

THE EXCEPTIONAL SUBSET OF A C_0 -CONTRACTION

BY

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ABSTRACT. Let T be a C_0 -operator acting on a (complex separable) Hilbert space \mathcal{K} ; i.e., T is a contraction on \mathcal{K} and it satisfies the equation $q(T) = 0$ for some inner function q , where $q(T)$ is defined in the sense of the functional calculus of B.Sz.-Nagy and C. Foiaş. Among all those inner functions q there exists a unique minimal function p defined by the conditions: (1) $p(T) = 0$; (2) if $q(T) = 0$, then p divides q . A vector $F \in \mathcal{K}$ is called *exceptional* if there exists an inner function r such that $r(T)F = 0$, but p does not divide r . The existence of nonexceptional vectors plays a very important role in the theory of C_0 -operators. The main result of this paper says that nonexceptional vectors actually exist; moreover, the exceptional subset of a C_0 -operator is a topologically small subset of \mathcal{K} .

Let H_K^2 be the H^2 Hardy class of analytic functions in the unit disc $D = \{z: |z| < 1\}$ with values in the complex separable Hilbert space K . If $F(z) = \sum_{n=0}^{+\infty} \psi_n z^n$ ($\psi_n \in K$) is an element of H_K^2 , then in the norm of K ,

$$\lim F(z) = F(e^{ix}), \quad z \rightarrow e^{ix} \text{ nontangentially,}$$

exists for almost every $e^{ix} \in \partial D = \{z: |z| = 1\}$. If $G(z) = \sum_{n=0}^{+\infty} \xi_n z^n$ is also an element of H_K^2 , then the inner product of F and G can be expressed as

$$\langle F, G \rangle = \sum_{n=0}^{+\infty} (\psi_n, \xi_n) = \int_{\partial D} (F(e^{ix}), G(e^{ix})) dm,$$

where (\cdot, \cdot) denotes the inner product of K and $dm = (1/2\pi)dx$ is the normalized Lebesgue measure on ∂D ([3], [4], [9]).

The shift operator S in H_K^2 is defined by

$$SF(z) = zF(z) = \sum_{n=0}^{+\infty} \psi_n z^{n+1}$$

and its adjoint $S^\#$ is given by

$$S^\# F(z) = \frac{F(z) - F(0)}{z} = \sum_{n=0}^{+\infty} \psi_{n+1} z^n.$$

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An *invariant subspace* is a subspace (i.e., a closed linear manifold) \mathfrak{M} of H_K^2 such that $S\mathfrak{M} \subset \mathfrak{M}$. If $K = \mathfrak{M}^\perp$, then K is invariant under $S^\#$ (we shall say that K is $S^\#$ -invariant) and the operator $T: K \rightarrow K$ defined by

$$(0.1) \quad TF = P(SF),$$

where P denotes the orthogonal projection of H_K^2 onto K , is a C_0 -contraction in the sense of [9]; i.e., $\|T\| = \|PS\| \leq 1$ and $T^{*n} \rightarrow 0$ strongly as n tends to ∞ . The first condition is clear and the second one follows from the fact that $T^* = S^\#|_K$ ($= S^\#$ restricted to K).

Moreover, we have

$$(0.2) \quad T^n P = PS^n, \quad \text{for } n = 0, 1, 2, \dots$$

Conversely, if T is a C_0 -contraction of the complex separable Hilbert space K , then T is unitarily equivalent to an operator of the type described above and (0.2) holds. This is a special case of the functional model of Sz.-Nagy and Foiaş. In [9], these authors develop the following functional calculus:

In $u \in H^\infty$ (the Banach algebra of all bounded analytic functions in D) and $T = PS$, then the map $u \rightarrow u(T)$ defined by

$$u(T)F = P(uF) \quad (F \in K)$$

is an algebra homomorphism from H^∞ into the space of all bounded linear operators in K ; moreover, $\|u(T)\| \leq \|u\|_\infty$.

In this functional model each property of the invariant subspace \mathfrak{M} corresponds to a property of the operator T , and conversely. We are going to use these correspondences in order to obtain results about a particular class of contractions.

Using results in [9, Chapter III], the class C_0 , originally defined by means of a quite complicated functional calculus, can be regarded as the subset of all those C_0 -contractions such that $u(T) = 0$ for some nonidentically zero function $u \in H^\infty$. It is known [9, Proposition III. 4.4] that if T is a C_0 -operator, then there exists an inner function m_T (the *minimal function* of T) such that $u(T) = 0$ if and only if $u \in m_T H^\infty$.

The existence of a nonexceptional vector was proved in [10] for the case when K is a finite dimensional vector space, and this result plays a very important role in the theory of Jordan models [11].

Professor B. Sz.-Nagy notified us that he and C. Foiaş have recently found a proof of the existence of a nonexceptional vector in the general case, and they have used this fact to extend the theory of Jordan models [13]. These results are partially analyzed in §3, in connection with the study of the so-called weak contractions.

The results of this article are partially contained in the author's thesis [4].

I. Preliminary results.

1. In [4] we have analyzed the properties of three families of invariant subspaces, namely, those included in the following definitions.

Definition A. An invariant subspace \mathfrak{M} is called a *full-range invariant subspace* if given any weakly measurable function $G(e^{ix})$ defined on ∂D with values in K , the condition $G(e^{ix}) \perp F(e^{ix})$ in the sense of K (a.e.), for all $F \in \mathfrak{M}$, implies $G(e^{ix}) = 0$ almost everywhere.

It follows from the Beurling-Lax-Halmos theorem [3, Lectures VI and VII] that \mathfrak{M} is a full-range invariant subspace if and only if it can be expressed as

$$\mathfrak{M} = \{U(e^{ix})F(e^{ix}) = UF(e^{ix}): F \in H_K^2\},$$

where U is an *inner function-operator*; i.e., $U(e^{ix})$ is a (weakly) measurable function defined on ∂D with values in the set of all unitary operators in K such that $U(e^{ix})\phi \in H_K^2$, for all $\phi \in K$. Then U is uniquely determined by \mathfrak{M} up to right multiplication by a constant unitary operator. (Two inner function-operators are identified whenever they agree a.e. on ∂D [3].)

An inner function-operator $U(e^{ix})$ can be continued to an analytic operator valued function defined on D such that $\|U(z)\|_K \leq 1$, for all $z \in D$, and $U(e^{ix}) = (\text{strong}) \lim U(z)$, as $z \rightarrow e^{ix}$ nontangentially (see [4], [9]).

Definition B. An invariant subspace \mathfrak{M} is called an *IN-subspace* if there exists a (scalar) inner function q such that $qH_K^2 \subset \mathfrak{M}$ [2], [4].

Definition C. An invariant subspace $\mathfrak{M} = UH_K^2$ is called a *subspace of the determinant class* if it is a full-range invariant subspace and if $I - U^*(z)U(z)$ belongs to the (classical) trace class of operators in K , for all $z \in D$ [1], [4].

