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AUTOMORPHISMS OF $GL_n(R)$, R A LOCAL RING

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ABSTRACT. Let R denote a commutative local ring with maximal ideal m and residue field $k = R/m$. In this paper we determine the group automorphisms of the general linear group $GL_n(R)$ when $n \geq 3$ and the characteristic of k is not 2.

I. Introduction and history. Let R denote a ring and $GL_n(R)$ the general linear n by n group over R.

In 1928 Schreier and van der Waerden [9] determined the group automorphisms of $GL_n(R)$ where R is a commutative field. Later Dieudonné [2] described the automorphisms in the case that R was a division ring. Almost immediately Hua and Reiner [3] determined the automorphisms of $GL_n(R)$ for R the ring of rational integers. Reiner, together with Landin [8], in 1957 extended the results to noncommutative principal domains. Recently the automorphisms of the general linear group over integral domains were examined separately by T. O'Meara [6] and Yan Shi-Jian [10]. More precisely, O'Meara obtained the form of the automorphisms for $GL_n(R)$ and $SL_n(R)$ while Yan was concerned only with the group generated by the elementary transvections $GE_n(R)$.

If R is a commutative local ring with maximal ideal m and residue field $k = R/m$, we determine in this paper the structure of the automorphisms of $GL_n(R)$ when $n \geq 3$ and the characteristic of k is not 2.

The general method in each case is first to determine the images of involutions and then determine the images of transvections under automorphisms. Since our approach is classical the characteristic of k is needed to be not 2 in order that canonical forms for involutions may be found. Once the images of transvections were obtained O'Meara invoked the Fundamental Theorem of Projective Geometry in the field of quotients of the integral domain to obtain his results. In our study we have no field of quotients to appeal to and, since R is not a domain, it was suspected the presence of torsion elements would produce new automorphisms. Indeed, the principal battle to be fought is with zero divisors.

In determining the automorphisms we follow Yan's method which is highly computational and not as elegant as the approaches appealing to division rings or fields.

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If R/m has characteristic other than 2 and $n \geq 3$, we show that any automorphism of $GL_n(R)$ can be described in terms of a triple (X, ρ, α); X is a group endomorphism of the units of R, ρ is in $GL_n(R)$ and gives rise to an inner-automorphism, and α is a ring automorphism of R.

Recently, Ojanguren and Sridharen [7] developed a fundamental theorem of projective geometry for commutative rings. It may be possible to use this result and rework this paper utilizing the beautiful approach of O'Meara. Indeed, in [5] this was done in the solution of the analogous problem for the automorphisms of the symplectic group over a local ring. However, we found that the symplectic group possesses properties which permitted us to bypass the centralizer arguments used by O'Meara in establishing the projectivity. In the case of $GL_n(R)$ we were overwhelmed by problems with zero divisors in determining centralizers of transvections à la O'Meara.

Cohn [1] discusses in detail the difficulty of determining the automorphisms of $GL_2(R)$.

II. Preliminaries. We let R denote a local ring with maximal ideal m and residue field $k = R/m$. Let V denote a free R-module of R-dimension n with $n \geq 2$. The *general linear group* $GL(V)$ is the group of all invertible R-linear maps of V to V. Once a basis for V is fixed $GL(V)$ may be identified with the group $GL_n(R)$ of invertible $n \times n$ matrices over R. We shall work almost entirely with $GL_n(R)$. The *special linear group* $SL_n(R)$ is the subgroup of $GL_n(R)$ consisting of those elements having determinant 1. We denote by $M_n(R)$ the ring of $n \times n$ matrices over R.

The letter I denotes the identity matrix, E_{ij} denotes a *standard matrix unit* (i.e., zeros in all positions except the (i, j)-position where a 1 appears), $B_{ij}(\lambda) = I + \lambda E_{ij}$ denotes an *elementary transvection* (where $i \neq j$ and λ is in R) and $D_i(u) = I + (u - 1)E_{ii}$ (where u is a unit of R).

Observe $B_{ij}(\lambda)^{-1} = B_{ij}(-\lambda)$, $B_{ij}(\lambda_1)B_{ij}(\lambda_2) = B_{ij}(\lambda_1 + \lambda_2)$ and $\det(B_{ij}(\lambda)) = 1$. While $D_i(u)^{-1} = D_i(u^{-1})$, $D_i(u_1)D_i(u_2) = D_i(u_1u_2)$ and $\det(D_i(u)) = u$.

