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ON BOUNDED OSCILLATION AND ASYMPTOTIC EXPANSION OF CONFORMAL STRIP MAPPINGS

BY

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ABSTRACT. Relations between the boundary parameters ϕ_-, ϕ_+ of a strip $S = \{\phi_-(x) < y < \phi_+(x)\}$ and the values $f(x)$ of its canonical conformal mapping onto a horizontal strip $H = \{|v| < 1\}$ are studied. Bounded oscillation ($\max_y \operatorname{Re} f(x+iy) - \min_y \operatorname{Re} f(x+iy) = \omega(x) = O(1)$) is characterized in terms of ϕ_-, ϕ_+. A formal series expansion $v = \sum y^m a_{m,n}(x)$ is derived for the solution to the Dirichlet problem on S and its partial sums are used to obtain formulas for the asymptotic expansion of f in terms of ϕ_+, ϕ_-.

1. **Introduction.** The relationship between the geometry of a "strip" S and the behavior of its "conformal strip mapping" f from S onto a fixed unit strip H is one of the most beautiful examples of geometric function theory. Both facets of this geometric-analytic structure are illuminated in two fundamental papers by Ahlfors [1] in 1930 and Warschawski [12] in 1942. Although the subsequent development of this structure is too rich to describe here, the additional reference to the length-area methods of Eke [4] or the extremal length methods of Jenkins and Oikawa [7] on the one side and the quasiconformal distortion method used by Gol'dberg and Stročik [6] on the other side should provide a proper setting and foundation for the results in this paper. (We refer to Lelong-Ferrand [8] for additional historical references and basic tools.) (Gaier [5] has also recently used extremal length to study this problem.)

The main results, which are indicated by the title and stated precisely in §4, are based on two observations made with regard to the application of the two methods mentioned in the first paragraph. The first is that the "horizontal truncation" (see §5) used by Jenkins and Oikawa can also be used in a "vertical" sense. The result is a geometric characterization in terms of certain squares in S of a type of analytic behavior of f termed *bounded oscillation.* The second observation is that a naive algorithm can be used to supply formulas which represent a formal double series solution to the Dirichlet problem

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$v = \Sigma\, y^m a_{m,n}(x)$ in terms of the higher order derivatives of ϕ_-, ϕ_+ and that these can be used to obtain formulas for the *asymptotic expansion* of the conformal strip mapping f by using the quasiconformal distortion method. (The specific realization of this method here is the Theorem of Teichmüller, Wittich and Belinski.)

2. **Definitions.** In this paper we shall take the simplistic point of view that by definition a *strip* S has the form

$$S = \{x + iy : \phi_-(x) < y < \phi_+(x)\},$$

where the boundary parameters ϕ_-, ϕ_+ are upper, lower semicontinuous real valued functions. *A canonical conformal strip mapping* f is the unique analytic homeomorphism of S onto $H = \{u + iv : |v| < 1\}$ which carries the triple of prime ends $(-\infty, i\phi_+(0), +\infty)$ into $(-\infty, i, +\infty)$. It has a unique extension to a homeomorphism between the closed strips which we continue to denote by f. Since we are interested mainly in the behavior of f near $+\infty$ only the normalization $f(+\infty) = +\infty$ is actually relevant. That is, changes in S outside a neighborhood of $+\infty$ or in the normalization of f have the essential effect of adding a constant to the values of f near $+\infty$. Fundamental notations used by Ahlfors and Warschawski to describe the geometric and analytic connections between S and f are as follows:

$\theta(x) = \phi_+(x) - \phi_-(x)$,

$\psi(x) = \frac{1}{2}(\phi_+(x) + \phi_-(x))$,

$\overline{u}(x) = \overline{u}(x; f) = \sup_y \operatorname{Re} f(x + iy)$,

$\underline{u}(x) = \underline{u}(x; f) = \inf_y \operatorname{Re} f(x + iy)$,

$\omega(x) = \omega(x; f) = \overline{u}(x) - \underline{u}(x)$.

Since we refer to ω as the *(real) oscillation* of f, *(absolute) bounded oscillation (near* $+\infty$*)* means $\omega(x) = O(1)$ (near $+\infty$). An alternate point of view, trivially equivalent but instructive, is that bounded oscillation means $\operatorname{Re} f(z) = \operatorname{Re} f(x + iy(x)) + O(1)$ for some path $x + iy(x)$. In this sense the strip acts like an asymptotic track for the growth of f and ω is a bound on the error of the asymptotic estimate $\operatorname{Re} f(x + iy(x))$. We are also interested in two other cases where the growth of ω is of the order $\omega(x) = o(1)$, $\omega(x) = o(\underline{u}(x))$, which we term *(absolute, relative) negligible oscillation.*

Throughout most of this paper we shall operate under the additional simplifying assumption that S is symmetric in both axes, which means

(Y) $\theta(x) \equiv \theta(-x)$ (y-axis),

(X) $\psi(x) \equiv 0$ (x-axis).

Symmetry in the y-axis affects the asymptotic behavior of f at $+\infty$ by at most an additive constant. But the symmetry in the x-axis has a much deeper effect because it restricts S near $+\infty$ and hence the adoption of this last restriction requires a little justification. Its obvious advantage, simplification of formulas and some technical details in the proofs, has the effect of facilitating comprehension of the structure. Besides we are biased by our own application of the theory to the classification of Riemann surfaces [10] because there a knowledge of the symmetric case was sufficient. On the other hand we need only cite an example, such as Drasin's application of the theory to the study of asymptotic curves in Denjoy's conjecture [3], to show that a knowledge of the asymmetric case is sometimes necessary. However, it is our contention that the only essential new feature of the asymmetric case is the effect of its nontrivial inclination and that even this complication should have a routine resolution once the theory is known in the symmetric case. Conjectures aside, our methods do apply in the asymmetric case too, but for the reasons cited above we defer discussion of the necessary adjustments in the results in that case to the last section.

In the symmetric case (Y) and (X) we consider $S_\theta = \{0 < y < \theta(x)\}$ and the conformal mapping $f(z) = w$ onto $H = \{0 < v < 1\}$ which leaves the prime ends $-\infty$, 0, $+\infty$ fixed.

3. **The asymptotic expansions of Warschawski and Gol'dberg-Stročik.** In this section we review a continuous version of the theory of Warschawski and Gol'dberg-Stročik in the symmetric case. This is necessary to provide perspective and motivation for the complex form of our formulas.