Denote by \mathcal{F}^\sim , $(\text{IN})^\sim$, and $(\det)^\sim$ the classes of all invariant subspaces corresponding to the *Definitions* A, B, C, resp. Then we have: $(\det)^\sim \subset (\text{IN})^\sim \subset \mathcal{F}^\sim$ ((\det) , (IN) and \mathcal{F} will denote the corresponding families of inner function-operators). If $\dim K < \infty$, then $(\det) = (\text{IN}) = \mathcal{F}$; on the other hand, if K is infinite dimensional, then $(\det) \neq (\text{IN}) \neq \mathcal{F}$ [4].

If $\mathfrak{M} = UH_K^2 \in (\det)^\sim$, then there exists a constant unitary operator X such that $U(z)X$ belongs to the classical determinant class of operators in K ; in what follows, we are always going to assume that, if $\mathfrak{M} = UH_K^2$ belongs to the determinant class, then the inner function-operator U is chosen so that $[\det U(z)]$ is well defined for all $z \in D$ (see [1], [4]); moreover, it will be assumed that $\det U(z)$ (which is an inner function) is normalized in the sense that its first nonzero Taylor coefficient is positive. We shall make the same assumption for any inner function.

Let $\mathfrak{M} = UH_K^2 \in \mathcal{F}^\sim$; if $K_1 = \mathfrak{M} \cap K$, then we can write $K = K_0 \oplus K_1$ (orthogonal direct sum) and

$$(1.1) \quad \mathfrak{M} = H_{K_1}^2 \oplus [\mathfrak{M} \cap H_{K_0}^2] = H_{K_1}^2 \oplus U_0 H_{K_0}^2,$$

where U_0 is an inner function-operator with respect to the subspace K_0 ; moreover, $\|U_0(0)\phi\|_K < \|\phi\|_K$, for all $\phi \in K_0$, $\phi \neq 0$.

It follows from [9, Proposition VI, 3.5] that $\mathfrak{M} \in \mathcal{F}^\sim$ if and only if $T \in C_{00}$; moreover, the inner function-operator $U_0(z)$ is equal (up to constant unitary factors, on both sides) to the characteristic function $\theta_T(z)$ of $T (= \mathbf{PS})$, i.e.

$$(1.2) \quad U_0(z) = \theta_T(z) = [-T + zD_T (I - zT^*)^{-1}D_T] \Big|_{\mathfrak{D}_T},$$

where $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$, $\mathfrak{D}_T = \overline{D_T(K)}$, $\mathfrak{D}_{T^*} = \overline{D_{T^*}(K)}$ (the upper bar denotes topological closure; $\theta_T(z): \mathfrak{D}_T \rightarrow \mathfrak{D}_{T^*}$, for each $z \in D$).

Let \mathfrak{M} be an invariant subspace and let $F \in H_K^2$. Following M. J. Sherman [7], we shall say that \mathfrak{M} contains the direction of F if, for some scalar function $f \neq 0$, $fF \in \mathfrak{M}$. Then, we have the following

Proposition 1.1 ([4, Chapter I]; see also [7]). (i) Let $\mathfrak{M} = UH_K^2 \in (\text{IN})^\sim$; then there exists an inner function q such that $rH_K^2 \subset \mathfrak{M}$ (r an inner function) if and only if $r \in qH^\infty$ (i.e., if and only if q divides r ; this fact will be denoted by $r < q$).

(ii) Similarly, if \mathfrak{N} is an invariant subspace containing the direction of $F \in H_K^2$, then there exists an inner function p such that $pF \in \mathfrak{N}$ and $f \in H^\infty$, $fF \in \mathfrak{N} \Rightarrow f \in pH^\infty$.

The function q of (i) is precisely the minimal function m_T of the introduction, and it will be called the minimal inner function (mif) of \mathfrak{M} (or U , or T). The function p of (ii) is the minimal inner function of F with respect to \mathfrak{N} (p is the mif (\mathfrak{N}) of F).

A relation between q , U and T is given by the following:

Theorem 1.2 ([4, Chapter V], [9, Chapter VI]). Let $U \in (\text{IN})$ and let q be its mif. The domain of analyticity of $U(z)$ ($U^{-1}(z)$, resp.) is the same as the domain of analyticity of $q(z)$ ($q^{-1}(z)$, resp.); moreover, U and q (U^{-1} and q^{-1} , resp.) have the same kind of isolated singularities.

In particular, the set $D \cap \sigma(T)$ (T defined as usual, $\sigma(T)$ denotes the spectrum of T) is precisely the set of all zeroes of $q(z)$.

Now we can state the following correspondences:

Proposition 1.3. The invariant subspace \mathfrak{M} belongs to the class:

- (i) \mathcal{F}^\sim if and only if $T \in C_{00}$,
- (ii) $(\text{IN})^\sim$ if and only if $T \in C_0$,
- (iii) $(\det)^\sim$ if and only if $T \in C_{00}$ and it is a weak contraction (i.e., $\text{trace}(I - T^*T) < \infty$ and $D \not\subset \sigma(T)$).

Proof. The statements (i) and (ii) are clear from the previous results. Observe that $(I - T^*T): K \rightarrow K = \mathcal{D}_T \oplus \mathcal{D}_T^\perp$ is a nonnegative hermitian operator with kernel equal to \mathcal{D}_T^\perp ; thus we have

$$\begin{aligned} \operatorname{tr}(I_K - T^*T) &= \operatorname{tr}([I_K - T^*T]|_{\mathcal{D}_T}) = \operatorname{tr}(I_{\mathcal{D}_T} - \theta_T^*(0)\theta_T(0)) \\ &= \operatorname{tr}(I_{K_0} - U_0^*(0)U_0(0)) = \operatorname{tr}(I_K - U^*(0)U(0)). \end{aligned}$$

On the other hand, the characteristic function $T_\alpha = (T - \alpha)(I - \bar{\alpha}T)^{-1}$ is equal to $\theta_{T_\alpha}(z) = \theta_T((z + \alpha)/(1 + \bar{\alpha}z))$, for any $\alpha \in D$ (cf. [9, Chapters VI and VIII]) and T_α is a weak contraction if and only if T is so.

Using these facts, the following statements are clearly equivalent:

- (1) T is a weak C_{00} -contraction;
- (2) $U \in \mathcal{F}$ and $\operatorname{tr}(I - U^*(0)U(0)) < \infty$;
- (3) $U \in \mathcal{F}$ and $\operatorname{tr}(I - U^*(\alpha)U(\alpha)) < \infty$, for all $\alpha \in D$;
- (4) $\mathfrak{M} \in (\det)^\sim$.

In fact, (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (3) follow immediately from the definitions and previous arguments. It is also clear that (3) implies that $T \in C_{00}$ and $\operatorname{tr}(I - T^*T) < \infty$. Since $U \in (\det) \subset (\operatorname{IN})$, it follows from Theorem 1.2 that $D \not\subset \sigma(T)$. Hence, T is a weak C_{00} -contraction. Q.E.D.