Let R^* denote the units of R.

Proposition 2.1. *If R is a local ring, then $GL_n(R)$ is generated by*

$$\{B_{ij}(\lambda), D_k(\mu) \mid \lambda \in R,\ \mu \in R^*,\ i \neq j \text{ and } 1 \leq i, j, k \leq n\}.$$

The proof of the above is straightforward. Indeed, one obtains from it the facts that if A is in $GL_n(R)$ then A can be written as $BD_n(u)$ where B is a product of elementary transvections and $u = \det A$ and, further, that $SL_n(R)$ is generated by all elementary transvections.

For the remainder of this section we assume that R/m is a field of characteristic other than 2. Denote the characteristic of R/m by $\chi(R/m)$. An element A of $GL_n(R)$ is called an *involution* if $A^2 = I$ and an *idempotent* if $A^2 = A$.

With each involution A we can associate two submodules of V:

$$N(A) = \{X \text{ in } V \mid A(X) = -X\}$$

and

$$P(A) = \{X \text{ in } V \mid A(X) = X\}.$$

Clearly $P(A) \cap N(A) = 0$. If X is in V then

$$X = \tfrac{1}{2}(X - A(X)) + \tfrac{1}{2}(X + A(X)).$$

Thus $V = N(A) \oplus P(A)$. But then $N(A)$ and $P(A)$, being direct summands of V, are projective. Since projective modules over local rings are free, we may select a free basis for $N(A)$ and $P(A)$ so that relative to this basis the involution has the form $-I_t \oplus I_{n-t}$ where I_s is an $s \times s$ identity block. This gives the following proposition.

Proposition 2.2. *Let R be a local ring with $\chi(R/m) \neq 2$. If A is an involution in $GL_n(R)$, then there is a unique integer t such that A is similar to the matrix $-I_t \oplus I_{n-t}$.*

Such an involution as above is said to be of *type* $(t, n-t)$. Let E_n denote the collection of all matrices in $GL_n(R)$ having 1 or -1 in any combination on the main diagonal and zeros elsewhere. An element of E_n is an involution and any two elements of E_n commute.

The proof of the following result is a straightforward induction on the cardinality of r of $\{A_i\}_{i=1}^r$.

Proposition 2.3. *Let R be a local ring with $\chi(R/m) \neq 2$. If $\{A_i\}_{i=1}^r$ is a collection of pairwise commutative involutions, then there is a P in $GL_n(R)$ for which $P^{-1}A_iP$ is in E_n for all $i = 1, 2, \cdots, r$.*

Corollary 2.4. *There are at most $\binom{n}{t}$ elements in any collection of pairwise commutative involutions of type $(t, n-t)$. In any set of pairwise commutative involutions there are at most 2^n elements.*

If A is an involution and $\chi(R/m) \neq 2$ then $B = \tfrac{1}{2}(I + A)$ is an idempotent. From a set of 2^n pairwise commuting involutions we obtain 2^n commuting idempotents. An argument similar to the above will show that any idempotent can be transformed under an inner-automorphism to a diagonal form with diagonal elements 0 and 1.

III. **The automorphisms of** $GL_n(R)$. Let J_{ij} denote the diagonal matrix with -1 in both the (i, i)-position and the (j, j)-position and 1 elsewhere. The matrix J_{ij} is an involution of type $(2, n-2)$, $J_{ij}J_{jk} = J_{ik}$ and $\{J_{ij} \mid 1 \leq i < j \leq n,\ i \neq j\}$ is a set of $\binom{n}{2}$ pairwise commutative involutions.

The next result is surprisingly useful.

Theorem 3.1. *Let R be a local ring. Then an element A of $GL_n(R)$ has a unit in each row and column.*

Proof. Observe the determinant of A is a unit and is given by a Laplace expansion about any row or column. Thus any row or column must contain a unit.

Note. We assume throughout this section that $n \geq 3$ and that R is a local ring with the characteristic of R/m not 2.

