A_0, R) = \{u \in H_m(\mathbf{R}^n; \mathbf{C}^k) \colon \text{ support } A_0 u \subseteq B_R^n\}.$$

Note that $N(A)$ is contained in $M(A_0, R)$ for every A in $E(A_0, R)$. The lemmas that follow show that the restriction of an operator in $E(A_0, R)$ to $M(A_0, R)$ be-haves in several ways as if the independent variables were restricted to a bounded subset of $\mathbf{R}^n$.

Lemma 1. *There exists a positive continuous real-valued function $C(R)$, defined for all positive R and depending on A_0, A_∞, n, and m as well as R, which is $O(R^m)$ for large R and which is such that the estimate $\|u\| \leq C(R)\|A_\infty u\|$ holds for every u in $M(A_0, R)$.*

Proof. The exact approach taken to the proof depends on whether $(m - n/2 + \epsilon)$ is positive or negative. In the following, for any positive R, u denotes an element of $M(A_0, R)$ and $A_\infty u$ is denoted by f.

Case 1. If $(m - n/2 + \epsilon)$ is positive, denote by p the largest nonnegative integer less than $(m - n/2 + \epsilon)$. Note that $(1 + |x|)^{m-n/2+\epsilon}|f(x)|$ is integrable over $\mathbf{R}^n$, since

$$\int_{\mathbf{R}^n} (1 + |x|)^{m-n/2+\epsilon}|f(x)|\, dx$$

$$\leq \int_{|x|\leq R} (1 + |x|)^{m-n/2+\epsilon}|f(x)|\, dx$$

$$+ \int_{|x|\geq R} (1 + |x|)^{m-n/2+\epsilon}|(A_\infty - A_0)u(x)|\, dx$$

$$\leq \left\{\int\!\!\int_{|x|\leq R} (1 + |x|)^{2m-n+2\epsilon}\, dx\right\}^{1/2} \|f\|$$

$$+ \left\{\int\!\!\int_{|x|\geq R} (1 + |x|)^{2m-n+2\epsilon}\phi(x)^2\, dx\right\}^{1/2} \|u\|_m.$$

A particular consequence of this is that $(-ix)^\alpha f(x)$ is absolutely integrable over $\mathbf{R}^n$ whenever $|\alpha| \leq p$. Thus, if $|\alpha| \leq p$, then

$$\frac{\partial^\alpha}{\partial\xi^\alpha}\hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{-i\xi x}(-ix)^\alpha f(x)\, dx$$

is the Fourier transform of an absolutely integrable function and, hence, is continuous. Now $|\xi|^{-2m}|\hat{f}(\xi)|^2$ can be bounded by a constant multiple of the integrable function $|\hat{u}(\xi)|^2$ and, therefore, must be integrable. It follows that $(\partial^\alpha/\partial\xi^\alpha)\hat{f}(0) = 0$ whenever $|\alpha| \leq p$. If $p = 0$, this is implied by the continuity of $\hat{f}(\xi)$; if p is positive, this is a consequence of the formula

$$\hat{f}(\xi) = \hat{f}(0) + \sum_{j=1}^{p-1} \sum_{i_1, \cdots, i_j = 1}^{n} \frac{1}{j!} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_j} \frac{\partial^j}{\partial \xi_{i_1} \cdots \partial \xi_{i_j}} \hat{f}(0)$$

$$+ \sum_{i_1, \cdots, i_p = 1}^{n} \frac{1}{(p-1)!} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_p} \int_0^1 \frac{\partial^p}{\partial \xi_{i_1} \cdots \partial \xi_{i_p}} \hat{f}(\xi t)(1-t)^{p-1} dt.$$

Now $0 < m - n/2 + \epsilon - p < 1$, and so for $|\alpha| = p$

$$\left| \frac{\partial^\alpha}{\partial \xi^\alpha} \hat{f}(\xi) \right|$$

$$\leq \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \hat{f}(\xi) - \frac{\partial^\alpha}{\partial \xi^\alpha} \hat{f}(0) \right| = \left| (2\pi)^{-n/2} \int_{\mathbf{R}^n} (e^{-i\xi x} - 1)(-ix)^\alpha f(x)\, dx \right|$$

$$\leq (2\pi)^{-n/2} \int_{|\xi| \cdot (1+|x|) \geq 1} |e^{-i\xi x} - 1| |x|^p |f(x)|\, dx$$

$$+ (2\pi)^{-n/2} \int_{|\xi| \cdot (1+|x|) \leq 1} |\xi| |x| \left[\sum_{j=1}^{\infty} \frac{(|\xi||x|)^{j-1}}{j!} \right] |x|^p |f(x)|\, dx$$

$$\leq (2\pi)^{-n/2} \left(2 + \sum_{j=1}^{\infty} \frac{1}{j!} \right) |\xi|^{m-n/2+\epsilon-p} \int_{\mathbf{R}^n} (1+|x|)^{m-n/2+\epsilon} |f(x)|\, dx.$$