Remark. The implication: T is a weak C_{00} -contraction $\Rightarrow \mathfrak{M} \in (\det)^\sim$, is contained in [9]. The proof of [9, Theorem VIII. 1.1] is precisely the construction of the determinant of $\theta_T(z)!$.

2. The next result is immediate ([3], [9]):

Lemma 1.4. Let \mathfrak{M} be an invariant subspace and let T be defined by (0.1). Then, to each T -invariant subspace $K_0 \subset K$ corresponds exactly one invariant subspace \mathfrak{M}_0 containing \mathfrak{M} , and conversely. The relation between K_0 and \mathfrak{M}_0 is given by

- (1) $K_0 = K \cap \mathfrak{M}_0$;
- (2) $\mathfrak{M}_0 = \mathfrak{M} \oplus K_0 = \mathfrak{M} \times K_0$ (Cartesian product).

Proposition 1.5 ([4, Chapter II], [9, Chapter VII]). Let $\mathfrak{M} = UH_K^2 \in (\operatorname{IN})^\sim$ with $\operatorname{mif} q$. For any function p such that $q < p$, define

$$(1.3) \quad \mathfrak{M}_p = \{F \in H_K^2: (q/p)F \in \mathfrak{M}\} = U_p H_K^2.$$

Then

- (i) $\mathfrak{M} \subset \mathfrak{M}_p \in (\operatorname{IN})^\sim$ and the mif of \mathfrak{M}_p is equal to p .
- (ii) If $q = \prod_n q_n$ is any factorization of q into pairwise coprime inner factors, then $\mathfrak{M} = \bigcap_n \mathfrak{M}_{q_n}$ and $K = \mathfrak{M}^\perp = \bigvee_n K_{q_n}$, where $K_{q_n} = \mathfrak{M}_{q_n}^\perp$, $K_{q_n} \cap K_{q_m} = \{0\}$ if $n \neq m$ ($\bigvee_n K_{q_n}$ denotes the closed subspace spanned by the K_{q_n} 's).

The most interesting case corresponds to the canonical factorization (see [4], [5]):

$$(1.4) \quad q = bds = \left(\prod_k b_k^{m_k} \right) \left(\prod_j d_j \right) s,$$

where

(1) b is the Blaschke product of q , b_k is the elementary Blaschke factor corresponding to the zero $\lambda_k \in D$ ($\lambda_k \neq \lambda_b$ if $k \neq b$) and m_k is the order of this zero,

(2) $d_j(z) = \exp\{l_j(z + e^{i\theta_j})/(z - e^{i\theta_j})\}$, $0 \leq \theta_j < 2\pi$, $\theta_j \neq \theta_b$ if $j \neq b$, $l_j > 0$, $\sum_j l_j < \infty$, and $d = \prod_j d_j$ is the discrete singular factor of q ; finally,

(3) $s(z) = \exp\{\int_0^D (z + e^{i\nu})/(z - e^{i\nu}) d\nu(y)\} = e(z, \nu)$, where $\nu(y)$ is a measure given by a continuous nondecreasing function, which is singular with respect to Lebesgue measure in $[0, 2\pi]$. $s(z)$ is called the continuous singular factor of q .

As in [4], [7], the least common multiple (greatest common divisor, resp.) of two inner functions, p and q , will be denoted by $p \wedge q$ ($p \vee q$, resp.).

Lemma 1.6 ([4, Chapter II]). Let $\mathfrak{M} \in \mathcal{F}^-$, $G \in H_K^2$ and assume that \mathfrak{M} contains the direction of G with $\text{mif}(\mathfrak{M})$ of G equal to $q = pr$, where p and r are inner functions. Then

$$\mathfrak{M} = UH_K^2 \subset \mathfrak{M}_p = \{F \in H_K^2: rF \in \mathfrak{M}\} \subset H_K^2$$

and the $\text{mif}(\mathfrak{M}_p)$ of G is equal to r .

Moreover, if $\mathfrak{M}_p = U_p H_K^2$ and $\mathfrak{M} = U_p V_p H_K^2$, $U_p, V_p \in \mathcal{F}$ (to every invariant subspace containing \mathfrak{M} there corresponds a factorization of U of this type; see [3], [8]), then $V_p \in (\text{IN})$ and its mif is equal to p . Thus, the lattice of invariant subspaces

$$(1.5) \quad \{\mathfrak{N}: \mathfrak{M} \subset \mathfrak{N} \subset H_K^2\}$$

contains a sublattice isomorphic to the lattice of inner functions

$$(1.6) \quad \{p: q < p < 1\}.$$

Proposition 1.7. Let $\mathfrak{M} = UH_K^2 \in (\det)^-$ be an invariant subspace whose mif q coincides with the determinant of U . Then the lattices (1.5) and (1.6) of Lemma 1.6 are identical.

Proof. The proof can be obtained by a modification of the argument given in [10, Theorem 1], or by the following argument based on results in [4, Chapter II]: Let $\mathfrak{N} = VH_K^2$ be any invariant subspace containing \mathfrak{M} and let $U = VW$ be the corresponding factorization of the inner function-operator U . Then $V, W \in (\det)$; moreover, if p, p' (r, r' , resp.) are the mif and the determinant of V (W , resp.),

then $p' = ps$ ($r' = rt$, resp.; s and t are scalar inner functions) and we have

$$q = \det U = (\det V)(\det W) = p'r' = (pr)(st).$$

On the other hand, $prH_K^2 \subset V(rH_K^2) \subset VWH_K^2 = UH_K^2$. Therefore, $pr < q < (pr)(st)$. We conclude that $st = 1$, and $p = \text{mif of } V = \det V$, $r = \text{mif of } W = \det W$.

By Proposition 1.5, the mif of $\mathfrak{M}_p = \{F \in H_K^2: rF \in \mathfrak{M}\} = U_p H_K^2$ is equal to p ; since the mif of \mathfrak{N} is also equal to p , it is not hard to verify that

$$\mathfrak{N} \subset \mathfrak{M}_p = \mathfrak{M} \vee pH_K^2 = \bigcap \{\mathfrak{N}': \mathfrak{M} \subset \mathfrak{N}', pH_K^2 \subset \mathfrak{N}'\}.$$

The first part of the proof and the above inclusion imply that $p = \det U_p < \det V = p$.

Hence $\det U = \det V$ and $\mathfrak{N} = \mathfrak{M}_p$. Q.E.D.

II. Exceptional subsets.

1. **Definition D.** Let $\mathfrak{M} \in (\text{IN})^\sim$ with mif q . The set

$$\mathfrak{E}(\mathfrak{M}) = \{F \in H_K^2: q_F \neq q, q_F \text{ is the mif } \mathfrak{M} \text{ of } F\}$$

will be called the *exceptional subset* of H_K^2 with respect to \mathfrak{M} .