Theorem 3.2. *Let Λ be an automorphism of $GL_n(R)$. Then there is a Q in $GL_n(R)$ such that $\Lambda J_{ij} = Q^{-1}J_{ij}Q$ for all $i \neq j$.*

Proof. We initially consider involutions of type $(1, n-1)$, say $P_1, \cdots, P_n$. These are n similar pairwise commuting involutions. Thus $\Lambda P_1, \cdots, \Lambda P_n$ are n similar pairwise commuting involutions of type $(t, n-t)$, say. We want to conclude that $t = 1$ or $t = n-1$. Trivially, $t \neq 0$ and $t \neq n$. Suppose $1 < t < n-1$ (observe we may also assume $n > 3$ since the case $n = 3$ is trivial). Then $\binom{n}{1} < \binom{n}{t}$. Thus, there exists at least one involution of type $(t, n-t)$ which is not in the set $\Lambda P_1, \cdots, \Lambda P_n$, which is similar to each ΛP_i and which commutes with each ΛP_i. But then $\Lambda^{-1}B$ commutes with each P_i, is distinct from each P_i and is similar to each P_i. This is impossible since $\{P_i\}$ is a maximal collection of commuting involutions of type $(1, n-1)$. Thus $t = 1$ or $t = n-1$. Then, there is in invertible Q with $Q(\Lambda P_i)Q^{-1} = \alpha P_i$ for $1 \leq i \leq n$ where $\alpha = \pm I_n$.

Now note that $J_{ij} = P_iP_j$. Thus

$$\Lambda J_{ij} = \Lambda(P_iP_j) = \Lambda(P_i)\Lambda(P_j) = Q^{-1}(\alpha P_j)(\alpha P_j)Q = Q^{-1}J_{ij}Q.$$

If Λ is an automorphism of $GL_n(R)$ then $\overline{\Lambda}$ is an automorphism where $\overline{\Lambda}(A) = Q\Lambda(A)Q^{-1}$ and $\overline{\Lambda}(J_{ij}) = J_{ij}$. Thus, we may replace in our consideration Λ by $\overline{\Lambda}$; that is, we assume $\Lambda(J_{ij}) = J_{ij}$.

The following is immediate from (3.1).

Lemma 3.3. *If a, b, c and d are elements of the local ring R and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = -I$, $\begin{bmatrix} a & -b \\ c & -d \end{bmatrix}^2 = I$, then $a = d = 0$ and $c = -b^{-1}$.*

Let $S_{i,i+1}$ denote the permutation matrix

$$S_{i,i+1} = I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus I_{n-i-1}.$$

Note that if $DS_{i,i+1} = S_{i,i+1}D$ where $D = \mathrm{diag}[a_1, \cdots, a_n]$ then $a_i = a_{i+1}$.

Theorem 3.4. *Let Λ be an automorphism of $GL_n(R)$. Then there is a Q in $GL_n(R)$ with*

$$Q\Lambda(S_{i,i+1})Q^{-1} = eI_{(i-1)} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus eI_{(n-i-1)}$$

for $1 \leq i \leq n-1$ *where* $e = \pm 1$.

Proof. We assume $\Lambda J_{i,i+1} = J_{i,i+1}$. Since $J_{i,i+1}$ commutes with S_{12} for $i = 1$ or $i \geq 3$ we have $J_{i,i+1} = \Lambda J_{i,i+1}$ commutes with ΛS_{12}.

Thus if $n = 3$ or $n \geq 5$, then $\Lambda S_{12} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \oplus [a_3, \cdots, a_n]$ or, if $n = 4$, $\Lambda S_{12} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$. In the case $n = 4$ the identities

$$(1) \qquad (\Lambda S_{12})^2 = \Lambda J_{12} = J_{12}, \qquad (\Lambda S_{12} \Lambda J_{23})^2 = I$$

imply that $x = y = 0$ and $w^2 = z^2 = 1$. Thus, for $n \geq 3$, $\Lambda S_{12} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \oplus [a_3, \cdots, a_n]$. Applying equations (1) again we obtain $c = b^{-1}$, $a = d = 0$ and $a_i = \pm 1$.

In general, for $i = 1, \cdots, n-1$,

$$\Lambda S_{i,i+1} = [a_1^{(i)}, \cdots, a_{i-1}^{(i)}] \oplus \begin{bmatrix} 0 & b_i \\ -b_i & 0 \end{bmatrix} \oplus [a_{i+1}^{(i)}, \cdots, a_n^{(i)}]$$

where $a_j^{(i)} = \pm 1$.