Since $\left(2 + \sum_{j=1}^{\infty} 1/j! \right) = (1 + e) < 4$, this inequality and (when $p > 0$) the formula

$$\hat{f}(\xi) = \sum_{i_1, \cdots, i_p = 1}^{n} \frac{1}{(p-1)!} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_p} \int_0^1 \frac{\partial^p}{\partial \xi_{i_1} \cdots \partial \xi_{i_p}} \hat{f}(\xi t)(1-t)^{p-1} dt$$

yield the estimate

$$|\hat{f}(\xi)| \leq 4(2\pi)^{-n/2} \frac{n^p}{p!} |\xi|^{m-n/2+\epsilon} \int_{\mathbf{R}^n} (1+|x|)^{m-n/2+\epsilon} |f(x)|\, dx$$

valid for $p \geq 0$. Now, noting that

$$\|u\|_m \leq \binom{m+n}{m}^{1/2} \left\{ \|u\| + \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \|A_\infty u\| \right\}$$

for u in $H_m(\mathbf{R}^n; \mathbf{C}^k)$, one has

$$\int_{\mathbf{R}^n} (1 + |x|)^{m - n/2 + \epsilon} |f(x)| \, dx$$

$$= \int_{|x| \leq R} (1 + |x|)^{m - n/2 + \epsilon} |f(x)| \, dx$$

$$+ \int_{|x| \geq R} (1 + |x|)^{m - n/2 + \epsilon} |(A_\infty - A_0) u(x)| \, dx$$

$$\leq \int_{|x| \leq R} (1 + |x|)^{m - n/2 + \epsilon} |f(x)| \, dx$$

$$+ \int_{|x| \geq R} (1 + |x|)^{m - n/2 + \epsilon} \phi(x) \left(\sum_{|\alpha| \leq m} \left| \frac{\partial^\alpha}{\partial x^\alpha} u(x) \right|^2 \right)^{1/2} dx$$

$$\leq \left\{ \int_{|x| \leq R} (1 + |x|)^{2m - n + 2\epsilon} dx \right\}^{1/2} \|f\|$$

$$+ \left\{ \int_{|x| \geq R} (1 + |x|)^{2m - n + 2\epsilon} \phi(x)^2 dx \right\}^{1/2} \binom{m+n}{m}^{1/2} \|u\|$$

$$+ \left\{ \int_{|x| \geq R} (1 + |x|)^{2m - n + 2\epsilon} \phi(x)^2 dx \right\}^{1/2} \binom{m+n}{m}^{1/2} \left[\max_{|\xi| = 1} |A_\infty(\xi)^{-1}| \right] \|A_\infty u\|$$

$$\leq C_1(R) \|f\| + C_2(R) \|u\|$$

where

$$C_1(R) = \left\{ \int_{|x| \leq R} (1 + |x|)^{2m - n + 2\epsilon} dx \right\}^{1/2}$$

$$+ \left\{ \int_{|x| \geq R} (1 + |x|)^{2m - n + 2\epsilon} \phi(x)^2 dx \right\}^{1/2} \binom{m+n}{m}^{1/2} \left[\max_{|\xi| = 1} |A_\infty(\xi)^{-1}| \right]$$

and

$$C_2(R) = \left\{ \int_{|x| \geq R} (1 + |x|)^{2m - n + 2\epsilon} \phi(x)^2 dx \right\}^{1/2} \binom{m+n}{m}^{1/2}.$$

(Note that $C_1(R)$ and $C_2(R)$ are continuous functions of R, and that $C_1(R)$ is $O(R^{m+\epsilon})$ for large R and $C_2(R)$ approaches zero as R grows large.) Substituting this into the estimate bounding $|\hat{f}(\xi)|$ gives the estimate

$$|\hat{f}(\xi)| \leq 4 (2\pi)^{-n/2} (n^p/p!) |\xi|^{m - n/2 + \epsilon} [C_1(R) \|f\| + C_2(R) \|u\|].$$

