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A PROOF THAT $\mathcal{C}^2$ AND $\mathcal{T}^2$ ARE DISTINCT MEASURES(1)

BY

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ABSTRACT. We prove that there exists a nonempty family X of subsets of $\mathbf{R}^3$ such that the two-dimensional Carathéodory measure of each member of X is less than its two-dimensional $\mathcal{T}$ measure. Every member of X is the Cartesian product of 3 copies of a suitable Cantor type subset of $\mathbf{R}$.

1. **Introduction.** To any positive integers m, n with $m \le n$ there correspond several m-dimensional measures over $\mathbf{R}^n$. These measures are studied extensively in [3]. We consider two of them, the m-dimensional Carathéodory measure, denoted by $\mathcal{C}^m$, and the m-dimensional $\mathcal{T}$ measure, denoted by $\mathcal{T}^m$. It is known that $\mathcal{C}^m(S) \le \mathcal{T}^m(S)$ for all $S \subset \mathbf{R}^n$ [3, 2.10.34], and $\mathcal{C}^m(S) = \mathcal{T}^m(S)$ if $m = 1$, $m = n$, or S is m rectifiable [3, 2.10.35, 3.2.26].

In this paper we prove (Theorem 3.4) that there exists a nonempty family X of subsets of $\mathbf{R}^3$ such that $\mathcal{C}^2(S) < \mathcal{T}^2(S)$ for all $S \in X$. A precise definition of X is given in §2, using the method of [3, 2.10.28], but roughly each member of X is the Cartesian product of 3 copies of a suitable Cantor type subset of $\mathbf{R}$. We obtain Theorem 3.4 directly from Theorems 3.2, 3.3. A key step in the proof of Theorem 3.2 depends in turn on Lemma 3.1.

2. **Preliminaries.** In general we adopt in this paper the notation and terminology of [3]. Presented in this section are modifications and additional definitions that we use.

For $S \subset \mathbf{R}^n$ let $S - S = \{x - y\colon\ x, y \in S\}$.

For $a, b \in \mathbf{R}^n$ define $[a, b]$ to be the closed line segment with endpoints a, b.

For $\varnothing \ne S \subset \mathbf{R}^n$ let

$$c^2(S) = \sup\{\mathcal{L}^2[p(S)]\colon p \in O^*(n, 2)\},$$
$$t^2(S) = (\pi/4)\sup\{|(a_1 - b_1) \wedge (a_2 - b_2)|\colon a_1, b_1, a_2, b_2 \in S\}.$$

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These are the gauge functions used in defining $\mathcal{C}^2$ and $\mathcal{T}^2$ respectively [3, 2.10.1, 2.10.3, 2.10.4].

The following series of definitions culminate in the definition of X and E. For any sequence $\nu = (\nu_1, \nu_2, \nu_3, \cdots)$ of integers greater than 1 we denote

$$H_0(\nu) = \{[0, 1]\},$$
$$H_k(\nu) = \bigcup \{\Phi(J, \nu_k): J \in H_{k-1}(\nu)\} \quad \text{for } k \geq 1,$$

where

$$\Phi(J, \nu_k) = \{[\inf J + (i-1)p, \inf J + (i-1)p + q]: i = 1, \cdots, \nu_k\}$$

with $p = (1 - \nu_k^{-3/2})(\nu_k - 1)^{-1} \operatorname{diam} J$, $q = \nu_k^{-3/2} \operatorname{diam} J$; then we define

$$A(\nu) = \bigcap_{k=0}^{\infty} \bigcup H_k(\nu).$$

Finally, we let $X = \{A(\nu) \times A(\nu) \times A(\nu)$: ν is a bounded sequence$\}$ and $E = A(\nu) \times A(\nu) \times A(\nu)$ be any fixed element of X.

3. **Principal results.** We proceed to prove that $\mathcal{C}^2(E) < \mathcal{T}^2(E)$.

3.1. **Lemma.** *If B is a compact convex subset of $\mathbf{R}^2$, $Y = \operatorname{Bdry} B$, $d = \operatorname{diam} B$, and there exists a one-dimensional vector subspace L of $\mathbf{R}^2$ and a positive real number k such that $\mathcal{H}^1(L \cap Y) \geq kd$, then*

$$\mathcal{L}^2(B) \leq t^2(B)/(1 + 2^{-10}k^4). \tag{1}$$

Proof. We can assume that $\mathcal{L}^2(B) > 0$, since (1) clearly holds when $\mathcal{L}^2(B) = 0$.