Define

$$Q = \operatorname{diag}\left[\prod_{t=1}^{n-1} b_t^{-1}, \prod_{t=2}^{n-1} b_t^{-1}, \cdots, b_{n-1}^{-1}, 1\right]$$

and observe $QJ_{i,i+1}Q^{-1} = J_{i,i+1}$. Thus $J_{i,i+1}$ is fixed under conjugation by Q. Further conjugation of $S_{i,i+1}$ by Q yields

$$Q\Lambda(S_{i,i+1})Q^{-1} = [a_1^{(i)}, \cdots, a_{i-1}^{(i)}] \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus [a_{i+1}^{(i)}, \cdots, a_n^{(i)}].$$

In a reasonably straightforward calculation one uses that $S_{i,i+1}$ commutes with $\{S_{j,j+1} \mid 1 \leq j \leq i-2,\ i+2 \leq j \leq n-1\}$ and thus $Q\Lambda(S_{i,i+1})Q^{-1}$ commutes with $\{Q\Lambda(S_{j,j+1})Q^{-1} \mid 1 \leq j \leq i-2,\ i+2 \leq j \leq n-1\}$ to show $a_1^{(i)} = a_2^{(i)} = \cdots = a_{i-1}^{(i)}$ and $a_{i+1}^{(i)} = a_{i+2}^{(i)} = \cdots = a_n^{(i)}$. Since $(S_{i-1,i}S_{i,i+1})^3 = I$ implies $(Q\Lambda(S_{i-1,i})\Lambda(S_{i,i+1})Q^{-1})^3 = I$, another calculation gives $a_{i-1}^{(i)} = a_{i+1}^{(i)}$. Which completes the proof.

Since the above inner-automorphism by Q fixes J_{ij} we may now assume (similar to the comment before (3.3)) that the automorphism Λ satisfies $\Lambda J_{ij} = J_{ij}$ and $\Lambda S_{i,i+1} = eI_{i-1} \oplus \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \oplus eI_{n-i-1}$ where $e = 1$ or $e = -1$.

The most involved step in the determination of Λ is in the next theorem.

Theorem 3.5. *If Λ is the automorphism of $GL_n(R)$ described above, then either $\Lambda B_{ij}(1) = B_{ij}(1)$ for all (i, j) or $\Lambda B_{ij}(1) = B_{ji}(-1)$ for all (i, j).*

Proof. Since $B_{12}(1)$ commutes with J_{12} and $J_{i,i+1}$ for $i \geq 3$ when $n \neq 4$, we must have $\Lambda B_{12}(1) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \oplus [a_3, \cdots, a_n]$ and, if $n = 4$, $\Lambda B_{12}(1) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$. We first show if $n \neq 4$ then $a_3 = a_4 = \cdots = a_n$. Since $B_{12}(1)$ commutes with $S_{i,i+1}$

for $i \geq 3$ we have that $\Lambda B_{12}(1)$ commutes with $\Lambda S_{i,i+1}$ for $i \geq 3$. In particular, $\Lambda B_{12}(1)\Lambda S_{34} = \Lambda S_{34}\Lambda B_{12}(1)$ implies

$$e\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} 0 & a_3 \\ -a_4 & 0 \end{bmatrix} \oplus e[a_5, \cdots, a_n]$$

$$= e\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} 0 & a_4 \\ -a_3 & 0 \end{bmatrix} \oplus e[a_5, \cdots, a_n]$$

and hence $a_3 = a_4$. Similarly we conclude $a_3 = a_4 = \cdots = a_n$.

Thus $B_{12}(1) = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \oplus fI_{n-2}$. We claim that $f = e$. To show this note

$$\Lambda(B_{12}(1)J_{23})^2 = I, \tag{2}$$

$$\Lambda(S_{12}B_{12}(1))^3 = I, \tag{3}$$

$$\Lambda(S_{23}^{-1}B_{12}(1)S_{23}) = \Lambda B_{13}(1). \tag{4}$$

Then (2) implies $f^2 = 1$ while (3) gives $(ef)^3 = 1$. Hence $ef = 1$ and since $e = \pm 1$, we have that $f = \pm 1$ and $e = f$.