Then

$$\|u\| = \|\hat{u}\| = \left\{\int_{\mathbf{R}^n} |A_\infty(\xi)^{-1} \hat{f}(\xi)|^2 \, d\xi\right\}^{1/2}$$

$$\leq \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|\right]\left\{\int_{\mathbf{R}^n} |\xi|^{-2m} |\hat{f}(\xi)|^2 \, d\xi\right\}^{1/2}$$

$$\leq \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|\right]\left\{\int_{|\xi|\geq R-1} |\xi|^{-2m} |\hat{f}(\xi)|^2 \, d\xi\right\}^{1/2}$$

$$+ \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|\right]\left\{\int_{|\xi|\leq R-1} |\xi|^{-2m} |\hat{f}(\xi)|^2 \, d\xi\right\}^{1/2}.$$

$$\leq \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|\right] R^m \|f\|$$

$$+ \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|\right] 4(2\pi)^{-n/2}\frac{n^p}{p!}[C_1(R)\|f\| + C_2(R)\|u\|]$$

$$\times \left\{\int_{|\xi|\leq R-1} |\xi|^{-2m+2m-n+2\epsilon} d\xi\right\}^{1/2}$$

$$\leq C_3(R)\|f\| + C_4(R)\|u\|$$

where

$$C_3(R) = \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|\right]\left[R^m + 4(2\pi)^{-n/2}\frac{n^p}{p!}\left(\frac{A_n}{2\epsilon}\right)^{1/2} C_1(R) R^{-\epsilon}\right]$$

and

$$C_4(R) = \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|\right] 4(2\pi)^{-n/2}\frac{n^p}{p!}\left(\frac{A_n}{2\epsilon}\right)^{1/2} \cdot C_2(R) R^{-\epsilon}$$

and A_n is the area of the unit sphere in $\mathbf{R}^n$. (Note that now $C_3(R)$ and $C_4(R)$ are continuous functions of R, and that $C_3(R)$ is $O(R^m)$ for large R and $C_4(R)$ approaches zero as R grows large.) Let R_0 be sufficiently large that $C_4(R) < 1$ whenever $R \geq R_0$, and define for all R

$$C(R) = \begin{cases} C_3(R)/(1 - C_4(R)) & \text{if } R \geq R_0, \\ C_3(R_0)/(1 - C_4(R_0)) & \text{if } R \leq R_0. \end{cases}$$

The nonnegative real-valued function $C(R)$ is continuous in R and $O(R^m)$ for large R. Furthermore, since $M(A_0, R)$ is contained in $M(A_0, R_0)$ whenever $R \leq R_0$, it is clear that for all R the estimate $\|u\| \leq C(R)\|A_\infty u\|$ holds for all u in $M(A_0, R)$. This proves the lemma in the case $(m - n/2 + \epsilon) > 0$.

Case 2. If $(m - n/2 + \epsilon)$ is negative, then for any pair K_1 and K_2 of positive numbers satisfying $K_1 < K_2$, one has

$$\int_{K_1 \leq |\xi| \leq K_2} |\xi|^{-2m} |\hat{f}(\xi)|^2 \, d\xi$$

$$\leq 2 \int_{K_1 \leq |\xi| \leq K_2} |\xi|^{-2m} \left| (2\pi)^{-n/2} \int_{(1+|x|) \geq K_2^{-1}} e^{-i\xi x} f(x) \, dx \right|^2 d\xi$$

$$+ 2 \int_{K_1 \leq |\xi| \leq K_2} |\xi|^{-2m} \left| (2\pi)^{-n/2} \int_{(1+|x|) \leq K_2^{-1}} e^{-i\xi x} f(x) \, dx \right|^2 d\xi$$

$$\leq 2 K_1^{-2m} \int_{\mathbf{R}^n} \left| (2\pi)^{-n/2} \int_{(1+|x|) \geq K_2^{-1}} e^{-i\xi x} f(x) \, dx \right|^2 d\xi$$

$$+ 2 (2\pi)^{-n} \int_{K_1 \leq |\xi| \leq K_2} |\xi|^{-2m}$$

$$\cdot \left[\int_{(1+|x|) \leq K_2^{-1}} |\xi|^{m-n/2+\epsilon} (1+|x|)^{m-n/2+\epsilon} |f(x)| \, dx \right]^2 d\xi$$

$$\leq 2 K_1^{-2m} \int_{(1+|x|) \geq K_2^{-1}} |f(x)|^2 \, dx$$

$$+ 2 (2\pi)^{-n} \left\{ \int_{K_1 \leq |\xi| \leq K_2} |\xi|^{-n+2\epsilon} \, d\xi \right\} \left\{ \int_{\mathbf{R}^n} (1+|x|)^{m-n/2+\epsilon} |f(x)| \, dx \right\}^2.$$