Theorem 2.1. Let $\mathfrak{M} \in (\text{IN})$ with mif $q = bds = \prod_n q_n$ (where the q_n 's are nonconstant pairwise coprime inner functions). Then

- (i) $\mathfrak{E}(\mathfrak{M}) = \bigcup_n \mathfrak{E}(\mathfrak{M}_{q_n})$ (\mathfrak{M}_{q_n} defined by (1.3));
- (ii) $\mathfrak{E}(\mathfrak{M}) = K_{\sigma} \times \mathfrak{M}$ (Cartesian product), where $K_{\sigma} = \mathfrak{E}(\mathfrak{M}) \cap K = P[\mathfrak{E}(\mathfrak{M})]$;
- (iii) if $s = 1$, then $\mathfrak{E}(\mathfrak{M})$ is a countable union of proper invariant subspaces and K_{σ} is a countable union of proper T -invariant subspaces of K ;
- (iv) $\mathfrak{E}(\mathfrak{M})$ is a first category F_{σ} subset of H_K^2 and K_{σ} is a first category F_{σ} subset of K .

Note. We shall see later that the statement (iii) is sharp.

Proof. (i) Let $F \in \mathfrak{E}(\mathfrak{M})$ and let p be its mif(\mathfrak{M}); then $q < p$ and $q \neq p$. Therefore, there exists a factor q_n of q such that q_n does not divide p ; let $p = p_n r_n$, where $p_n = p \vee q_n$. Then $q_n < p_n$, $q_n \neq p_n$ and $q_n \wedge r_n = 1$.

Therefore, the w^* -closed invariant subspace of H^∞ , $\mathfrak{N} = \{f \in H^\infty: f(p_n F) \in \mathfrak{M}_{q_n}\}$, contains q_n and r_n ; hence (see [3, Lecture IV]) it also contains $r_n \vee q_n = 1$. Thus, $F \in \mathfrak{E}(\mathfrak{M}_{q_n})$ and we conclude that

$$\mathfrak{E}(\mathfrak{M}) \subset \bigcup_n \mathfrak{E}(\mathfrak{M}_{q_n}).$$

Conversely, if $p_n F \in \mathfrak{M}_{q_n}$, where $q_n < p_n$ for all n , and $p_m \neq q_m$, for some m , then

$$\left(\prod_n p_n\right) F \in \mathfrak{M} \Rightarrow \frac{qp_m}{q_m} F \in \mathfrak{M} \Rightarrow F \in \mathfrak{E}(\mathfrak{M}) \Rightarrow \mathfrak{E}(\mathfrak{M}) \supset \bigcup_n \mathfrak{E}(\mathfrak{M}_{q_n}).$$

(ii) Let $F = F_0 + F_1 \in H_K^2$, where $F_0 = PF \in \mathcal{K}$ and $F_1 = (I - P)F \in \mathfrak{M}$, and let p be an inner function.

Then, $pF \in \mathfrak{M}$ if and only if $pF_0 \in \mathfrak{M}$ (since \mathfrak{M} is an invariant subspace, it is clear that $pF_1 \in \mathfrak{M}$).

Hence if $F \in \mathcal{E}(\mathfrak{M})$, then $PF \in \mathcal{E}(\mathfrak{M}) \cap \mathcal{K}$, and therefore, $P[\mathcal{E}(\mathfrak{M})] \subset \mathcal{E}(\mathfrak{M}) \cap \mathcal{K}$.

On the other hand, if $F \in \mathcal{E}(\mathfrak{M}) \cap \mathcal{K}$ and $G \in \mathfrak{M}$, then $F + G$ has the same $\text{mif}(\mathfrak{M})$ as F ; hence $F + G \in \mathcal{E}(\mathfrak{M})$ from which the result follows.

(iii) If $s = 1$, then $q = (\prod_k b_k^{m_k})(\prod_j d_j)$ [as in (1.4)]. According to (i),

$$\mathcal{E}(\mathfrak{M}) = \left[\bigcup_k \mathcal{E}(\mathfrak{M}_{b_k^{m_k}}) \right] \cup \left[\bigcup_j \mathcal{E}(\mathfrak{M}_{d_j}) \right].$$

Observe that

$$\mathcal{E}(\mathfrak{M}_{b_k^{m_k}}) = \{F \in H_K^2; b_k^{m_k-1}F \in \mathfrak{M}_{b_k^{m_k}}\} = \mathfrak{M}_k''.$$

By Proposition 1.5 the mif of $\mathfrak{M}_{b_k^{m_k}}$ is equal to $b_k^{m_k}$ and the mif of \mathfrak{M}_k'' is equal to b_k ; therefore, \mathfrak{M}_k'' is a *proper* invariant subspace of H_K^2 .

Similarly, the mif of \mathfrak{M}_{d_j} is equal to d_j ($d_j \neq 1$) and the mif of

$$\mathfrak{M}_{j,n}'' = \{F \in H_K^2; d_j^{(1-1/n)}F \in \mathfrak{M}_{d_j}\}$$

is equal to $d_j^{1/n} \neq 1$; therefore $\mathfrak{M}_{j,n}''$ is a *proper* invariant subspace of H_K^2 .

Since $\mathcal{E}(\mathfrak{M}_{d_j})$ is clearly equal to $\bigcup_{n=1}^{\infty} \mathfrak{M}_{j,n}''$, we conclude that

$$\mathcal{E}(\mathfrak{M}) = \left[\bigcup_k \mathfrak{M}_k'' \right] \cup \left[\bigcup_j \bigcup_{n=1}^{\infty} \mathfrak{M}_{j,n}'' \right]$$

has the desired property.

The statement relative to \mathcal{K}_σ follows from (ii) and the previous argument.

(iv) We are going to prove that $\mathcal{E}(\mathfrak{M})$ is always an F_σ subset of H_K^2 .

According to (i),

$$\mathcal{E}(\mathfrak{M}) = \mathcal{E}(\mathfrak{M}_b) \cup \mathcal{E}(\mathfrak{M}_d) \cup \mathcal{E}(\mathfrak{M}_s)$$

and we have already proved that $\mathcal{E}(\mathfrak{M}_b)$ and $\mathcal{E}(\mathfrak{M}_d)$ are actually first category F_σ subsets of H_K^2 .

Let $s(z) = e(z, \mu)$; then $\mathcal{E}(\mathfrak{M}_s) = \bigcup_{n=1}^{\infty} \mathcal{R}_{s,n}$, where $\mathcal{R}_{s,n} = \{F \in H_K^2; e(z, \nu)F \in \mathfrak{M}_s, \text{ for some measure } \nu \leq \mu, \text{ such that } \|\nu\| \leq \|\mu\| - 1/n\}$.