The equations (2) and (3) and (4) determine $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$. From (2) one obtains $[\begin{smallmatrix} a & -b \\ c & -d \end{smallmatrix}]^2 = I$ while (4) gives

$$\Lambda B_{12}(1)\Lambda(S_{23}^{-1}B_{12}(1)S_{23}) = \Lambda(S_{23}^{-1}B_{12}(1)S_{23})\Lambda B_{12}(1)$$

which when computing upper-left 3 by 3 blocks yields

$$\begin{bmatrix} a^2 & be & aeb \\ ac & de & bec \\ c & 0 & ed \end{bmatrix} = \begin{bmatrix} a^2 & ab & b \\ ec & ed & 0 \\ aec & bec & ed \end{bmatrix}.$$

Since e is a unit we have $bc = 0$ and $a^2 = d^2 = 1$. A direct computation using $[\begin{smallmatrix} c & d \\ -a & -b \end{smallmatrix}]^3 = I$ from equation (3) implies that either b or c is a unit and thus the other (since $bc = 0$) is 0.

Now if $c = 0$, from (3), $bad = 1$ and since $a = d = \pm 1$ it is concluded that $b = 1$. Similarly, if $b = 0$ then $c = -1$.

Thus we have shown that either $\Lambda B_{12}(1) = eI_n + E_{12}$ or $\Lambda B_{12}(1) = eI_n - E_{21}$ and, using (4), that either $\Lambda B_{13}(1) = eI_n + eE_{13}$ or $\Lambda B_{13}(1) = eI_n - eE_{31}$, respectively.

We now claim that $e = 1$. Observe $S_{12}^{-1}B_{13}(1)S_{12} = B_{23}(1)$. Thus either $\Lambda B_{23}(1) = eI_n + E_{23}$ or $\Lambda B_{23}(1) = eI_n - E_{32}$, respectively. Since the commutator $[\Lambda B_{12}(1), \Lambda B_{23}(1)] = \Lambda B_{13}(1)$, we have that $e^4 = e$ and hence $e = 1$.

It now remains in the case $n \neq 4$ to determine the images of the other commutators.

Assume $\Lambda B_{12}(1) = B_{12}(1)$ and note $\Lambda B_{21}(1) = \Lambda(S_{12}^{-1}B_{12}(-1)S_{12}) = B_{21}(1)$. Thus, by induction, suppose that $\Lambda B_{1i}(1) = B_{1i}(1)$ and $\Lambda B_{i1}(1) = B_{i1}(1)$. Then $S_{i,i+1}^{-1}B_{1i}(1)S_{i,i+1} = B_{1,i+1}(1)$ and $S_{i,i+1}^{-1}B_{i1}(1)S_{i,i+1} = B_{i+1,1}(1)$ imply $\Lambda B_{1,i+1}(1) = B_{1,i+1}(1)$ and $\Lambda B_{i+1,1}(1) = B_{i+1,1}(1)$. Thus, if $\Lambda B_{12}(1) = B_{12}(1)$, then $\Lambda B_{1i}(1) = B_{1i}(1)$ and $\Lambda B_{i1}(1) = B_{i1}(1)$ for $i = 2, 3, \cdots, n$.

Now using $[B_{ij}(1), B_{jk}(1)] = B_{ik}(1)$, $i \neq j \neq k$, we have $\Lambda B_{ij}(1) = B_{ij}(1)$ for all i and j.

If $\Lambda B_{12}(1) = B_{21}(-1)$ an argument analogous to the above gives $\Lambda B_{ij}(1) = B_{ji}(-1)$ for all i and j.

This completes the proof for the case $n \neq 4$.

We now let $n = 4$. The computation is lengthy but not difficult. Thus we merely sketch the steps.

Since $\Lambda B_{12}(1)$ commutes with J_{12} we have

$$\Lambda B_{12}(1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

Since $B_{12}(1)$ and S_{34} commute we obtain $x = -y$ and $z = w$. By equation (2) it may be shown that either b or c is a unit, $x = 0$ and $a = d = w$. Finally use equation (4) and the fact that $\Lambda B_{12}(1)$ commutes with $\Lambda B_{13}(1)$ to show that if b is a unit then $c = 0$ and conversely. At this step $\Lambda B_{12}(1)$ has one of the following forms: either $\Lambda B_{12}(1) = [\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}] \oplus [a, a]$ or $\Lambda B_{12}(1) = [\begin{smallmatrix} a & 0 \\ c & a \end{smallmatrix}] \oplus [a, a]$. The remaining conclusions for the case $n = 4$ may be argued as in the case $n > 4$.