If K_2 is sufficiently small that $K_2^{-1} \geq R + 1$, then

$$\int_{(1+|x|) \geq K_2^{-1}} |f(x)|^2 \, dx \leq \int_{(1+|x|) \geq K_2^{-1}} |(A_\infty - A_0) u(x)|^2 \, dx$$

$$\leq \int_{(1+|x|) \geq K_2^{-1}} \phi(x)^2 \left(\sum_{|\alpha| \leq m} \left| \frac{\partial^\alpha}{\partial x^\alpha} u(x) \right|^2 \right) dx$$

$$\leq \left[\sup_{(1+|x|) \geq K_2^{-1}} \phi(x) \right]^2 \|u\|_m^2$$

$$\leq \left[\sup_{(1+|x|) \geq K_2^{-1}} \phi(x) \right]^2 \binom{m+n}{m} \left\{ \|u\| + \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \|A_\infty u\| \right\}^2.$$

Furthermore, there is the previously derived estimate

$$\int_{\mathbf{R}^n} (1 + |x|)^{m-n/2+\epsilon} |f(x)| \, dx \le C_1(R) \|f\| + C_2(R) \|u\|$$

where

$$C_1(R) = \left\{ \int_{|x| \le R} (1 + |x|)^{2m-n+2\epsilon} \, dx \right\}^{1/2}$$

$$+ \left\{ \int_{|x| \ge R} (1 + |x|)^{2m-n+2\epsilon} \phi(x)^2 \, dx \right\}^{1/2} \binom{m+n}{m}^{1/2} \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right]$$

and

$$C_2(R) = \left\{ \int_{|x| \ge R} (1 + |x|)^{2m-n+2\epsilon} \phi(x)^2 \, dx \right\}^{1/2} \binom{m+n}{m}^{1/2}.$$

Then for K_2 sufficiently small that $K_2^{-1} \ge R + 1$,

$$\int_{K_1 \le |\xi| \le K_2} |\xi|^{-2m} |\hat{f}(\xi)|^2 \, d\xi$$

$$\le 2 K_1^{-2m} \left[\sup_{(1+|x|) \ge K_2^{-1}} \phi(x) \right]^2 \binom{m+n}{m} \left\{ \|u\| + \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \|f\| \right\}^2$$

$$+ 2 (2\pi)^{-n} (A_n/2\epsilon) \{ K_2^{2\epsilon} - K_1^{2\epsilon} \} \{ C_1(R) \|f\| + C_2(R) \|u\| \}^2$$

where A_n is again the area of the unit sphere in $\mathbf{R}^n$. If $R \ge 1$, it follows that

$$\int_{|\xi| \le (2R)^{-1}} |\xi|^{-2m} |\hat{f}(\xi)|^2 \, d\xi = \sum_{j=1}^{\infty} \int_{(2^{j+1}R)^{-1} \le |\xi| \le (2^j R)^{-1}} |\xi|^{-2m} |\hat{f}(\xi)|^2 \, d\xi$$

$$\le \sum_{j=1}^{\infty} 2 (2^{j+1} R)^{2m} \left[\sup_{(1+|x|) \ge 2^j R} \phi(x) \right]^2 \binom{m+n}{m} \left\{ \|u\| + \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \|f\| \right\}^2$$

$$+ \sum_{j=1}^{\infty} 2 (2\pi)^{-n} \left(\frac{A_n}{2\epsilon} \right) [(2^j R)^{-2\epsilon} - (2^{j+1} R)^{-2\epsilon}] \{ C_1(R) \|f\| + C_2(R) \|u\| \}^2$$

$$\le 2^{2m+1} \left[\sup_{(1+|x|) \ge 2R} (1 + |x|)^{m+\epsilon} \phi(x) \right]^2 \left(\sum_{j=1}^{\infty} 2^{-2j\epsilon} \right) R^{-2\epsilon} \binom{m+n}{m}$$