Let $\mathcal{Q}_{s,n}$ be the set of all those measures on ∂D satisfying the above conditions. $\mathcal{Q}_{s,n}$ is clearly a convex w^* -compact set of measures; hence if $F_k \in \mathcal{R}_{s,n}$, $k = 1, 2, \dots$, $F_k \rightarrow F_0$ in H_K^2 and $\nu_k \in \mathcal{Q}_{s,n}$ is the measure associated to F_k , then we can assume (passing if necessary to a subsequence) that $\nu_k \rightarrow \nu_0 \in \mathcal{Q}_{s,n}$ in the w^* -topology. Then,

$$\begin{aligned} \|e(z, \nu_k)F_k - e(z, \nu_0)F_0\| &\leq \| [e(z, \nu_k) - e(z, \nu_0)]F_0 \| + \| e(z, \nu_k)[F_k - F_0] \| \\ &= \| [e(z, \nu_k) - e(z, \nu_0)]F_0 \| + \| F_k - F_0 \|. \end{aligned}$$

By hypothesis, $\|F_k - F_0\| \rightarrow 0$, as $k \rightarrow \infty$. On the other hand, $e(z, \nu_k) \rightarrow e(z, \nu_0)$ uniformly on compact subsets of D , and therefore, $e(z, \nu_k) \rightarrow e(z, \nu_0)$ weakly in H^2 (actually, the sequence converges in H^2 -norm). Since $e(z, \nu_k)$, $e(z, \nu_0)$ are inner functions, we conclude (cf. [4, Chapter II]) that

$$\|e(z, \nu_k)F_k - e(z, \nu_0)F_0\| \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Thus $e(z, \nu_0)F_0 \in \mathfrak{M}$, $F_0 \in \mathfrak{R}_{s,n}$, and we conclude that $\mathfrak{R}_{s,n}$ is a closed subset of H_K^2 .

Hence $\mathfrak{G}(\mathfrak{M}_s)$ (and, therefore, $\mathfrak{G}(\mathfrak{M})$) is a F_σ subset of H_K^2 .

In order to prove that $\mathfrak{G}(\mathfrak{M}_s)$ is a first category subset we only need to show that $H_K^2 \setminus \mathfrak{G}(\mathfrak{M}_s)$ is dense in H_K^2 .

We shall need the following

Lemma 2.2 (M. J. Sherman, [8]). *Let \mathfrak{M} be an invariant subspace containing the directions of $F_1, F_2 \in H_K^2$ and let q_1, q_2 be the mif(\mathfrak{M}) of F_1, F_2 , resp. Then, the mif(\mathfrak{M}) of $F = F_1 + \lambda F_2$ is equal to $q = q_1 \wedge q_2$ for all $\lambda \in \mathbb{C}$, except at most for a denumerable subset.*

Corollary 2.3. (i) *If $\mathfrak{L} = \bigvee \{F_1, F_2, \dots, F_N\}$, and \mathfrak{M} contains the direction of $F_j \in H_K^2$ with mif(\mathfrak{M}) equal to p_j , $j = 1, 2, \dots, N$, then there exists $F \in \mathfrak{L}$ whose mif(\mathfrak{M}) is equal to $p = p_1 \wedge p_2 \wedge \dots \wedge p_N$.*

(ii) *If K is infinite dimensional and $\mathfrak{M} \in (\text{IN})^\sim$ with mif q satisfying the condition $q(0) > 0$, then for every $\epsilon > 0$ there exists a function $F_\epsilon \in H_K^2$ whose mif(\mathfrak{M}), q_ϵ , satisfies: $q_\epsilon(0) < q(0) + \epsilon$.*

Furthermore, the set $G_\epsilon = \{F \in H_K^2: q_F(0) < q(0) + \epsilon\}$ (where q_F denotes the mif(\mathfrak{M}) of F) is dense in H_K^2 .

Proof. (i) This follows from Lemma 2.2 by induction on N .

(ii) Let $\{\phi_n\}_{n=1}^\infty$ be an ONB of K , and let $K_N = \bigvee \{\phi_1, \dots, \phi_N\}$. Then the mif of $\mathfrak{M}_N = \mathfrak{M} \cap H_{K_N}^2$ (considered as an invariant subspace of $H_{K_N}^2$) is equal to $p_{(N)} = p_1 \wedge p_2 \wedge \dots \wedge p_N$.

By induction on N we can obtain vectors $\psi_N \in K_N$ such that $p_{(N)}$ is the mif(\mathfrak{M}_N) = mif(\mathfrak{M}) of ψ_N , $N = 1, 2, \dots$.

It is clear that $|p_{(N)}(z)|$ is nonincreasing for each $z \in D$ and that $p_{(N)}$ converges to q in L^2 -norm and uniformly on compact subsets of D [4, Chapter I]. Therefore, given any $\epsilon > 0$ there exists N such that $p_N(0) < q(0) + \epsilon$; then ψ_N (chosen as above) satisfies our requirements.

From Lemma 2.2, the mif(N) of $F_\lambda = \psi_N + \lambda F$ (for any fixed $F \in H_K^2$) is equal to $q_{F_\lambda} = p_{(N)} \wedge q_F$, for all but a countable subset of λ 's. Hence

$$q_{F_\lambda}(0) \leq p_{(N)}(0) < q(0) + \epsilon.$$

This proves (ii). Q.E.D.

We are going to complete the proof of Theorem 2.1. To this end observe that Corollary 2.3, (ii) implies that $H_K^2 \setminus \mathcal{R}_{s,n}$ is dense in H_K^2 . This proves that $\mathcal{E}(\mathcal{M}_s)$ is a first category F_σ subset of H_K^2 .

As in (iii), the statement relative to K_σ follows from (ii) and the previous argument.

The proof is complete now.

Remarks. (a) The statement of Lemma 2.2 cannot be improved. In fact, if the inner function q admits the expression $q = \prod_{n=1}^{\infty} q_n$ (where the q_n 's are nonconstant pairwise coprime inner functions), then we can construct an invariant subspace $\mathcal{M} \in (\text{IN})^\sim$ with mif q , so that the set of exceptional functions $\omega_k = \phi_1 + \lambda_k \phi_2$ (where ϕ_1, ϕ_2 denote the constant functions $F_j(e^{ix}) \equiv \phi_j$, $\phi_j \in K$, $j = 1, 2$) of the lemma is in one-to-one correspondence with an arbitrary given, denumerable family of complex numbers $\{\lambda_k\}_{k=1}^{\infty}$. More generally, if $\dim K > 1$ and $\{K_j\}_{j=1}^{\infty}$ is any family of proper closed subspaces of K , then the mif of $\mathcal{M} = \bigcap_{j=1}^{\infty} \mathcal{M}_j$, where

$$\mathcal{M}_j = [P_j + (I - P_j)q_j]H_K^2$$

(P_j denotes the orthogonal projection of K onto K_j), are equal to q , q_j , resp, and the exceptional subset $\mathcal{E}(\mathcal{M})$ satisfies the condition

$$\mathcal{E}(\mathcal{M}) \cap K = \left[\bigcup_{j=1}^{\infty} \mathcal{E}(\mathcal{M}_j) \right] \cap K = \bigcup_{j=1}^{\infty} K_j.$$