Note $B_{ji}(-1)$ is the inverse transpose of $B_{ij}(1)$. For a matrix A in $GL_n(R)$ let $(A^t)^{-1} = (A^{-1})^t$ be denoted by A^*.

Theorem 3.6. *If Λ: $GL_n(R) \longrightarrow GL_n(R)$ is a group automorphism, then there is a ring automorphism σ: $R \longrightarrow R$ and a P in $GL_n(R)$ such that either*

$$\Lambda A = P^{-1}A^{\sigma}P \quad \textit{for all } A \textit{ in } SL_n(R)$$

or

$$\Lambda A = P^{-1}(A^{\sigma})^{*}P \quad \textit{for all } A \textit{ in } SL_n(R).$$

Proof. By comment following (2.1), $SL_n(R)$ is generated by elementary transvections; thus, it suffices to find a ring automorphism σ and an invertible matrix P such that either $P\Lambda B_{ij}(\lambda)P = B_{ij}(\lambda^{\sigma})$ for all $B_{ij}(\lambda)$ or $P\Lambda B_{ij}(\lambda)P = B_{ji}(-\lambda^{\sigma})$ for all $B_{ij}(\lambda)$. Assume by (3.5) and (3.4) that $P\Lambda B_{ij}(1)P^{-1} = B_{ij}(1)$ and $P\Lambda S_{i,i+1}P^{-1} = S_{i,i+1}$. For λ in R since $B_{12}(\lambda)$ commutes with $B_{12}(1)$ and

$B_{ij}(1)$, $3 \le i, j \le n$, note that $P\Lambda B_{12}(\lambda)P^{-1} = \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \oplus fI_{n-2}$. Since $S_{23}^{-1}B_{12}(\lambda)S_{23} = B_{13}(\lambda)$,

$$\text{(5)} \qquad P\Lambda B_{13}(\lambda)P^{-1} = \begin{bmatrix} a & 0 & b \\ 0 & f & 0 \\ 0 & 0 & a \end{bmatrix} \oplus fI_{n-3}.$$

The commutator relation $[B_{12}(\lambda), B_{23}(1)] = B_{13}(\lambda)$ gives $P[\Lambda B_{12}(\lambda), \Lambda B_{23}(1)]P^{-1} = P\Lambda B_{12}(\lambda)P^{-1}$ and

$$\text{(6)} \qquad P\Lambda B_{13}(\lambda)P^{-1} = \begin{bmatrix} 1 & 0 & bfa^{-2} \\ 0 & 1 & 1-fa^{-1} \\ 0 & 0 & a^{-2} \end{bmatrix} \oplus I_{n-3}.$$

Now define $\sigma: R \longrightarrow R$ by $\lambda^\sigma = \beta$ if $P\Lambda B_{12}(\lambda)P^{-1} = I_n + \beta E_{12}$. Clearly, $P\Lambda B_{ij}(\lambda)P^{-1} = B_{ij}(\lambda^\sigma)$ and σ is additive. If $\lambda^\sigma = 0$, then $\Lambda B_{12}(\lambda) = I_n$ and $\lambda = 0$. Thus σ is injective. To show σ is multiplicative, consider

$$\begin{aligned} B_{13}([\lambda_1\lambda_2]^\sigma) &= P\Lambda B_{13}(\lambda_1\lambda_2)P^{-1} \\ &= P[\Lambda B_{12}(\lambda_1), \Lambda B_{23}(\lambda_2)]P^{-1} = B_{13}(\lambda_1^\sigma\lambda_2^\sigma). \end{aligned}$$

Thus σ is a ring monomorphism. It remains to show σ is surjective. We clearly have that $\Lambda[SL_n(R)]$ is in $SL_n(R)$ and (see Klingenberg [4]) has order ideal R. Further, $\Lambda[SL_n(R)]$ is normal in $GL_n(R)$ so by Klingenberg's results on normal subgroups [4], $\Lambda[SL_n(R)] = SL_n(R)$. Thus if r is any element of R, there is a product $B = \Pi B_{ij}(\lambda_{ij})$ of elementary transvections such that $P\Lambda(B)P^{-1} = B_{12}(r)$. That is, r is a finite sum of finite products of λ_{ij}^σ. Thus σ is surjective and hence a ring automorphism of R.