I (Warschawski). Suppose that $\theta(x)$ has a continuous derivative θ' which satisfies the two conditions

(1)
$$\theta'(x) = o(1),$$

(2)
$$\int_0^\infty \frac{(\theta'(t))^2}{\theta(t)}\, dt < \infty.$$

Strips with condition (1) are called *L-strips* (*of inclination zero*) by Warschawski. He shows that on such strips both

(3)
$$\omega(x; f) = o(1),$$

(4)
$$\operatorname{Im} f(z) = y/\theta(x) + o(1).$$

Under the weaker condition of *bounded* θ*-slope*, which means $\theta'(x) = O(1)$, he establishes an upper bound on Re $f(z)$ and under the assumptions (1), (2), he combines this with (3), (4) and the Ahlfors Distortion Theorem [1] to obtain the fundamental formula (in the symmetric case) that

$$f(z) = \int_0^x \frac{dt}{\theta(t)} + \lambda + i\frac{y}{\theta(x)} + o(1). \tag{5}$$

As we pointed out in [10] this can be proved by considering the family of vertical line segments cut off by S and by using the extremal length method of Jenkins-Oikawa (even in the asymmetric case).

II (Gol′dberg-Stročik). Suppose that θ has a second continuous derivative $\theta''(x)$ which satisfies the three conditions

$$\int_0^\infty |\theta''(t)|\, dt < \infty, \tag{6}$$

$$\liminf \theta(x)\theta''(x) > -2 \quad (\text{at } +\infty), \tag{7}$$

$$\int_0^\infty \theta(t)[\theta''(t)]^2\, dt < \infty. \tag{8}$$

Then the result is that, for $x \geq x_0$,

$$f(z) = \int_0^\tau \frac{\theta'(t)\, dt}{\theta(t) \arctan \theta'(t)} + \lambda + i\eta(z) + o(1), \tag{9}$$

where $\tau = \tau(x, y)$, $\eta = \eta(x, y)$ are (for $x \geq x_0$) the unique solutions to the system of equations

$$\begin{aligned} \theta(\tau)\theta'(\tau)^{-1}(1 + \theta'(\tau)^2)^{1/2} \sin(\eta \arctan \theta'(\tau)) &= y, \\ \tau - \theta(\tau)\theta'(\tau)^{-1} + \theta(\tau)\theta'(\tau)^{-1}(1 + \theta'(\tau)^2)^{1/2} \cos(\eta \arctan \theta'(\tau)) &= x. \end{aligned} \tag{10}$$

Condition (6) is essentially stronger than Warschawski's condition (1) in that it implies θ' has total bounded variation. This implies in particular that

$$\lim \theta'(x) = \tan \alpha \quad (\text{at } +\infty). \tag{11}$$

If $\tan \alpha = 0$, this is Warschawski's condition. If $\tan \alpha \neq 0$ then the simple change of variables $z = e^\zeta$, $\zeta = \log z$, will carry the tail end of this strip onto the tail end of an L-strip. Hence the behavior of f in S_θ in that case would be given, because of (3), on circles. Indeed, consideration of the family of circles orthogonal to $\{y = 0\}$ will lead to the formula of Gol′dberg and Stročik just as the straight lines do in Warschawski's case. Also, an asymmetric version could be proved in the case of Gol′dberg and Stročik,

$$\lim \phi_+'(x) = \tan \alpha_+, \quad \lim \phi_-'(x) = \tan \alpha_-,$$

by their method without essential change.

Condition (7) is used to ensure that the circles do not intersect and the unique solution of (10) for $x \geq x_0$. If $\alpha_+ \equiv \alpha_-$ (mod π) then their formula reduces to the Warschawski formula. If $\alpha_+ \not\equiv \alpha_-$ (mod π) then the Warschawski convergence condition (2) in the ζ-plane transfers to the condition in the z-plane

$$\int_0^\infty \frac{\theta'(t)^2}{\theta(t)} \frac{dt}{t} < \infty. \tag{12}$$

Clearly (2) may hold even if θ'' fails to exist. On the other hand Gol'dberg and Stročik give an example where their conditions (6)–(8) hold while Warschawski's condition (2) fails. In addition, the quasiconformal distortion method used by Gol'dberg and Stročik is extremely simple, natural and elegant.

4. **Statement of the main results.** a. To characterize bounded oscillation in the symmetric case consider, for each real value $t \geq t_0$, the largest two squares $\square_-(t)$, $\square_+(t)$ which lie in S_θ, have sides parallel to the x- and y-axes and whose right, left vertical sides abut along the line $l_t = \{\mathrm{Re}\ z = t,\ 0 < \mathrm{Im}\ z \leq \theta(t)\}$ (see Figure 1),

$$\square_+(t) = \{t < \mathrm{Re}\ z < t + a_+,\ 0 < \mathrm{Im}\ z < a_+\},$$
$$\square_-(t) = \{t - a_- < \mathrm{Re}\ z < t,\ 0 < \mathrm{Im}\ z < a_-\},$$

where

$$a_+ = \sup\{s : \theta(x) \geq s, \text{ for } t \leq x \leq t + s\},$$
$$a_- = \sup\{s : \theta(x) \geq s, \text{ for } t - s \leq x \leq t\}.$$

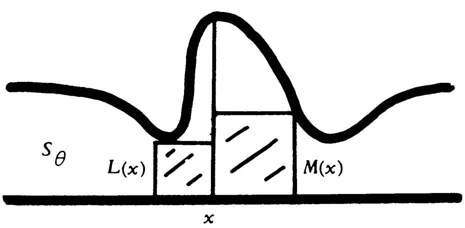


Figure 1

Let $M(t)$, $L(t)$ denote the larger, smaller of the lengths of their sides $M(t) = \max(a_+, a_-)$, $L(t) = \min(a_+, a_-)$. The characterization is then stated as

Theorem 1. $\omega(x) = O(1) \Leftrightarrow M(x)/L(x) = O(1)$.

One interpretation is that the condition is a weak form of bounded difference quotients of $\theta(x)$. Indeed, we note that $\theta'(x) = O(1)$ guarantees this condition by the mean value theorem. Of course, the converse is not valid even if $\theta'(x)$ does exist. We also remark that the characterizing condition differs in an essential way from this in the asymmetric case.