(b) If $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2 \in (\text{IN})^\sim$, the mif of \mathcal{M}_j is q_j , $j = 1, 2$, but $q_1 \vee q_2 \neq 1$, then $\mathcal{E}(\mathcal{M}) \subset \mathcal{E}(\mathcal{M}_1) \cup \mathcal{E}(\mathcal{M}_2)$ and the inclusion can be strict, in general. For example, if $K = \mathbb{C}^2$,

$$\mathcal{M}_1 = \begin{bmatrix} e^{ix} & 0 \\ 0 & 1 \end{bmatrix} H_K^2, \quad \mathcal{M}_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{ix} \end{bmatrix} H_K^2,$$

then $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2 = e^{ix} H_K^2$ has exceptional set $\mathcal{E}(\mathcal{M}) = \mathcal{M}$, but $\mathcal{E}(\mathcal{M}_1) \cup \mathcal{E}(\mathcal{M}_2) = \mathcal{M}_2 \cup \mathcal{M}_1 \neq \mathcal{M}$.

2. We are going to complete our results about exceptional subsets.

Proposition 2.4. Let $\mathcal{M} \in (\text{IN})^\sim$ with mif $q = b.r$. Then

- (i) if $F \in \mathcal{E}(\mathcal{M})$, then closure $\{f: f \in H^\infty\} \subset \mathcal{E}(\mathcal{M})$;
- (ii) $\mathcal{E}(\mathcal{M})$ is always a (not necessarily denumerable) union of invariant subspaces;
- (iii) if r is nonconstant, then $\mathcal{E}(\mathcal{M})$ is dense in H_K^2 (and K_σ is dense in K);
- (iv) $H_K^2 = \bigvee [H_K^2 \setminus \mathcal{E}(\mathcal{M})]$, $K = \bigvee [K \setminus K_\sigma]$.

Moreover, if q is not a power of an elementary Blaschke factor, then

$$(2.1) \quad H_K^2 = \bigvee \mathfrak{E}(\mathfrak{M}), \quad \mathcal{K} = \bigvee \mathcal{K}_\sigma,$$

(v) $\mathcal{K}_\sigma = \{0\}$ if and only if there exist $\lambda \in D$ and a subspace $K_0 \subset K$, $K_0 \neq \{0\}$ such that

$$\mathcal{K} = \{(1 - \bar{\lambda}z)^{-1}\phi : \phi \in K_0\}.$$

In this case $\mathfrak{M} = [(I - P_0) + (\lambda - z)/(1 - \bar{\lambda}z)P_0]H_K^2$, where P_0 is the orthogonal projection of K onto K_0 .

Proof. (i) Since \mathfrak{M} is invariant

$$q_F F \in \mathfrak{M} \Rightarrow f(q_F F) = q_F(fF) \in \mathfrak{M}, \quad \text{for all } f \in H^\infty.$$

Now the result follows from the fact that \mathfrak{M} is closed.

(ii) This follows immediately from (i). In a more explicit form, we have

$$\mathfrak{E}(\mathfrak{M}) = \bigcup \{\mathfrak{M}_p : q\bar{p} \text{ is a proper divisor of } q\}.$$

(iii) Let $r = e(z, \mu)$; then

$$\mathfrak{E}(\mathfrak{M}) \supset \mathfrak{E}(\mathfrak{M}_r) \supset \bigcup_{n=2}^{\infty} [e(z, \mu/n)H_K^2],$$

and this union of invariant subspaces is dense in H_K^2 .

(iv) The first part is an immediate consequence of Theorem 2.1. Assume that $q(z)$ is not of the form $b^N(z, \lambda)$, for some $\lambda \in D$ and some nonnegative integer N , then we have two different cases:

(1) $\sigma(T)$ contains more than one point. Then q can be factored as $q = p \cdot r$, where p and r are nonconstant coprime inner functions.

Let $\mathfrak{M}_p, \mathfrak{M}_r$ be defined by (1.3). Then $\mathfrak{M}_p \cup \mathfrak{M}_r \subset \mathfrak{E}(\mathfrak{M})$ and

$$\mathfrak{M}_p \vee \mathfrak{M}_r = [\mathfrak{M}_p^\perp \cap \mathfrak{M}_r^\perp]^\perp = H_K^2.$$

(2) $\sigma(T)$ consists of one isolated point of ∂D ; in this case $q(z) = d(z, l, \theta)$ for some $l > 0$ and $e^{i\theta} \in \partial D$. According to (iii) $\mathfrak{E}(\mathfrak{M})$ is dense in H_K^2 and, by Theorem 2.1, (ii) \mathcal{K}_σ is dense in \mathcal{K} . This proves (2.1).

(v) Assume that $\mathcal{K}_\sigma = \{0\}$. Then the argument of (iv)(1) implies that q cannot be factored in a nontrivial way; i.e., $q = 1$ or $q(z) = b(z, \lambda)$, for some $\lambda \in D$. In the first case, $\mathcal{K} = \{0\}$ and $\mathcal{K}_\sigma = \emptyset$; the second case and the remaining statements are contained in [4, Chapter III].

From Theorem 2.1 we get

Corollary 2.5. Let $\mathfrak{M} = UH_K^2 \in \mathcal{F}^\sim$ and let q be an inner function. The following two statements are equivalent.

- (i) *There exists $F \in H_K^2$ whose $\text{mif}(\mathfrak{M})$ is q ;*
 (ii) *U can be factored as $U = W \cdot B$, where $W \in \mathcal{F}$ and B is an IN-operator with $\text{mif } q$.*

Proof. (i) \Rightarrow (ii). This follows from Lemma 1.6.

(ii) \Rightarrow (i). Assume that $U = W \cdot B$ and take $F = W \cdot G$, where $G \in H_K^2$ is any function such that its $\text{mif}(B)$ is equal to q . Then the $\text{mif}(\mathfrak{M})$ of F is also q .

3. Most of the previous results are still valid when H_K^2 is replaced by some suitable (closed) subspace; namely

Proposition 2.6. *Let $\mathfrak{M} \in (\text{IN})^\sim$ and let \mathcal{L} be any closed subspace of H_K^2 such that*

$$(2.2) \quad K \subset \bigcup_{n=0}^{\infty} S^n(\mathcal{L}).$$

Then $\mathcal{E}''(\mathfrak{M}) = \mathcal{E}(\mathfrak{M}) \cap \mathcal{L}$ is a first category F_σ -subset of \mathcal{L} .

Note. Observe that $\bigcup_{n=0}^{\infty} S^n(K) = H_{K_0}^2$ (defined by (1.1)).

Proof. It is clear from Theorem 2.1 that $\mathcal{E}''(\mathfrak{M})$ is an F_σ -subset of \mathcal{L} .