$$\times \left\{ \|u\| + \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \|f\| \right\}^2$$

$$+ 2 (2\pi)^{-n} (A_n/2\epsilon)(2R)^{-2\epsilon} \{ C_1(R) \|f\| + C_2(R) \|u\| \}^2$$

$$\le \{ C_3(R) \|f\| + C_4(R) \|u\| \}^2$$

where $C_3(R)$ and $C_4(R)$ are now taken to be

$$C_3(R) = 2^{m+1/2} \left[\sup_{(1+|x|) \geq 2R} (1 + |x|)^{m+\epsilon} \phi(x) \right] \left(\sum_{j=1}^{\infty} 2^{-2j\epsilon} \right)^{1/2}$$

$$\cdot R^{-\epsilon} \binom{m+n}{m}^{1/2} \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right]$$

$$+ 2^{1/2}(2\pi)^{-n/2}(A_n/2\epsilon)^{1/2}(2R)^{-\epsilon} C_1(R)$$

and

$$C_4(R) = 2^{m+1/2} \left[\sup_{(1+|x|) \geq 2R} (1 + |x|)^{m+\epsilon} \phi(x) \right] \left(\sum_{j=1}^{\infty} 2^{-2j\epsilon} \right)^{1/2} R^{-\epsilon} \binom{m+n}{m}^{1/2}$$

$$+ 2^{1/2}(2\pi)^{-n/2}(A_n/2\epsilon)^{1/2}(2R)^{-\epsilon} C_2(R).$$

(Note that $C_3(R)$ and $C_4(R)$ are continuous functions of R, and that $C_3(R)$ is $O(R^m)$ for large R and $C_4(R)$ approaches zero as R grows large.) Then as before,

$$\|u\| \leq \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \left\{ \int_{|\xi| \geq R^{-1}} |\xi|^{-2m} |\hat{f}(\xi)|^2 d\xi \right\}^{1/2}$$

$$+ \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \left\{ \int_{|\xi| \leq R^{-1}} |\xi|^{-2m} |\hat{f}(\xi)|^2 d\xi \right\}^{1/2}$$

$$\leq \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] R^m \|\hat{f}\| + \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \{C_3(R)\|f\| + C_4(R)\|u\|\}$$

$$\leq \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] \{R^m + C_3(R)\} \|f\| + \left[\max_{|\xi|=1} |A_\infty(\xi)^{-1}| \right] C_4(R)\|u\|.$$

Let R_0 be sufficiently large that $[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|] C_4(R) < 1$ whenever $R \geq R_0$, and define for all R

$$C(R) = \frac{[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|] \{R^m + C_3(R)\}}{1 - [\max_{|\xi|=1} |A_\infty(\xi)^{-1}|] C_4(R)} \quad \text{if } R \geq R_0,$$

$$= \frac{[\max_{|\xi|=1} |A_\infty(\xi)^{-1}|] \{R_0^m + C_3(R_0)\}}{1 - [\max_{|\xi|=1} |A_\infty(\xi)^{-1}|] C_4(R_0)} \quad \text{if } R \leq R_0.$$

The nonnegative real-valued function $C(R)$ is continuous in R and $O(R^m)$ for

large R. Furthermore, since $M(A_0, R)$ is contained in $M(A_0, R_0)$ whenever $R \leq R_0$, it is clear that for all R the estimate $\|u\| \leq C(R)\|A_\infty u\|$ holds for all u in $M(A_0, R)$. This completes the proof of the lemma.

Lemma 2. *For any positive R, every subset of $M(A_0, R)$ which is bounded in $H_m(\mathbf{R}^n; \mathbf{C}^k)$ is relatively compact in $L_2(\mathbf{R}^n; \mathbf{C}^k)$.*

Proof. Consider first the following

Claim. There exists an R_0 such that whenever $R \geq R_0$, there is a positive constant c for which the estimate $\|u\| \leq c\|A_0 u\|$ holds for all u in $M(A_0, 2R)$ having support in $\mathbf{R}^n - B_R^n$.