The sufficiency follows from a persistent use and analysis of the Jenkins-Oikawa approach. Necessity follows from a normal family argument and the domain convergence theorem of Carathéodory.

b. To state the next result we assume that θ is infinitely (and hence continuously) differentiable. A naive and formal algorithm discussed in §6 leads to

Theorem 2. *Let θ, f have derivatives of all orders and define the series*

$$H_f(z) = \sum_{m,\, n=0}^{\infty} y^{2m+1} a_{2m+1,\, n}(x), \tag{13}$$

where the $a_{2m+1,n}$ *are defined recursively by the formulas*

$$a_{1,0} = f/\theta \qquad (m = n = 0), \tag{14}$$

$$a_{2m+1,\, n} = - D^2 a_{2m-1,\, n}/(2m)(2m+1) \qquad (m \geq 1), \tag{15}$$

$$a_{1,\, n} = - \sum_{k=1}^{n} \theta^{2k} a_{2k+1,\, n-k} \qquad (n \geq 1), \tag{16}$$

for $D^k = \partial^k/\partial x^k$ *and explicitly by the formula*

$$a_{2m+1,\, n} = \left[\frac{(-1)^{m+1} D^{2m}}{(2m+1)!} \sum_I \prod_j \frac{(-1)^{k_j+1}}{(2k_j+1)!} \, \theta^{2k_j} D^{2k_j} \right] \left(\frac{f}{\theta} \right),$$

$I = \{(k_1, k_2, \cdots, k_n): 0 \leq k_j \leq n,\ k_1 + k_2 + \cdots + k_n = n\}$, *where the product* Π_j *is noncommutative.*

Then if $H = H_f$ *converges absolutely and uniformly on compact subsets of the closed strip* $\mathrm{cl}\, S_\theta = \{0 \leq y \leq \theta(x)\}$, *it is the solution to the Dirichlet problem in* S_θ *with boundary values* f, 0.

This is the symmetric Dirichlet problem. The proof is strictly formal and holds whenever the pointwise limit operator $\lim_{N\to\infty} H_N(z) = H(z)$ exists on the *triangular partial sums*

$$H_N = \sum_{0 \leq m+n \leq N} y^{2m+1} a_{2m+1,\, n}, \tag{17}$$

is differentiable and commutes with the Laplacian operator Δ. As a consequence of the formula and the extremal length method we obtain the following

Corollary. *If* θ *has* $2N$ *continuous derivatives then with* $f \equiv 1$ *let* $v_N = H_N$. *If* θ *also has the bounded oscillation condition of Theorem* 1 *then*

$$\mathrm{Re}\, f(z) \leq \iint_{\{0 \leq s \leq x,\ 0 \leq t \leq \theta(x)\}} |\mathrm{grad}\, v_N|^2 \, ds\, dt + K.$$

Moreover, if $M(x)/L(x) \leq B$ *then* $K \leq 8B^2$.

We remark that with $v_1 = y/\theta$ this corollary, coupled with the Jenkins-Oikawa extremal length proof of the Ahlfors Distortion Theorem, gives an incisive proof of Warschawski's Theorem. We imagine that it is also possible to improve the Gol'dberg-Stročik hypothesis by applying this type of argument.

c. We now assume that θ has $2N+2$ continuous derivatives. We let $f \equiv 1$. Then $a_{1,0} = 1/\theta$ and $a_{2m+1,n}$ are defined above and we let

(18) $$v_N(z) = \sum_{0 \le m+n \le N} y^{2m+1} a_{2m+1,n},$$

(19) $$u_N(z) = \int_0^z \frac{\partial}{\partial y}(v_{N+1} - a_{1,N+1})\,dx - \frac{\partial v_N}{\partial x}\,dy,$$

(20) $$f_N(z) = u_N + iv_N,$$

(21) $$\mu_N(z) = (\partial f_N/\partial \bar{z})/(\partial f_N/\partial z),$$

(22) $$J_N(z) = |\partial f_N/\partial z|^2 - |\partial f_N/\partial \bar{z}|^2,$$

(23) $$k_N(t) = \sup_{x \ge t} |\mu_N(z)|,$$

(24) $$I_N(t) = \iint_{\{x \ge t,\ 0 \le y \le \theta(x)\}} |\mu_N| J_N\,dx\,dy.$$

(We tacitly assume $\partial f_N(z)/\partial z \neq 0$ when we write $\mu_N(z)$.)

With this notation we can state

Theorem 3. *If θ has $2N+2$ continuous derivatives, satisfies the symmetry conditions* (X), (Y) *and satisfies the conditions* $\lim_{z \to +\infty} u_N(z) = +\infty$,

(25) $$k_N(0) = \sup |\mu_N(z)| < 1,$$

(26) $$I_N(0) = \iint_{S_\theta} |\mu_N|\, J_N < \infty,$$

then

(27) $$f(z) = f_N(z) + \lambda_N + \epsilon_N(z),$$

where $|\lambda_N| \le I_N(0)$, $|\epsilon_N(z)| \le \Phi(x;\, k_N,\, I_N)$, *where* $\Phi = \Phi(x;\, k_N,\, I_N)$ *is the explicit function* (51) *in our statement of the Theorem of Teichmüller, Wittich, Belinski and has the following two properties*:

(28) $$\lim_{x \to \infty} \Phi(x;\, K_N,\, I_N) = 0 \qquad (k_N,\, I_N \text{ fixed}),$$

(29) $$\lim_{k_N(x)+I_N(x) \to 0} \Phi(t;\, k_N,\, I_N) = 0 \qquad (x,\, t_1,\, t_2 \text{ fixed},\ x \le t_1 \le t \le t_2).$$

In addition if

(30) $$\omega(x,\, f_N) = o(1),$$

then the formula reduces to

(31) $$\operatorname{Re} f(z) = \int_0^x \sum_{k=0}^{N} a_{1,k}(t)\,dt + \lambda_N + o(1).$$

We remark that the L-strip hypothesis $\theta' = o(1)$ implies (30). The proof of Theorem 3 is to process the formulas by the quasiconformal distortion method.

5. **Proof of Theorem 1.** The sufficiency is a variation on the theme of Jenkins and Oikawa. Much of their argument is repeated here for completeness. The necessity is a Carathéodory domain convergence argument.

a. Consider the quadrangle

$$Q_\theta(t) = \{0 < x < t,\ 0 < y < \theta(x)\}$$

and the associated family $\Gamma_Q = \Gamma(Q_\theta(t))$ of arcs γ which lie in $Q = Q_\theta(t)$ and separate its vertical sides. Let the module of Γ_Q be denoted by any of the symbols

$$m(t) = m_\theta(t) = m(Q_\theta(t)) = m(\Gamma_Q).$$

It follows from the conformal invariance and monotoneity properties of modules that bounded oscillation is equivalent to the condition

$$\text{Re } f(z) = m_\theta(x) + O(1), \tag{32}$$

where $\omega(x)$ bounds the error. We prepare for our proof by noting that

$$m_\theta(t) > 1 \quad (\text{for } t > t_0) \tag{33}$$

because $\lim m_\theta(t) = \infty$. In many cases this follows from the useful inequality, proved in [6] that

$$m_\theta(x) \geq \int_0^x \frac{dt}{\theta(t)}. \tag{34}$$

In general if θ has one finite value, say $\theta(0)$, then let $A_Q = \{c_x \colon \theta(0) \leq x \leq t\}$, $t > \theta(0)$, $c_s = \{|z| = s,\ 0 \leq \arg z \leq \pi/2\}$ and obtain, from $m\Gamma_Q \geq mA_Q$, the inequality $m_\theta(t) \geq (2/\pi)\log(t/\theta(0))$, from which (33) follows.