Let $\{F_k\}_{k \in J}$ be on ONB of \mathcal{L} and let p_k be the $\text{mif}(\mathfrak{M})$ of F_k . If $p_{(N)} = p_1 \wedge p_2 \wedge \dots \wedge p_N$, then the limit

$$p(z) = \lim_{(N)} p_{(N)}(z) \quad (N \rightarrow \infty)$$

defines an inner divisor of the $\text{mif } q$ of \mathfrak{M} .

Since $K \subset \bigcup_{n=0}^{\infty} \{S^n(F_k) : k \in J\}$ and, clearly, $p(S^n F_k) \in \mathfrak{M}$ for all $k \in J$ and all $n = 0, 1, 2, \dots$, it follows that $pK \subset \mathfrak{M}$; therefore, $pH_K^2 \subset \mathfrak{M}$, i.e., q divides p . Hence $q = p$.

On the other hand, a standard argument shows that the set $\mathcal{L}_{(N)} = \{F \in \mathcal{L} : q_F \text{ divides } p_{(N)}\}$ is dense in \mathcal{L} .

Using these two facts it is not hard to conclude (by a formal repetition of the proof of Theorem 2.1) that the complement of $\mathcal{E}''(\mathfrak{M})$ in \mathcal{L} is a dense subset of \mathcal{L} .

In [7], M. J. Sherman conjectured that any invariant subspace containing all directions is necessarily an IN-subspace. This result was recently proved by Sz.-Nagy and Foiaş ([12]). Using Sz.-Nagy and Foiaş techniques, Sherman proved a stronger result: The conclusion remains valid if we merely assume that \mathfrak{M} contains all the directions corresponding to a subspace \mathcal{L} such that $H_K^2 = \bigcup_{n=0}^{\infty} S^n(\mathcal{L})$ [8]. A slightly stronger form of this theorem is given by the following:

Theorem 2.7. *Let \mathfrak{M} be an invariant subspace containing all the directions of the functions $F \in \Omega$, where Ω is a second category subset of a subspace \mathcal{L} satisfying (2.2). Then \mathfrak{M} is an IN-subspace.*

Proof. It is clear from the definition of \mathcal{L} and the properties of Ω that

$$(2.3) \quad K = \mathcal{M}^\perp = \bigvee \left\{ \bigvee_{n=0}^{\infty} P S^n F = \bigvee_{n=0}^{\infty} T^n P F : F \in \Omega \right\}.$$

Let $K_F = \bigvee_{n=0}^{\infty} T^n P F$, $\mathcal{M}_F = K_F^\perp$ ($F \in \Omega$). By hypothesis, $\mathcal{M}_F \in (\text{IN})^\sim$ and (2.3) implies that $\mathcal{M} = \bigcap \{ \mathcal{M}_F : F \in \Omega \}$.

Let q_F be the $\text{mif}(\mathcal{M})$ of $F \in \Omega$. If there exists an inner function q such that $q < q_F$, for all $F \in \Omega$, then it is not hard to see that $q\Omega \subset \mathcal{M}$, and this implies (by the definition of \mathcal{L}) that $qH_K^2 \subset \mathcal{M}$, i.e., $\mathcal{M} \in (\text{IN})^\sim$.

Assume that there is no such a function q ; then we can choose a sequence $\{F_n\} \subset \Omega$ such that, if p_n is the $\text{mif}(\mathcal{M})$ of F_n , then $p_{(N)} = p_1 \wedge p_2 \wedge \dots \wedge p_N$ converges to zero uniformly on compact subsets of D . By our hypothesis on Ω , it is not hard to deduce from Proposition 2.6 that the set

$$\Omega' = \Omega \setminus \left[\bigcup_{n=1}^{\infty} \mathcal{E}(\mathcal{M}_{F_n}) \right]$$

cannot be empty!

Pick any function $F \in \Omega'$. It follows from the definition of Ω' that $p_{(N)}$ divides the $\text{mif}(\mathcal{M})$ of F for all values of N , which is clearly impossible. Therefore, $\mathcal{M} \in (\text{IN})^\sim$.

If we omit the condition (2.2), then we have

Theorem 2.8. *Let \mathcal{M} be an invariant subspace containing all the directions of the functions $F \in \Omega$, where Ω is a second category subset of a subspace \mathcal{L} . Then there exists an inner function p satisfying the conditions:*

- (1) $p\mathcal{L} \subset \mathcal{M}$.
- (2) $q\mathcal{L} \subset \mathcal{M}$ (q an inner function) $\Rightarrow p$ divides q .
- (3) The set $\{F \in \mathcal{L} : p \text{ does not divide } q_F = \text{the } \text{mif}(\mathcal{M}) \text{ of } F\}$ is a first category F_σ -subset of \mathcal{L} .

Proof. Let $\mathcal{N} = \bigvee_{n=0}^{\infty} S^n(\mathcal{L})$. \mathcal{N} is an invariant subspace and therefore, by the Beurling-Lax-Halmos theorem (see [3, Lecture V]) can be written as $\mathcal{N} = AH_{K_1}^2$, where K_1 is a suitable complex separable Hilbert space and A is a function defined on ∂D (a.e.) whose values are isometries of K_1 into K and satisfying the condition: $A\psi \in H_{K_1}^2$, for all $\psi \in K_1$.

The operator A commutes with the multiplication by e^{ix} and it maps $H_{K_1}^2$ isometrically onto \mathcal{N} . Hence there exist subspaces $\mathcal{M}_1, \mathcal{L}_1$ contained in $H_{K_1}^2$ such that $A: \mathcal{M}_1 \rightarrow \mathcal{M} \cap \mathcal{N}$; $A: \mathcal{L}_1 \rightarrow \mathcal{L}$ (isometrically onto).

Moreover, if $F = AF_1$ ($F_1 \in H_{K_1}^2$), then $pF \in \mathcal{M}$ if and only if $pF_1 \in \mathcal{M}_1$ and the subset $\Omega_1 = \{F_1 \in H_{K_1}^2 : AF_1 \in \Omega\}$ is a second category subset of \mathcal{L}_1 .

Then, by Theorem 2.7, \mathcal{M}_1 is an IN-subspace of $H_{K_1}^2$. Let p be the mif of

\mathfrak{M}_1 ; then p clearly satisfies the conditions (1) and (2). Furthermore, $\{F \in \mathcal{L}: p \text{ does not divide } q_F\} = A[\mathcal{E}(\mathfrak{M}_1)]$; hence p also satisfies (3).

An example. Theorem 2.7 is almost sharp. Let $\mathfrak{M} = UH_K^2$ be the full-range invariant subspace defined by the "diagonal" operator U , where $U\phi_n = z^n\phi_n$ ($\{\phi_n\}_{n=1}^\infty$ is an ONB of K). Then \mathfrak{M} contains the analytic direction of all analytic polynomials whose coefficients are finite linear combinations of the ϕ_n 's (the set of all these polynomials is dense in H_K^2). However, \mathfrak{M} is not an IN-subspace.