Choose a one-dimensional vector subspace V of $\mathbf{R}^2$ perpendicular to L, and let B' be the subset of $\mathbf{R}^2$ obtained by applying Steiner symmetrization [3, 2.10.30] to B with respect to V. Let $Y' = \operatorname{Bdry} B'$. Then by [3, 2.10.30, proof in 2.10.32] B' is a compact convex set, $\mathcal{L}^2(B') = \mathcal{L}^2(B)$, $t^2(B') \leq t^2(B)$ and $\mathcal{H}^1(L \cap Y') \geq kd$.

We now proceed to prove the existence of $G \subset \mathbf{R}^2$ satisfying

$$\mathcal{L}^2(B') \leq \mathcal{L}^2(G)/(1 + 2^{-10}k^4), \tag{2}$$
$$t^2(G) \leq t^2(B'). \tag{3}$$

This will establish the lemma, since (2) and (3) combined with the relations $\mathcal{L}^2(B) = \mathcal{L}^2(B')$, $\mathcal{L}^2(G) \leq t^2(G)$ [3, 2.10.32], and $t^2(B') \leq t^2(B)$ yield (1).

Let a_1 be the midpoint of $L \cap Y'$. Choose orthonormal basis vectors e_1, e_2 for $\mathbf{R}^2$ so that $\mathbf{R}e_2 = L$, and $(x - a_1) \cdot e_1 \geq 0$ for all $x \in B'$. Let $m_1, m_2 \in L \cap Y'$ be such that $m_2 - a_1 = a_1 - m_1 = kde_2/4$. Choose $b_1, b_2 \in Y'$ satisfying

$$(a_1 - b_1) \cdot e_2 = (b_2 - a_1) \cdot e_2 = \sup\{(x - a_1) \cdot e_2 : x \in B'\}$$

and $(b_2 - b_1) \cdot e_1 = 0$. Let

$$z = \operatorname{diam}(B' \cap \{x: (x - b_1) \cdot e_2 = 2^{-6}k^2 d\}).$$

Choose $m_3 \in \mathbf{R}^2$ with $a_1 - m_3 = kze_1/4$. Then let

$$F = B' \cap \{x: (x - b_1) \circ e_2 \geq 2^{-6}k^2 d \text{ and } (b_2 - x) \circ e^2 \geq 2^{-6}k^2 d\}$$

and G be the convex hull of $F \cup \{m_3\}$.

We now verify (2). Since $B' \sim G$ is contained in the union of two rectangles with dimensions z and $2^{-6}k^2 d$, $\mathcal{L}^2(B' \sim G) \leq 2^{-5}k^2 dz$, while, since the interior of the convex hull of $\{m_1, m_2, m_3\}$ is contained in $G \sim B'$, $\mathcal{L}^2(G \sim B') \geq 2^{-4}k^2 dz$; hence

$$\mathcal{L}^2(G) - \mathcal{L}^2(B') \geq 2^{-5}k^2 dz. \tag{4}$$

Choose $a_2 \in Y'$ with $(a_2 - a_1) \cdot e_2 = 0$, $a_2 \neq a_1$. Then B' is contained in a rectangle of side lengths $|a_2 - a_1|$ and $|b_2 - b_1|$; consequently,

$$\mathcal{L}^2(B') \leq |a_2 - a_1| \cdot |b_2 - b_1|. \tag{5}$$

Take $i = 1, 2$. Let $w_i = [a_i, b_1] \cap \{x: (x - b_1) \cdot e_2 = 2^{-6}k^2 d\}$, $s = (a_i - b_1) \cdot e_2 / [w_i - b_1) \cdot e_2]$. We see from our construction that $|s| \leq 2^5 k^{-2}$, since $|(a_i - b_1) \cdot e_2| \leq d/2$ and $(w_i - b_1) \cdot e_2 = 2^{-6}k^2 d$. Furthermore, $(a_i - b_1) = s(w_i - b_1)$, and by subtraction $a_2 - a_1 = s(w_2 - w_1)$. Therefore, $|a_2 - a_1| \leq 2^5 k^{-2}|w_2 - w_1|$. We combine this result with the inequalities (4), $|w_2 - w_1| \leq z$, $|b_2 - b_1| \leq d$ and (5) to obtain

$$\mathcal{L}^2(G) - \mathcal{L}^2(B') \geq 2^{-5}k^2 dz \geq 2^{-10}k^4 |a_2 - a_1| \cdot |b_2 - b_1| \geq 2^{-10}k^4 \mathcal{L}^2(B')$$

and then by addition (2).