We now can determine the action of an automorphism on $GL_n(R)$. We state the complete hypothesis.

Theorem 3.7. *Let R be a local ring with the characteristic of R/m other than 2. Suppose $n \ge 3$ and $\Lambda: GL_n(R) \longrightarrow GL_n(R)$ is a group automorphism. Then there is a P in $GL_n(R)$, a ring automorphism $\sigma: R \longrightarrow R$ and a group morphism $\chi: R^* \longrightarrow R^*$ such that either*

$$\Lambda(A) = \chi(\det A)P^{-1}A^\sigma P \quad \textit{for all } A \textit{ in } GL_n(R)$$

or

$$\Lambda(A) = \chi(\det A)P^{-1}(A^\sigma)^*P \quad \textit{for all } A \textit{ in } GL_n(R).$$

Proof. If A is in $GL_n(R)$ then $A = D_n(r)B$ where $\det A = r$ and B is a product of elementary transvections. Since ΛB has been determined it is only necessary to compute $\Lambda D_n(r)$. We assume $\Lambda B_{ij} = P^{-1}B_{ij}(\lambda^\sigma)^*P$. The other case is similar. For any (i, j), $D_n(r)B_{ij}(1)D_n(r^{-1})$ is an element of $SL_n(R)$ and, thus,

$$P(\Lambda D_n(r)\Lambda B_{ij}(1)\Lambda D_n(r^{-1}))P^{-1} = (D_n(r)B_{ij}(1)D_n(r^{-1}))^{\sigma *} = D_n(r^\sigma)^{-1}B_{ji}(-1)D_n(r^\sigma).$$

Then,

$$D_n(r^\sigma)P\Lambda D_n(r)P^{-1}B_{ji}(-1) = B_{ji}(-1)D_n(r^\sigma)P\Lambda D_n(r)P^{-1}.$$

Therefore $D_n(r^\sigma)P\Lambda D_n(r)P^{-1}$ commutes with $B_{ji}(-1)$ for $i \neq j$ and hence is a scalar matrix. That is, $\Lambda D_n(r) = (\text{scalar})P^{-1}D_n(r^\sigma)^{-1}P$. Define $\chi\colon R^* \longrightarrow R^*$ by

$$\chi(r) = \text{scalar associated with } \Lambda D_n(r).$$

Since $\Lambda(D_n(r_1)D_n(r_2)) = \Lambda D_n(r_1r_2)$ it follows that $\chi(r_1)\chi(r_2) = \chi(r_1r_2)$ and χ is a group morphism. This completes the proof.

The ring morphisms $\rho_t\colon R \longrightarrow R/m^t$ for $t = 1, 2, 3, \cdots$ induce natural group morphisms $h_t\colon GL_n(R) \longrightarrow GL_n(R/m^t)$. For each t,

$$h_t^{-1}(\text{center of } GL_n(R/m^t))$$

is the *general congruence subgroup modulo* m^t, denoted $GC_n(R, t)$. The *special congruence subgroup* $SC_n(R, t)$ *modulo* m^t is the set of P in $GL_n(R)$ with $h_t(P) = I$ and $\det(P) = 1$. If $R = Z/Zp^s$ (rational integers modulo a prime power) the above congruence groups have been shown to be characteristic under the automorphisms of $GL_n(R)$ by involved combinatorial arguments.

Corollary 3.8. *Let R be a local ring with the characteristic of R/m other than 2. Then if $n \geq 3$, the subgroups $GC_n(R, t)$ and $SC_n(R, t)$ are characteristic in $GL_n(R)$ for $t \geq 1$.*

Proof. Assume $\Lambda(A) = \chi(\det A)P^{-1}A^\sigma P$ for all A in $GL_n(R)$ (the other case is similar). If A is in $GC_n(R, t)$ then $A = rI + N$ where N is an $n \times n$ matrix with elements in m^t. Direct computation shows that $\Lambda(A)$ is in $GC_n(R, t)$. Finally, Klingenberg [4] shows that $SC_n(R, t) = [GL_n(R), GC_n(R, t)]$. The result follows.

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