b. We shall prove in this subsection that

$$m_\theta(t) \leq \underline{u}(t) - \overline{u}(0) + 2 \text{ if } t \geq t_0. \tag{35}$$

To prove this we recall from [7, argument on p. 665 with $a = 1$] that

$$\text{if } \underline{u}(t) - \overline{u}(0) \leq -1, \text{ then } m_\theta(t) \leq 1. \tag{36}$$

But we have assumed (33) that $m_\theta(t) > 1$ so $\underline{u}(t) - \overline{u}(0) > -1$ and by [7, formula (2), p. 665] our formula (35) holds.

c. The second half of the sufficiency is proved via the same approach (which was referred to as the "truncation trick" in the introduction) but in the domain, not in the range. We truncate vertically as well as horizontally. The upper inequality is

(37) $$\overline{u}(t) - \underline{u}(0) \le m_\theta(t) + 4 \sum_{s=0,\, t} [M(s)/L(s)]^2,$$

where $M(s)$, $L(s)$ were defined in §4. To prove it let

(38) $$T_\theta(t) = \{z \in S_\theta : \underline{u}(0) \le \operatorname{Re} f(z) \le \overline{u}(t)\}.$$

Then by conformal invariance

$$m(T_\theta(t)) = m(f(T_\theta(t))) = \overline{u}(t) - \underline{u}(0).$$

Let us define *end boxes* and *end metrics* by setting

$$B_s = \{s - a_- < \operatorname{Re} z < s + a_+,\ 0 \le \operatorname{Im} z \le 2M(s)\},$$

$$\rho_s(z) = \begin{cases} 1/L(s), & \text{if } z \in B_s, \\ 0, & \text{elsewhere}, \end{cases}$$

for $s = 0,\ t$. Then if $\rho_*(z)|dz|$ denotes the length-normalized extremal metric for $m_\theta(t)$ we set

(39) $$\sigma(z) = \max(\rho_0(z), \rho_*(z), \rho_t(z))$$

and prove the following

Lemma. *$\sigma(z)|dz|$ is an admissible metric in the length normalized module problem for arcs in T which join $\{y = 0\}$ to $\{y = \theta(x)\}$.*

Proof. For every arc γ in T which joins $\{y = 0\}$ to $\{y = \theta(x)\}$ we must prove that

(40) $$\int_\gamma \sigma(z)\,|dz| \ge 1.$$

If $\gamma \le Q$ then $\sigma \ge \rho_*$ implies (40). Thus we may as well assume that γ must meet $T - Q$. That is, γ meets $\{\operatorname{Re} z < 0\}$ or $\{\operatorname{Re} z > t\}$ and because of the symmetry of the argument here we restrict our attention to the latter case. It is useful to let c_-, c_+ denote points in $[t - a_-, t]$, $[t, t + a_+]$ for which $\theta(c_-) \le M(t)$, $\theta(c_+) \le M(t)$ and to define the following six arcs (see Figure 2):

$$\alpha_- = \{x = t - a_-,\ 0 \le y \le 2M(t)\} \cup \{t - a_- \le t \le c_-,\ y = 2M(t)\},$$

$$\beta_- = \{x = c_-,\ \theta(c_-) \le t \le 2M(t)\},$$

$$\gamma_- = \{c_- \le x \le t,\ y = 2M(t)\} \cup \{x = t,\ 0 \le y \le 2M(t)\};$$

$$\gamma_+ = \{x = t,\ 0 \le y \le 2M(t)\} \cup \{t \le x \le c_+,\ y = 2M(t)\},$$

$$\beta_+ = \{x = c_+,\ \theta(c_+) \le y \le 2M(t)\},$$

$$\alpha_+ = \{c_+ \le x \le t + a_+,\ y = 2M(t)\} \cup \{x = t + a_+,\ 0 \le y \le 2M(t)\}.$$

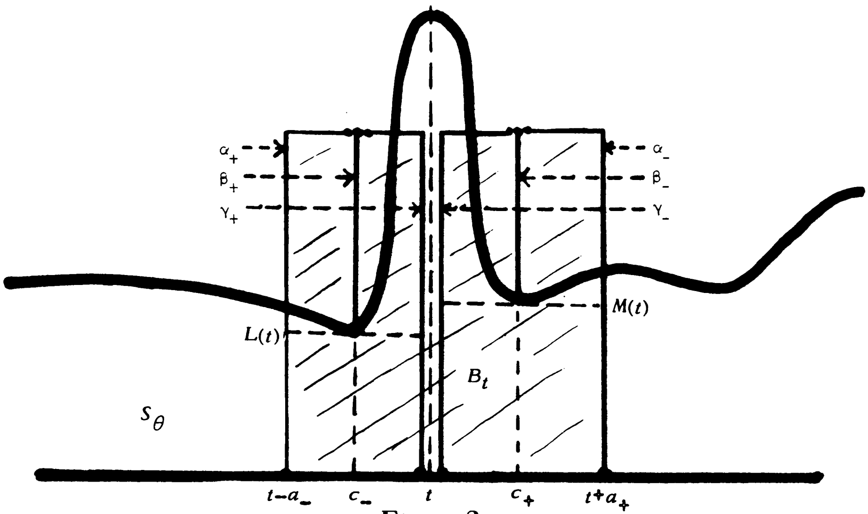


Figure 2

We then consider four cases according to the location of the initial point $\gamma(0) \in \{y = 0\}$.

I. $\gamma(0) \leq t - a_-$.

II. $t - a_- < \gamma(0) \leq t$.

III. $t \leq \gamma(0) < t + a_+$.

IV. $t + a_+ \leq \gamma(0)$.

In cases I, II, γ begins in $\{\text{Re } z \leq t\}$ and we assumed that γ must meet $\{\text{Re } z \geq t\}$. In cases IV, III, γ begins in $\{\text{Re } z \geq t\}$ and it must meet $\{\text{Re } z \leq t\}$ because the continuum $\{z \in S_\theta \leq \text{Re } f(z) = \overline{u}(t)\}$ blocks every other path from $\{y = 0\}$ to $\{y = \theta(x)\}$ in T (in these two cases). From this point on the argument is symmetric ("in the line $\{\text{Re } z = t\}$") with case I corresponding to case IV and case II to case III. Hence we need only present cases I and II.