To establish (3) we need only show that

$$t^2(F \cup \{m_3\}) \leq t^2(B'), \tag{6}$$

since $t^2(F \cup \{m_3\}) = t^2(G)$ by [3, 2.10.3]. We now proceed to prove (6) by the following method:

Let

$$Q = [(F \cup \{m_3\}) - (F \cup (m_3\})] \times [(F \cup \{m_3\}) - (F \cup \{m_3\})].$$

To each ordered pair $(v_1, v_2) \in Q$, $v_1 = p_1 e_1 + p_2 e_2$, $v_2 = q_1 e_1 + q_2 e_2$, we associate $(v_1^*, v_2^*) \in Q$ by means of a map f such that $(v_1^*, v_2^*) = f(v_1, v_2)$ satisfies the three conditions, $v_1^* \in F - F$, $v_2 \in F - F$ implies $v_2^* \in F - F$, and $|v_1^* \wedge v_2^*| \geq |v_1 \wedge v_2|$. The existence of such a map f will prove (6), since $(v_2^{**}, v_1^{**}) =$

$f(v_2^*, v_1^*)$ will then satisfy $v_2^{**}, v_1^{**} \in F - F \subset B' - B'$ and $|v_2^{**} \wedge v_1^{**}| \geq |v_1 \wedge v_2|$. To define f and show the required conditions are satisfied we will consider the following cases and subcases:

Case I. $v_1 \in F - F$.

Let $v_1^* = v_1$ and $v_2^* = v_2$.

Case II. $v_1 \notin F - F$ *and* $p_1 \geq 0$.

We note that $v_1 = x - m_3$ for a unique x in F. Let $r = z/d$, $u = kd/4$. We then consider four subcases:

Case II.A. $|q_1| \geq r|q_2|$ *and* $q_1(p_1q_2 - p_2q_1) \geq 0$.

Let $v_1^* = x - m_2$, $v_2^* = v_2$. Then, since $x - m_2 = (x - m_3) + (m_3 - m_2) = (p_1 - ru)e_1 + (p_2 - u)e_2$, we deduce that

$$|v_1^* \wedge v_2^*| = |p_1q_2 - p_2q_1 + uq_1 - ruq_2| \geq |p_1q_2 - p_2q_1| = |v_1 \wedge v_2|.$$

Case II.B. $|q_1| \geq r|q_2|$ *and* $q_1(p_1q_2 - p_2q_1) \leq 0$.

Let $v_1^* = x - m_1 = (x - m_3) + (m_3 - m_1)$, $v_2^* = v_2$, and proceed as in Case II.A.

Case II.C. $|q_1| \leq r|q_2|$ *and* $p_2 \geq 0$.

Let $v_1^* = x - m_1$, $v_2^* = rq_2e_1 - q_2e_2$. Note that $v_2^* \in F - F$ by the construction of F and the definition of r. Furthermore,

$$|v_1^* \wedge v_2^*| = |p_1q_2 + rp_2q_2| \geq |p_1q_2 - p_2q_1| = |v_1 \wedge v_2|.$$

Case II.D. $|q_1| \leq r|q_2|$ *and* $p_2 \leq 0$.

Let $v_1^* = x - m_2$, $v_2^* = rq_2e_1 + q_2e_2$, and proceed as in Case II.C.

Case III. $v_1 \notin F - F$ *and* $p_1 \leq 0$.

Let $f(v_1, v_2) = f(-v_1, v_2)$.

Thus the existence of the required map f has been shown, (6) has been established, and the proof of the lemma is complete.

3.2. **Theorem.** *There exists $s < 1$ such that if K is a closed subset of E and M is the convex hull of K, then $c^2(M) \leq st^2(M)$.*

Proof. Choose any $\theta \in \mathbf{O}^*(3, 2)$. Let $\theta(M) = B$, $d = \operatorname{diam} B$. We can assume by excluding the trivial case when B is a single point that $d > 0$. Denote by Y the boundary of B in $\theta(\mathbf{R}^3)$. Let $S = M \cap \theta^{-1}(Y)$. Note that S is a continuum.