Proof of claim. For any R and all u in $M(A_0, 2R)$ having support in $\mathbf{R}^n - B_R^n$, one has the estimate

$$\|u\| \leq C(2R)\|A_\infty u\| \leq C(2R)\|A_0 u\| + C(2R)\|(A_\infty - A_0)u\|$$

$$\leq C(2R)\|A_0 u\| + C(2R)\left(\sup_{|x| \geq R} \phi(x)\right)\|u\|_m$$

$$\leq C(2R)\|A_0 u\| + C(2R)\left(\sup_{|x| \geq R} \phi(x)\right)C_1\|u\| + C(2R)\left(\sup_{|x| \geq R} \phi(x)\right)C_2\|A_0 u\|$$

$$= C(2R)\left[1 + C_2\left(\sup_{|x| \geq R} \phi(x)\right)\right]\|A_0 u\| + C(2R)\left(\sup_{|x| \geq R} \phi(x)\right)C_1\|u\|$$

where $C(2R)$ is the function described in Lemma 1 and where C_1 and C_2 are the constants appearing in the estimate $\|u\|_m \leq C_1\|u\| + C_2\|A_0 u\|$ on $H_m(\mathbf{R}^n; \mathbf{C}^k)$. Since $C(R)$ is $O(R^m)$ for large R and since $\phi(x)(1 + |x|)^{m+\epsilon}$ approaches zero as $|x|$ grows large there exists an R_0 such that $C(2R)(\sup_{|x| > R} \phi(x))C_1 < 1$ whenever $R \geq R_0$. Then for $R \geq R_0$, the estimate

$$\|u\| \leq \frac{C(2R)[1 + C_2(\sup_{|x| \geq R} \phi(x))]}{1 - C(2R)C_1(\sup_{|x| \geq R} \phi(x))}\|A_0 u\|$$

holds for all u in $M(A_0, 2R)$ having support in $\mathbf{R}^n - B_R^n$, and the claim is proved.

Now since $M(A_0, R)$ is contained in $M(A_0, R_0)$ for every $R \leq R_0$, the lemma will be proved if it can be shown to hold true for all R greater than or equal to the R_0 of the above claim. Suppose that $R \geq R_0$ is given. To prove that every subset of $M(A_0, R)$ which is bounded in $H_m(\mathbf{R}^n; \mathbf{C}^k)$ is relatively compact in $L_2(\mathbf{R}^n; \mathbf{C}^k)$, it suffices to show that an arbitrary sequence $\{u_i\}$ in $M(A_0, R)$ which is bounded in $H_m(\mathbf{R}^n; \mathbf{C}^k)$ contains a subsequence which is Cauchy in $L_2(\mathbf{R}^n; \mathbf{C}^k)$. Given such a sequence, let ψ be a scalar-valued infinitely-differentiable function on $\mathbf{R}^n$ satisfying the following:

(i) $\psi(x) = 1$ for $|x| \leq R$.

(ii) $\psi(x) = 0$ for $|x| \geq 2R$.

Now for all i

$$\|\psi u_i\|_m \leq \text{const } \|u_i\|_m \leq \text{const}$$

and the functions ψu_i have compact support, so it follows from the Rellich Compactness Theorem [2, p. 169] that there is a subsequence $\{\psi u_{i_j}\}$ which is Cauchy in $L_2(\mathbf{R}^n; \mathbf{C}^k)$. It remains to find a Cauchy subsequence of $\{(1 - \psi)u_{i_j}\}$. Since each u_{i_j} is in $M(A_0, R)$, $(1 - \psi)A_0 u_{i_j}$ is identically zero on $\mathbf{R}^n$. It is then apparent that the functions $A_0[(1 - \psi)u_{i_j}]$ are bounded in $H_1(\mathbf{R}^n; \mathbf{C}^k)$ and have support in B_{2R}^n. Then the Rellich Compactness Theorem implies that there exists a subsequence $A_0[(1 - \psi)u_{i_{j_1}}]$ which is Cauchy in $L_2(\mathbf{R}^n; \mathbf{C}^k)$. But the functions $[(1 - \psi)u_{i_{j_1}}]$ and their differences are in $M(A_0, 2R)$ and have support in $\mathbf{R}^n - B_R^n$. Since $R \geq R_0$, it follows from the claim that the sequence $\{(1 - \psi)u_{i_{j_1}}\}$ is itself Cauchy in $L_2(\mathbf{R}^n; \mathbf{C}^k)$. Therefore, the sequence $\{u_{i_{j_1}} = \psi u_{i_{j_1}} + (1 - \psi)u_{i_{j_1}}\}$ is a subsequence of $\{u_i\}$ which is Cauchy in $L_2(\mathbf{R}^n; \mathbf{C}^k)$, and the lemma is proved.