I. $\gamma(0) \leq t - a_-$. Since γ must meet $\{\text{Re } z \geq t\}$ it must contain a subarc in $B_t \cap \{\text{Re } z \leq t\} - \beta_-$ which connects α_- to γ_-. Since this subarc's Euclidean length is at least $L(t)$ we have (40).

II. $t - a_- < \gamma(0) \leq t$. There are four subcases.

A. An initial arc of γ connects $\{y = 0\}$ to α_- in $\{\text{Re } z < t\} \cap B_t$. Because γ eventually meets $\{\text{Re } z > t\}$ the argument of case I applies to this γ.

B. An initial subarc of γ meets $\{\text{Re } z \geq t\}$ and then connects to α_- in B_t. This subarc has Euclidean length at least $L(t)$.

C. An initial subarc of γ connects $\{y = 0\}$ to $\{\text{Im } z = 2M(t)\} \cup \{\text{Re } z = t + a_+\}$ in B_t and has Euclidean length at least $M(t) \geq L(t)$.

D. γ is contained in B_t and hence must have Euclidean length at least $L(t)$. Q.E.D.

d. To prove the necessity of Theorem 1 suppose $M(t_n)/L(t_n) \nearrow \infty$ for some sequence $t_n \nearrow \infty$. Then we may assume for definiteness that $M(t_n) = a_+(t_n)$, $L(t_n) = a_-(t_n)$ since the argument is symmetric. Let x_n be chosen such that $(x_n, L(t_n)) = (x_n, \theta(x_n))$ for $t_n - L(t_n) \le x_n \le t_n$. Then for $n \ge n_0$ define

$$F_n(\zeta) = f(M(t_n)\zeta + x_n) - f(M(t_n)\tfrac{1}{2} + x_n), \qquad D_n = \{\zeta = (z - x_n)/M(t_n) : z \in S_\theta\}.$$

These domains D_n, their F_n-images H and the mappings F_n can be extended by reflection in the x-axis which we continue to denote by D_n, F_n, H. Now every domain D_n $(n \ge n_0)$ contains the disk $\Delta = \{|\zeta - \tfrac{1}{2}| < \tfrac{1}{4}\}$ and omits the points $(0, 1)$, $(0, -1)$. But $F_n(D_n) = H$, $F_n(\tfrac{1}{2}) = 0$ and hence $\{F_n^{-1}\}$ is a normal family. Thus by selecting a suitable subsequence we may assume that F_n^{-1} converges to F^{-1}. By Carathéodory's Theorem, $D_n = F_n^{-1}(H)$ converges to its kernel $D = F^{-1}(H)$ uniformly on compact subsets of D.

Now it follows that the limit function F and its limit domain have these properties:

$$D \subseteq \{\mathrm{Re}\ \zeta > 0\},$$
$$\{\xi = 0,\ |\eta| \le 1\} \subseteq \partial D,$$
$$F\{\xi = 0,\ 0 < \eta \le 1\} \subseteq \{\mathrm{Im}\ z = 1\},$$
$$F\{\xi = 0,\ -1 \le \eta < 0\} \subseteq \{\mathrm{Im}\ z = -1\}.$$

Consequently $F(0) = -\infty$. Hence

$$\mathrm{Re}\ F(\xi) \longrightarrow -\infty, \text{ as } \xi \searrow 0,$$
$$\mathrm{Re}\ F(\xi + \tfrac{1}{2} i) \longrightarrow K_0 > -\infty, \quad \text{as } \xi \searrow 0.$$

Therefore $\omega(\xi, F) \to \infty$ and it follows that, given $M > 0$, $\omega(\xi_k; F) > 2M$ for some k, $\omega(\xi_k; F_n) > M$ for some $n = n(k)$, and hence $\omega(x_n + M(t_n)\xi_k; f) > M$ which proves the oscillation is unbounded.

e. Comment. The sufficiency is a "hard" analytic inequality proof, the necessity is a "soft" analytic proof. A "hard" analytic proof of the necessity would be more enlightening.

6. **Proof of Theorem 2—an algorithm. a.** Let $a_{1,0} = f/\theta$ and set $H_1 = ya_{1,0}$. Note that H_1 has boundary values f, 0 but is not quite harmonic. In fact if $D^k = \partial^k/\partial x^k$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ then $\Delta H_1 = \Delta(ya_{1,0}) = yD^2 a_{1,0}$. Therefore we add the term $-(y^3/3!)D^2 a_{1,0}$ to cancel this and note that

$$\Delta(ya_{1,0} - (y^3/3!)D^2 a_{1,0}) = -(y^3/3!)D^4 a_{1,0}.$$

Continuation of this process leads to the formal series

$$\sum_{m=0}^{\infty} y^{2m+1} a_{2m+1,0} = \sum_{m=0}^{\infty} (-1)^m \frac{y^{2m+1}}{(2m+1)!} D^{2m} a_{1,0}$$

which we call the *harmonic completion* of $ya_{1,0}$ and which is represented by the first column in the following array:

$$
\begin{array}{llll}
y\Big\{\dfrac{f}{\theta} & +\left[\dfrac{\theta^2}{3!}D^2\left(\dfrac{f}{\theta}\right)\right] & +\left[-\dfrac{\theta^4}{5!}D^4\left(\dfrac{f}{\theta}\right)+\dfrac{\theta^2}{3!}D^2\left(\dfrac{\theta^2}{3!}D^2\left(\dfrac{f}{\theta}\right)\right)\right] & +\cdots\Big\} \\
\quad\downarrow\!\uparrow \quad \nearrow & \quad\downarrow\!\uparrow \quad \nearrow & \quad\downarrow\!\uparrow \quad \nearrow & \\
-\dfrac{y^3}{3!}\Big\{D^2\left(\dfrac{f}{\theta}\right) & +\left[D^2\left(\dfrac{\theta^2}{3!}D^2\left(\dfrac{f}{\theta}\right)\right)\right] & +\cdots & \Big\} \\
\quad\downarrow\!\uparrow \quad \nearrow & \quad\downarrow\!\uparrow \quad \nearrow & & \\
+\dfrac{y^5}{5!}\Big\{D^4\left(\dfrac{f}{\theta}\right) & +\cdots & & \Big\} \\
\quad\downarrow\!\uparrow \quad \nearrow & & & \\
+\cdots & & &
\end{array}
\tag{41}
$$

Although the first column is formally harmonic, it may no longer have the correct boundary values even when it converges. Thus we add the first element in the second column $y(\theta^2/3!)D^2(f/\theta)$ in order to cancel boundary values of the second element in the first column. We then add the terms in the second column which are the terms in the harmonic completion of $y(\theta^2/3!)D^2(f/\theta)$. We then add the first element in the third column to cancel the boundary values of the third element in the first column and the second element in the second column. Continuation of this process leads to the array (41) which is formally harmonic and has the boundary values f, 0 on $\{y = \theta\}$, $\{y = 0\}$ if the double series converges. Double arrows indicate formal cancellation under the Laplacian. Diagonal arrows indicate formal cancellation of boundary values. This completes the proof of Theorem 2.

b. There are several remarks we wish to make. First, we shall ignore examination of the obvious question, when does the series converge. Second, the bracketed terms may be arranged in a different order and we may add the terms to cancel boundary values in a different order. Third, starting with any term $a(x, y)$ which satisfies $a(x, 0) \equiv 0$ we may proceed with a two-dimensional analog to this algorithm. Fourth, there are other boundary value problems which could be treated in this naive way.