The first main step in our proof will be to show that there exists a closed line segment $[a, b] \subset S$ such that $|a - b| = rd$, where r is a number depending only on the sequence ν. (The construction involved in establishing this is basically a generalization of a procedure in [4].) Choose $\alpha, \beta \in S$ with $|\alpha - \beta| \geq d$. Let i be such that $|\alpha_i - \beta_i| \geq |\alpha_j - \beta_j|$ for $j = 1, 2, 3$, where α_j is the jth coordinate of α. Then $|\alpha_i - \beta_i| \geq d/3^{1/2} > d/2$.

Let $Q(k) = \prod_{j=1}^{k} \nu_j^{-3/2}$. Then $H_k(\nu)$ is a disjointed family consisting of $Q(k)^{-2/3}$ closed intervals of length $Q(k)$, and $[0, 1] \sim \bigcup H_k(\nu)$ is the union of $Q(k)^{-2/3} - 1$ open intervals of length $[(1 - \nu_j^{-1/2})/(\nu_j - 1)]Q(j-1)$, where j ranges from 1 to k. Furthermore, since $1 - \nu_j^{-1/2} \geq 1 - 2^{-1/2} > 1/4$, $\nu_j - 1 < \nu_j^{3/2}$, it follows that

$$[(1 - \nu_j^{-1/2})/(\nu_j - 1)]Q(j-1) > \nu_j^{-3/2}Q(j-1)/4 = Q(j)/4 \geq Q(k)/4.$$

Therefore, if $J \subset [0, 1]$ is a closed interval such that $\operatorname{diam} J > 3Q(k)/2$, then there exists an open interval $U \subset J$ with

$$U \subset [0, 1] \sim \bigcup H_k(\nu) \subset [0, 1] \sim A(\nu)$$

and $\operatorname{diam} U > Q(k)/4$. Consequently, if we choose k satisfying $d > 3Q(k)$ and $d \leq 3Q(k-1)$, then, since $|\alpha_i - \beta_i| > d/2$, there exists an open interval $I \subset [\alpha_i, \beta_i]$ such that $I \subset [0, 1] \sim A(\nu)$ and

$$\operatorname{diam} I > Q(k)/4 = \nu_k^{-3/2}Q(k-1)/4 \geq \nu_k^{-3/2}d/12 \geq \xi^{-3/2}d/12,$$

where ξ is the least upper bound of the sequence ν. Let $r = \xi^{-3/2}/24$ and let G be the set of all x for which x_i is the midpoint of I. Observe that $S \cap G \neq \varnothing$, since α, β are on opposite sides of G, and S is connected. Choose $a \in S \cap G$. Then $\operatorname{distance}(a, K) > rd$, since $K \cap \{x\colon x_i \in I\} = \varnothing$. Let N be a supporting line of B at $\theta(a)$ and $D = \theta^{-1}(N)$. Then D is a supporting plane of M at a. Since M is the convex hull of K, $D \cap S$ is convex and $\operatorname{distance}(a, K) > rd$, it follows that there exists $b \in D \cap S$ with $[a, b] \subset D \cap S \subset S$, $|a - b| = rd$.

At this point we will divide the proof into cases and subcases in each of which it will be shown that there exists a number less than 1, depending only on ν, which multiplied by $t^2(M)$ is greater than or equal to $\mathfrak{L}^2(B)$. We will then let s be the largest of these numbers among all cases.

We first divide the remainder of the proof into two cases:

Case I. $|\theta(a) - \theta(b)| > 2^{-9}r^3d$.

We use Lemma 3.1 with k, $L \cap Y$ replaced by $2^{-9}r^3$, $[\theta(a), \theta(b)]$ to obtain that $\mathfrak{L}^2(B) \leq t^2(B)/(1 + 2^{-46}r^{12})$. Furthermore, we note that $t^2(B) \leq t^2(M)$, since $\|\bigwedge_2 \theta\| = 1$. We then conclude that

(7) Case I implies $\mathfrak{L}^2(B) \leq t^2(M)/(1 + 2^{-46}r^{12})$.

Case II. $\|\theta(a) - \theta(b)\| \leq 2^{-9}r^3d$.

Let $\lambda = b - a$. Choose orthonormal basis vectors e_1, e_2, e_3 for $\mathbf{R}^3$ so that $\operatorname{kernel}(\theta) = \mathbf{R}e_3$ and $\lambda \cdot e_1 = 0$. As a result of this choice and the fact that $r < 1$ it follows that $\lambda = p_2e_2 + p_3e_3$ with p_2, p_3 satisfying $|p_2| \leq 2^{-9}r^3d$, $|p_3| > (1 - 2^{-18})^{1/2}rd > rd/2$, $|p_2/p_3| < 2^{-8}$. Let m be the midpoint of $[\theta(a), \theta(b)]$. Choose $w \in S - S$, $w = q_1e_1 + q_2e_2 + q_3e_3$, satisfying $|\theta(w)| = d$.