The following lemma is a consequence of Lemma 2 and the elliptic estimate $\|u\|_m \leq C_1 \|u\| + C_2 \|Au\|$ on $H_m(\mathbf{R}^n; \mathbf{C}^k)$ for an operator A in $E(A_0, R)$. In the statement of the lemma, $N(A)^\perp$ denotes as usual the orthogonal complement of $N(A)$ in $L_2(\mathbf{R}^n; \mathbf{C}^k)$. The proof is a trivial generalization of the proof of Lemma 3 of [6].

Lemma 3. *For any positive R and any operator A in $E(A_0, R)$, there exists a positive constant c for which the estimate $\|u\| \leq c \|Au\|$ holds for every u in $M(A_0, R) \cap N(A)^\perp$.*

3. **Null-spaces of operators in $E(A_0, R)$.** Let there be given linear elliptic partial differential operators A_∞ and A_0 of order m as described in the preceding sections. Note that, for any positive R and any operators A and A' in $E(A_0, R)$, it follows from the elliptic estimate for such operators and from the boundedness of the coefficients of the operator $(A - A')$ that there exist positive constants c_1 and c_2 for which the estimate $\|(A - A')u\| \leq c_1 \|u\| + c_2 \|Au\|$ holds for all u in $H_m(\mathbf{R}^n; \mathbf{C}^k)$. In particular, the constants c_1 and c_2 in this estimate can be made arbitrarily small by taking the coefficients of A' sufficiently near those of A uniformly in $\mathbf{R}^n$. Theorem 1 below is a consequence of Lemma 2, which plays a role here analogous to that played by the Rellich Compactness Theorem in similar investigations in which the independent variables are restricted to a bounded subset of $\mathbf{R}^n$. Theorem 2 is deduced from Lemma 3 and Theorem 1 by using standard perturbation theory arguments that appear in [3]. For details of the proofs of Theorems 1 and 2, the reader is referred to the respective proofs of Theorems 1 and 2 of [6].

Theorem 1. *If A is an operator in $E(A_0, R)$ for some positive R, then the dimension of $N(A)$ is finite.*

Theorem 2. *If A and A' are operators in $E(A_0, R)$ for some positive R, and if A' is sufficiently near A in the sense that the constants c_1 and c_2 are sufficiently small in the estimate $\|(A - A')u\| \leq c_1\|u\| + c_2\|Au\|$ for u in $H_m(\mathbf{R}^n; \mathbf{C}^k)$, then the dimension of $N(A')$ is no greater than the dimension of $N(A)$.*

4. An operator with an infinte-dimensional null-space. The following example is intended to demonstrate that the preceding theorems are invalid unless it is assumed that the coefficients of the operators at hand approach constant limiting values at infinity with a certain rapidity. For a real number α, $0 < \alpha < \frac{1}{2}$, consider the operator $A_\alpha u(x) = A_\infty u(x) + B_\alpha u(x)$ acting on functions u in $H_1(\mathbf{R}^2, \mathbf{C}^2)$, where A_∞ is the Cauchy-Riemann operator

$$A_\infty u(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x_1} u(x) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x_2} u(x)$$

and where B_α is the operator defined by

$$B_\alpha u(x) = \frac{2\alpha}{(1 + x_1^2 + x_2^2)^{1-\alpha}} \begin{pmatrix} -x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} u(x).$$

Note that the coefficients of A_α approach those of A_∞ on the order of $|x|^{2\alpha-1}$ as $|x|$ grows large. (In order to satisfy the hypotheses of the preceding theorems, the coefficients of a first-order linear elliptic operator in $\mathbf{R}^2$ must approach constant limiting values at infinity on the order of $|x|^{-1-\epsilon}$ for some positive ϵ. Thus the preceding theorems are "within ϵ" of being the best possible results.) Now for each positive integer j, the function

$$u_j(x) = \exp\{-(1 + x_1^2 + x_2^2)^\alpha\} \begin{pmatrix} \operatorname{Re}(x_1 + ix_2)^j \\ \operatorname{Im}(x_1 + ix_2)^j \end{pmatrix}$$

is in $H_1(\mathbf{R}^2; \mathbf{C}^2)$ and is annihilated by A_α. Since the functions u_j are linearly independent, it follows that the null-space of A_α is infinite-dimensional.

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DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409