Fifth, if the series actually represents $H(x_0 + iy)$ at x_0, then as y varies its values depend only on the sequence of derivatives $\{D^k(f/\theta)(x_0)\}$. However, we know that in general, with proper interpretation

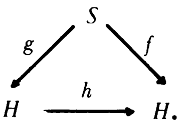
$$H = \int_{\partial S_\theta} f^* \, dg,$$

where g denotes the Green's function. Hence a change in f or θ on a neighborhood disjoint from x_0 should in general affect the value of H at $x_0 + iy$. Consequently we would not expect convergence unless f, θ were real analytic.

c. Proof of the corollary. By Theorem 1 we know that (32) holds. Now if $h \equiv 1$, $a_{1,0} = 1/\theta$ we denote the Nth triangular partial sum (18) by $v_N = H_N$. Since v_N has boundary values 1, 0 and is continuously differentiable we note that $|\text{grad } v_N|\,|dz|$ is an admissible metric in the length-normalized module problem for $\Gamma(Q_\theta(t))$ and the corollary follows.

7. **The quasiconformal approximation method.**

a. The method used by Gol′dberg and Stročik is essentially an application of the Theorem of Teichmüller, Wittich and Belinski, which we shall refer to as a particular application of the *quasiconformal approximation method.* Consider the following commutative diagram



We are given $f: S \to H$. We find an explicit quasiconformal function g such that $h = f \circ g^{-1}$ satisfies the hypothesis of the Theorem of Teichmüller, Wittich and Belinski. Its conclusion is that $h(\zeta) = \zeta + \lambda + O(1)$. Consequently $f = g + \lambda + O(1)$.

b. Our source for this theorem will be the book *Quasikonforme Abbildungen* by Lehto and Virtanen [9]. However, we want to sharpen its statement (as it appears there) for two reasons. First, we wish to make it a "hard" explicit theorem, so we must compute certain constants. Second, we wish to emphasize that as we take g closer to being conformal the corresponding constant $\lambda \to 0$ and the error term $\to 0$. This will not require a change in the proof.

We assume that $w = h(\zeta)$ is a quasiconformal mapping of H onto H with complex dilatation $\mu(\zeta)$ such that

$$h(+\infty) = +\infty, \tag{42}$$

$$h(-\bar{\zeta}) = -\bar{h}(\zeta), \tag{43}$$

$$k(t) = \sup_{|\xi|>t} |\mu(\zeta)| < 1 \qquad \text{(all } t), \tag{44}$$

$$I(t) = \iint_{|\xi|>t} |\mu(\zeta)|\, d\xi\, d\eta < 1 \quad \text{(all } t). \tag{45}$$

We shall let $K(t) = (1 + k(t))/(1 - k(t))$. We recall [7, p. 67] using $h(\zeta - s)$ that the oscillation satisfies

$$\omega(t, h) \le \Lambda(t) = \inf_{1 \ge s \ge t} \log \frac{\phi_{K(s)}(t/s)}{\phi_{1/K(s)}(t/s)} \tag{46}$$

where, for $0 < r < 1$, $K > 1$,

$$\phi_K(r) = l^{-1}(l(r)/K), \qquad l(r) = \pi L((1 - r^2)^{1/2})/(2L(r)),$$
$$L(r) = \int_0^1 [(1 - x^2)/(1 - r^2x^2)]^{-1/2}\, dx.$$

Hence if

$$k(t) = o(1) \tag{47}$$

then $\omega(t) = o(1)$. Now we can state

Theorem A (Teichmüller-Wittich-Belinski). *If h maps H onto H quasiconformally and satisfies conditions* (42)–(45) *then*

$$h(\zeta) = \zeta + \lambda + \epsilon(\zeta), \tag{48}$$

where

$$|\lambda| \le I(0), \tag{49}$$

$$|\epsilon(\zeta)| \le \Phi(\xi;\, k,\, I) \qquad (\xi \ge \xi_0), \tag{50}$$

where $\Phi = \Phi(\xi;\, k,\, I)$ has the properties (28), (29) *because it has the explicit expression*

$$\begin{aligned} \Phi &= C_{K(0)} \max\left(I(t),\, I(t)^{1/2},\, \Lambda(t),\, \Lambda(t)^{1/2}\right), \\ C_{K(0)} &= 7 + 10\pi + 32 e^{\pi/K(0)} \phi_{K(0)}(e^{-1})^K [1 + \phi_{K(0)}(e^{-1})^{2\pi}]. \end{aligned} \tag{51}$$

The function Φ is not the best possible, but does have the two properties we desire. The proof is in [9, pp. 230–244]. However, to obtain this explicit Φ we must first compute a Hölder constant "C" which appears in [9, (6.25), p. 238]. Use the fact that $T(\zeta) = (ie^{\pi\zeta} - 1)/(ie^{\pi\zeta} + 1)$ maps H onto $E = \{|z| < 1\}$ to obtain, from the Hölder inequality on the unit disk, the fact that

$$|Th(\zeta_1) - Th(\zeta_2)| \le 16 |T(\zeta_1) - T(\zeta_2)|^{1/K},$$

where here $K = K(0)$. As they pointed out in [9, p. 238] we may assume by using translations that $|\mathrm{Re}\ \zeta| \le 1$. Then using the basic error bounds

$$|\mathrm{Re}\ h(\zeta)| \le \log [\phi_K(e^{-1})]^{-1},$$
$$|ie^{\pi\zeta} + 1| \ge 1 \qquad (\zeta \in H),$$
$$|\zeta_1 - \zeta_2| e^{\min(|\zeta_1|, |\zeta_2|)} \le |e^{\zeta_1} - e^{\zeta_2}| \le |\zeta_1 - \zeta_2| e^{\max(|\zeta_1|, |\zeta_2|)},$$

we obtain

$$|h(\zeta_1) - h(\zeta_2)| \le M_K|\zeta_1 - \zeta_2|^{1/K},$$
$$M_K = 16e^{\pi/K}\phi_K(e^{-1})^{\pi}[1 + \phi_K(e^{-1})^{2\pi}], \quad |\zeta_1 - \zeta_2| \le 1, \ \zeta_1, \zeta_2 \in H,$$

where M_K is called the Hölder constant.