We consider now four subcases of Case II:

Case II.A. $|q_3| > d$.

Choose $z \in S - S$ satisfying $\theta(z) \cdot \theta(w) = 0$ and

$$|\theta(z)| = \sup\{|v| : v \in Y - Y \text{ and } v \cdot \theta(w) = 0\}.$$

Using [5, 1.15(7)] we obtain that

$$4t^2(M)/\pi \geq |w \wedge z| \geq [|w|/|\theta(w)|]|\theta(w) \wedge \theta(z)|$$
$$> 2^{1/2}|\theta(w) \wedge \theta(z)| = 2^{1/2}|\theta(w)| \cdot |\theta(z)| \geq 2^{1/2}\mathfrak{L}^2(B),$$

hence

(8) Case II.A implies $\mathfrak{L}^2(B) \leq 4t^2(M)/(2^{1/2}\pi)$.

Case II.B. $|q_3| \leq d$ *and* $\mathfrak{L}^2(B) < rd^2/4$.

We deduce that

$$4t^2(M)/\pi \geq |\lambda \wedge w| = [(p_2q_1)^2 + (p_3q_1)^2 + (p_2q_3 - p_3q_2)^2]^{1/2}$$
$$\geq [(p_3q_1)^2 + (p_3q_2)^2 - 2p_2p_3q_2q_3]^{1/2} = |p_3|[q_1^2 + q_2^2 - 2(p_2/p_3)q_2q_3]^{1/2}$$
$$> (rd/2)(d^2 - 2^{-7}d^2)^{1/2} > 3rd^2/8 > 3\mathfrak{L}^2(B)/2,$$

where the fifth relation in this chain follows from the conditions $|p_3| > rd/2$, $q_1^2 + q_2^2 = d^2$, $|p_2/p_3| < 2^{-8}$, $|q_2| \leq d$, $|q_3| \leq d$. Therefore,

(9) Case II.B implies $\mathfrak{L}^2(B) \leq 8t^2(M)/(3\pi)$.

Case II.C. $\mathfrak{L}^2(B) \geq rd^2/4$, *and* $|(x - m) \wedge v| \leq 4(1 - 2^{-8}r^2)t^2(B)/\pi$ *for all* $x \in B$, $v \in B - B$.

Let $\rho = 2^{-12}r^3d$, $W = B \cup \mathbf{B}(m, \rho)$. We take any $u_1, u_2 \in W - W$ and consider two possibilities:

First, if $u_1, u_2 \in B - B$, then clearly $(\pi/4)|u_1 \wedge u_2| \leq t^2(B)$.

On the other hand, suppose at least one of u_1, u_2, say u_1 for the sake of argument, is not in $B - B$. Then $u_1 = u_3 + u_4$, $u_2 = u_5 + u_6$, where $u_3 = x - m$ for some $x \in B$, $|u_4| \leq \rho$, $u_5 \in B - B$, $|u_6| \leq 2\rho$. We also note that $r < 1$, $|u_3 \wedge u_5| \leq 4(1 - 2^{-8}r^2)t^2(B)/\pi$, $rd^2/4 \leq \mathfrak{L}^2(B) \leq t^2(B)$ by [3, 2.10.32], and then obtain

$$|u_1 \wedge u_2| \leq |u_3 \wedge u_5| + |u_3| \cdot |u_6| + |u_4| \cdot |u_5| + |u_4| \cdot |u_6|$$
$$\leq |u_3 \wedge u_5| + 3\rho d + 2\rho^2 < |u_3 \wedge u_5| + 4\rho d < 4t^2(B)/\pi.$$

Consequently, $t^2(W) \leq t^2(B)$. Furthermore,

$$\mathfrak{L}^2(W) \geq \mathfrak{L}^2(B) + \pi\rho^2/2 \geq (1 + 2^{-25}r^6)\mathfrak{L}^2(B),$$

since $\mathfrak{L}^2(B) \leq \pi d^2$ by [3, 2.10.33]. In addition, $\mathfrak{L}^2(W) \leq t^2(W)$, $t^2(B) \leq t^2(M)$. We combine all these inequalities and conclude that

(10) Case II.C implies $\mathfrak{L}^2(\mathcal{B}) \le t^2(M)/(1+2^{-25}r^6)$.

Case II.D. $\mathfrak{L}^2(B) \ge rd^2/4$, *and there exists* $y \in Y$, $v_1, v_2 \in Y - Y$ *with* $v_1 = y - m$, *such that* $|v_1 \wedge v_2| > 4(1 - 2^{-8}r^2)t^2(B)/\pi$.

Take any $\tau \in S \cap \theta^{-1}\{y\}$. Then choose $\zeta \in \{a-\tau,\ b-\tau\}$, $\zeta = k_1e_1 + k_2e_2 + k_3e_3$, satisfying $|k_3| \ge rd/4$, and $\eta \in S - S$ satisfying $\theta(\eta) = v_2$. Let $v_3 = \theta(\zeta)$. We observe that $|v_1 - v_3| = |p_2|/2 \le 2^{-10}r^3d$, $t^2(B) \ge rd^2/4$, and then deduce

$$\begin{aligned}|v_3 \wedge v_2| &\ge |v_1 \wedge v_2| - |v_1 - v_3| \cdot |v_2|\\ &\ge 4(1-2^{-8}r^2)t^2(B)/\pi - 2^{-10}r^3d^2 > 4(1-2^{-7}r^2)t^2(B)/\pi.\end{aligned}$$

Furthermore, $|\zeta|/|v_3| > 1 + 2^{-6}r^2$, since $|k_3| \ge rd/4$. These last two results combined with [5, 1.15(7)] and the inequalities $r < 1$, $\mathfrak{L}^2(B) \le t^2(B)$ yield

$$\begin{aligned}t^2(M) &\ge (\pi/4)|\zeta \wedge \eta|\\ &\ge (\pi/4)(|\zeta|/|v_3|)|v_3 \wedge v_2| > (1+2^{-8}r^2)\mathfrak{L}^2(B).\end{aligned}$$

Therefore,

(11) Case II.D implies $\mathfrak{L}^2(B) \le t^2(M)/(1+2^{-8}r^2)$.

We now finish the proof of Theorem 3.2 by letting $s = 1/(1+2^{-46}r^{12})$ and then using (7), (8), (9), (10), (11) to conclude that $c^2(M) \le st^2(M)$.

3.3. **Theorem.** $0 < \mathcal{T}^2(E) < \infty$.

Proof. From [3, 2.10.28] we see that $\mathcal{H}^{2/3}[A(\nu)] = \alpha(2/3)2^{-2/3}$; consequently, repeated application of [3, 2.10.27] yields

$$\begin{aligned}\mathcal{H}^2(E) &\ge \alpha(2)[\alpha(4/3)\alpha(2/3)]^{-1}\mathcal{H}^{4/3}[A(\nu)\times A(\nu)]\mathcal{H}^{2/3}[A(\nu)]\\ &\ge \alpha(2)\alpha(2/3)^{-3}\mathcal{H}^{2/3}[A(\nu)]\times\mathcal{H}^{2/3}[A(\nu)]\times\mathcal{H}^{2/3}[A(\nu)] = \pi/4.\end{aligned}$$

Furthermore, $\mathcal{T}^2(E) \ge \mathcal{H}^2(E)/6$ by [3, 2.10.39, 2.10.6]. Therefore, $\mathcal{T}^2(E) \ge \pi/24$.

Let $P(j) = \prod_{i=1}^{j} \nu_i^3$. Given any $\delta > 0$ choose k so that $3^{1/2}P(k)^{-1/2} < \delta$. $H_k(\nu)\times H_k(\nu)\times H_k(\nu)$ covers E and consists of $P(k)$ cubes D_j of diameter $3^{1/2}P(k)^{-1/2}$. Therefore,

$$\sum_{j=1}^{P(k)} t^2(D_j) \le (\pi/4)\sum_{j=1}^{P(k)} (\operatorname{diam} D_j)^2 = (\pi/4)\sum_{j=1}^{P(k)} 3P(k)^{-1} = 3\pi/4$$

and hence $\mathcal{T}^2(E) \le 3\pi/4$.

3.4. **Main Theorem.** $\mathcal{C}^2(S) < \mathcal{T}^2(S)$ *for all* S *in* X.

Proof. This follows directly from Theorems 3.2, 3.3 and the definitions of $\mathcal{C}^2$ and $\mathcal{T}^2$.

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