Now if ϵ_1 is any number such that $|\mathrm{Re}\,(h(\zeta_1) - h(\zeta_2))| < \epsilon_1$, then they show [9, pp. 239–242] that $|\mathrm{Im}\,(h(\zeta_1) - h(\zeta_2))| < \epsilon_5$, where ϵ_5 is an explicit function of ϵ_1. We find that if $\delta = \max\,(I(t), I(t)^{1/2}, \Lambda(t), \Lambda(t)^{1/2})$ then

$$\epsilon_1 \le \delta, \quad \epsilon_2 \le \delta, \quad \epsilon_3 \le 2M_K\delta^{1/K} + 4\pi\delta + 2\delta,$$
$$\epsilon_4 = 2\pi(2(x - x_1 + \epsilon_1)(x - x_1 + I(t)/2\pi)^{1/2}) - 2\sqrt{2}\pi(x - x_1) \le 2\sqrt{2}\pi\delta,$$
$$\epsilon_5 = 2\epsilon_1 + 2\epsilon_3 + 2^{-1}\pi\epsilon_4 \le (6 + 10\pi + 4M_K)\delta^{1/K}.$$

c. The question of a necessary and sufficient condition for this behavior is open. Reich and Walczak have some partial results [11]. In particular, this condition $I(0) < \infty$ is not necessary.

8. **Proof of Theorem 3.** a. The problem is to find a function g in the commutative diagram of §7a for which $h = f \circ g^{-1}$ satisfies the hypothesis in Theorem A. We take f_N from formula (20) for g when θ is $2N + 2$ times differentiable and satisfies conditions (X), (Y), (25), (26).

Lemma 1. *f_N is a homeomorphism:*

This follows by the usual argument (i.e. Ahlfors [2, p. 73]) which we repeat here. The condition $k(0) < 1$ implies that f_N has a positive Jacobian on cl S_θ the closure of S_θ. Hence f_N is locally homeomorphic on cl S_θ and therefore globally homeomorphic on ∂S_θ, the boundary of S_θ. Moreover, $f_N \to \pm\infty$ at $\pm\infty$. Hence f_N makes S_θ a smooth unlimited covering surface over H and it follows from the monodromy theorem that h is a homeomorphism.

Thus $|\mu(z; f_N)| = |\mu(\zeta; f_N^{-1})|$ for $\zeta = f_N(z)$ implies that $h = f \circ f_N^{-1}$ is a $K(0)$-quasiconformal mapping of H onto H $(K(0) = (1 + k(0))/(1 - k(0)))$. It may be useful to note that Theorem 1 holds for quasiconformal mappings too. What we need here is

Lemma 2. *Let g be a K-quasiconformal mapping of S_θ onto H, which satisfies the symmetry conditions* (X), (Y). *Then*

$$\omega(x; g) \le 8K \sup[M(t)/L(t)]^2 + 2K^2. \tag{52}$$

Recall the sufficiency part of the proof of Theorem 1 in §5. First $mQ_\theta(t) \le Kmg(Q_\theta(t))$ because g is K-quasiconformal. So by the same proof, $mhQ_\theta(t) \le \underline{u}(t, h) - \overline{u}(0, h) + 2$ provided $\underline{u}(t, h) - \overline{u}(0, h) > -1$. But this follows for t large because if $\underline{u}(t, b) - \overline{u}(0, b) \leq 1$ then

$$1 \geq mb(Q_\theta(t)) \geq K^{-1} mQ_\theta(t),$$

which cannot hold for all t. On the other hand if $T = \{z \in S_\theta \leq \underline{u}(0) < \operatorname{Re} b < \overline{u}(t)\}$ then, as before,

$$\overline{u}(t) = \underline{u}(0) \leq KmT, \qquad mT \leq mQ + 4(M(0)/L(0))^2 + 4(M(t)/L(t))^2,$$

from which Lemma 2 follows.

b. To prove the theorem we know that $b = f \circ f_N^{-1}$ is $K(0)$-quasiconformal, because $|\mu(z; f_N)| = |\mu(\zeta; f_N^{-1})|$ for $\zeta = f_N(z)$. Also the integrability condition (26) of our hypothesis implies the integrability condition of the Theorem A. Therefore, its conclusion is valid and changing variables we find

$$f(z) = f_N(z) + \lambda_N + \epsilon_N(f_N(z)),$$

where

$$|\epsilon_N(f_N(z))| \leq \Phi(f_N(z); k_N, l_N), \quad |\lambda_N| \leq l_N(0).$$

Property (28) follows because $\lim \underline{u}_N(t, f_N) = \infty$. Lemma 2 can be used to establish the property (29). The specific formula (31), when condition (30) holds, follows because then we can choose the x-axis as the path of integration.

c. It is a trivial remark that one can obtain explicit estimates for expansion of K-quasiconformal mappings of S_θ onto H which satisfy the conditions

$$k(t) = O(1), \qquad \iint_{S_\theta} |\mu| J < \infty.$$

9. **An example.** Let $\theta(x)$ be any positive, symmetric, infinitely differentiable function which has the properties

$$\theta(x) = \theta(-x) > 0, \quad \tfrac{1}{4} < \theta(x) < 2, \quad \theta(x) = 1 + \tfrac{1}{2} \sin x^{1-(1/n)} \qquad (x \geq x_0),$$

for integer $n \geq 2$. We note first that for $x \geq x_0$

$$\theta'(x) = \tfrac{1}{2}(1 - 1/n)x^{-1/n} \cos x^{1-(1/n)},$$

$$\theta''(x) = -\tfrac{1}{2}(1 - 1/n)^2 x^{-2/n} \sin x^{1-(1/n)} - (\tfrac{1}{2}(1 - 1/n)/n)x^{-1-(1/n)} \cos x^{1-(1/n)},$$

$$\theta^{(k)}(x) = \pm \tfrac{1}{2}(1 - 1/n)^k x^{-k/n} \operatorname{trig} x^{1-(1/n)} + O(x^{-(k+1)/n}),$$

where $k \geq 2$ and trig denotes the function sine or cosine. Note that θ has the following conditions:

(53) $$\theta'(x) = o(1),$$

(54) $$\int_0^\infty \frac{(\theta'(t))^2}{\theta(t)}\, dt = \infty,$$

(55) $$\int_0^\infty |\theta''(t)|\, dt = \infty.$$

Consequently S_θ is an L-strip for which neither the hypothesis of Warschawski nor the hypothesis of Gol'dberg and Stročik holds. Indeed, we shall see that for this θ their formulas are not correct either. For if $n = 2$ then we note that

$$\int_0^\infty (|\theta'|^4 + |\theta''||\theta'|^2 + |\theta''|^2 + |\theta'\theta'''| + |\theta \quad |)\, dt < \infty.$$

Therefore, since (for $N = 2$)

$$\begin{aligned} |a_2| &= |\partial(v_3 - a_{1,2} - v_2)/\partial y| \\ &\leq \left|\frac{y^4}{4!} D^4\left(\frac{1}{\theta}\right)\right| + \left|\frac{y^2}{2!} D^2\left(\frac{\theta^2}{3!} D^2\left(\frac{1}{\theta}\right)\right)\right| \\ &\leq K(|\theta'|^4 + |\theta'|^2|\theta''| + |\theta''|^2 + |\theta'\theta'''| + |\theta''''|), \end{aligned}$$

$$|b_2| = |\partial(v_2)/\partial y| = 1/\theta + o(1), \qquad |c_2| = |\partial v_2/\partial x| = o(1).$$

Hence if x_0 is sufficiently large and θ is suitably chosen for $|x| \leq x_0$, then (25), (26) hold with $N = 2$. Consequently by (53) we know that $\omega(x; f_2) = o(1)$ and hence we have

$$\begin{aligned} f(z) &= \int_0^x \left[\frac{1}{\theta} + \frac{\theta^2}{3!} D^2\left(\frac{1}{\theta}\right)\right] dt + \lambda + i\frac{y}{\theta(x)} + o(1) \\ &= \int_0^x \left[\frac{1}{\theta} + \frac{1}{3}\frac{(\theta')}{\theta} - \frac{1}{6}\theta''\right] dt + \lambda + i\frac{y}{\theta(x)} + o(1). \end{aligned} \tag{56}$$

But the term $\int \theta''\, dt$ is significant, not bounded and part of the error. Hence our formulas differ from their predecessors in a significant way.

Similarly by choosing n larger we can obtain cases of L-strips where $f = f_N + \lambda + o(1)$ and the terms $\int a_{1,k}\, dt$ are significant in formula (31). Other examples could be given to show that in general the other terms in f_N are significant, because our formulas are applicable even in cases where S_θ is neither an L-strip nor an L-angle (for example when $\theta(x) = x + x^{1/2} \sin x^{1/2}$ for $x \geq x_0$).

10. **Asymmetry. a. Bounded oscillation.** In the general case we wish to discuss the effect of the introduction of the new element, nontrivial inclination, on our theorems. In Theorem 1 if $\theta(x) \equiv 1$, $\psi(x) = x^2$ then clearly real oscillation will be unbounded, while the size of maximal squares is bounded. Hence a new condition is needed.

Let $\square_-(t)$, $\square_+(t)$ denote the two largest squares, whose interiors lie in $\{\mathrm{Re}\ z < t\}$, $\{\mathrm{Re}\ z > t\}$, abut along the segment $\{x = t, \phi_-(x) < y < \phi_+(x)\}$ and have maximal distance between their centers. We denote that distance by $D(t)$ and again the larger and smaller sides of the squares by $M(t)$, $L(t)$. Let

$$\delta(t)^2 = \inf\{|\phi_\pm(x) - \phi_\mp(s)|^2 + (x-s)^2 \colon |x-t| \le M(t),\ -\infty < s < \infty\}$$

be the minimum of the squares of the distances between the arcs $\{\phi_\pm(x) = y \colon |x-t| \le M(t)\}$ and the opposite sides $\{\phi_\mp(x) = y\}$.

Theorem 1′. $\omega(x) = O(1) \Leftrightarrow (M(x) + D(x))/\delta(x) = O(1)$.

The proof of the sufficiency is now exactly as before. The only difference is that we use *end rectangles* instead of *end boxes* and that more cases must be considered in the table of crossings.

The necessity is more complicated. Suppose $(M(x_n) + D(x_n))/\delta(x_n) \to \infty$ for $x_n \nearrow \infty$. Then either,

(a) $M(x_n)/\delta(x_n) \to \infty$ or

(b) $M(x_n)/\delta(x_n) \le B$ and $D(x_n)/\delta(x_n) \to \infty$.

In case (a) we use the same argument as before. In case (b) let

$$F_n(\zeta) = f(M(x_n)\zeta + w_n) - f(w_n),$$

where $w_n = x_n + \frac{1}{2}\mathrm{Im}\,(z_-(x_n) + z_+(x_n))$ and $z_\pm(x_n)$ is the center of $\Box_\pm(x_n)$. We use a Carathéodory limit argument as before except that in the limit iH is mapped onto H. In other words, $F(\zeta) \equiv -i\zeta$ is the limit function and from that it is clear that $\omega(x_n, f) \to \infty$.

b. Dirichlet algorithm. We consider f, g infinitely differentiable as functions of $(x, \phi_+(x))$, $(x, \phi_-(x))$ and let

$$v_1(z) = \tfrac{1}{2} + (y - \psi(x))/\theta(x), \qquad H_1(z) = fv_1 + g(1 - v_1).$$

Then we proceed with the algorithm as before to obtain

$$H(z) = \sum_{m,n=0}^{\infty} a_{m,n}(x) y^m,$$

where H is the formal solution to the Dirichlet problem with boundary values f, g and where

$$a_{0,0} = (\tfrac{1}{2} - \psi/\theta)f + (\tfrac{1}{2} + \psi/\theta)g, \qquad m = 0,\ n = 0,$$

$$a_{1,0} = (f - g)/\theta, \qquad m = 1,\ n = 0,$$

$$a_{m,n} = -D^2 a_{m-2,n}/(m-1)m \qquad \text{if } m \ge 2,$$

$$a_{k,n} = -\sum_{\nu=1}^{n} \theta^{2\nu} a_{2\nu+k,\,n-\nu} \qquad \text{if } n \ge 1,\ k = 0, 1.$$

c. Asymptotic expansion. Only the formulas are changed in the statement and proof of Theorem 3′ in that

$$v_N = \sum_{0 \leq m+n \leq N} a_{2m,n}(x)y^{2m} + a_{2m+1,n}(x)y^{2m+1},$$

where $a_{m,n}$ are as above with $f \equiv 1$, $g \equiv 0$. In the case $\theta'(x) = \psi'(x) = o(1)$ (S is an L-strip of inclination zero), $\theta(x) \geq \delta > 0$ we can obtain a simpler formula by integrating along a path near the x-axis.

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