Now we are in a position to prove that the statement (iii) of Theorem 2.1 is sharp, i.e.

Proposition 2.9. *Let $\mathfrak{M} \in (\text{IN})^\sim$ with $\text{mif } q = bds$ and assume that $s(z)$ is non-constant. Then $\mathcal{E}(\mathfrak{M})$ cannot be written as a countable union of subspaces of H_K^2 .*

We shall need the following rather technical lemma:

Lemma 2.10. *Let μ be a positive Borel measure on ∂D containing no atoms and let $\{E_n\}_{n=1}^\infty$ be a sequence of μ -measurable subsets such that $\mu(E_n) > 0$. Then there exists a μ -measurable subset E such that (1) $\mu(E) > 0$; (2) $\mu(E_n \setminus E) > 0$, for all $n = 1, 2, \dots$.*

Proof. Since μ contains no atoms, we can, given the sequence $\{E_n\}_{n=1}^\infty$, obtain a new sequence of μ -measurable subsets $\{E'_n\}_{n=1}^\infty$ such that

- (a) $E'_n \subset E_n$ for each n ,
- (b) $0 < \mu(E'_n) < \epsilon/2^n$,

where ϵ is any positive number less than $\|\mu\|$.

Then the set $E = \partial D \setminus \bigcup_{n=1}^\infty E'_n$ satisfies our requirements.

Proof of Proposition 2.9. Let $s(z) = e(z, \mu)$ and assume that

$$\mathcal{E}(\mathfrak{M}_s) = \bigcup_{n=1}^\infty \mathcal{L}_n,$$

where \mathcal{L}_n is a subspace of H_K^2 ($n = 1, 2, \dots$).

It follows from Theorem 2.8 that, for each n , \mathcal{L}_n is contained in an IN-subspace of the form $\mathfrak{M}_n = \{F \in H_K^2: e(z, \mu_n) F \in \mathfrak{M}\}$, where μ_n is a measure on ∂D such that $d\mu_n = h_n(t)d\mu$, $0 \leq h_n(t) \leq 1$ (a.e., $d\mu$) and

$$\int_{\partial D} h_n(t) d\mu = \int_D d\mu_n = \|\mu_n\| < \|\mu\|.$$

Since $\mathfrak{M}_n \subset \mathcal{E}(\mathfrak{M}_s)$, we can assume that

$$\mathcal{E}(\mathfrak{M}_s) = \bigcup_{n=1}^\infty \mathfrak{M}_n.$$

Let $E_n = \{t \in [0, 2\pi]: h_n(t) \neq 1\}$; then $\{E_n\}_{n=1}^\infty$ is a sequence of μ -measurable subsets of ∂D satisfying the conditions of Lemma 2.10. Let E be a μ -

measurable subset of ∂D , chosen as in the lemma and let $d\nu = \chi_E(t) d\mu$, where $\chi_E(t)$ is the characteristic function of $\partial D \setminus E$.

Claim. If $\mathcal{N} = \{F \in H_K^2: e(z, \nu) F \in \mathcal{M}_s\}$ then

$$\mathcal{N} \not\subset \bigcup_{n=1}^{\infty} \mathcal{M}_n.$$

In fact, we have the following sequence of implications: $\mathcal{N} \subset \bigcup_{n=1}^{\infty} \mathcal{M}_n \Rightarrow \mathcal{N} \subset \mathcal{M}_{n_0}$, for some $n_0 \Rightarrow \nu < \mu_{n_0} \Rightarrow \chi_E(t) \leq b_{n_0}(t)$ (a.e., $d\mu$). However, by our choice of E , this last inequality is false for all values of n .

Therefore $\mathcal{N} \not\subset \bigcup_{n=1}^{\infty} \mathcal{M}_n$; since \mathcal{N} is clearly contained in $\mathcal{E}(\mathcal{M}_s)$, we conclude that $\mathcal{E}(\mathcal{M}_s) \neq \bigcup_{n=1}^{\infty} \mathcal{L}_n$, contradicting our assumption.

Since (by Theorem 2.1, (iii)) $\mathcal{E}(\mathcal{M}) = \mathcal{E}(\mathcal{M}_s) \cup \mathcal{E}'$, where \mathcal{E}' is a countable union of subspaces, we conclude that $\mathcal{E}(\mathcal{M})$ itself cannot be written as a countable union of subspaces.

4. We shall complete this section with two miscellaneous results.

Proposition 2.11. *Let $\mathcal{M} \in (\text{IN})^\sim$ with $\text{mif } q$ and let \mathcal{L} be a closed subspace of H_K^2 satisfying the condition (2.2). Then there exists an orthonormal basis $\{F_n\}$ of \mathcal{L} such that the $\text{mif}(\mathcal{M})$ of F_n is equal to q , for all $n = 1, 2, \dots$.*

Proof. According to Proposition 2.6, we can find a function $G_1 \in \mathcal{L}^1 = \{F \in \mathcal{L}: \|F\| = 1\}$ so that the $\text{mif}(\mathcal{M})$ of G_1 is q . Assume that $\{F_1, F_2, \dots, F_{n-1}, G_n\}$ is an orthonormal system such that the $\text{mif}(\mathcal{M})$ of F_j , $j = 1, 2, \dots, n-1$, and G_n are equal to q . Let $G'_{n+1} \in \mathcal{L}^1$, $G'_{n+1} \perp \bigvee \{F_1, F_2, \dots, F_{n-1}, G_n\}$. Applying Lemma 2.2 to the functions $(\alpha G_n + \beta G'_{n+1}), (\beta G_n - \alpha G'_{n+1})$, $\alpha, \beta \geq 0$, $\alpha^2 + \beta^2 = 1$, we can get a pair

$$F_n = \alpha_n G_n + \beta_n G'_{n+1}, \quad G_{n+1} = \beta_n G_n - \alpha_n G'_{n+1},$$

such that the $\text{mif}(\mathcal{M})$ of F_n, G_{n+1} are equal to q .

Let $\{E_n\}$ be any given ONB of \mathcal{L} and let G_1 be chosen as above. If $E_1 = \lambda G_1$ for some $\lambda \in \partial D$, take $G'_2 = E_2$; if E_1 and G_1 are linearly independent, then take

$$G'_2 = (E_1 - \langle E_1, G_1 \rangle G_1) \|E_1 - \langle E_1, G_1 \rangle G_1\|^{-1},$$

and so on. Now the result follows by induction on n .

The statement (i) of Proposition 2.4 clearly implies that if $F \in \mathcal{K}_\sigma$ then it cannot be a cyclic vector for T . On the other hand, B. Sz.-Nagy and C. Foiaş have proved [13] that if $F \in \mathcal{K} \setminus \mathcal{K}_\sigma$ is a cyclic vector for T , then every element of $\mathcal{K} \setminus \mathcal{K}_\sigma$ is a cyclic vector; hence we have

Corollary 2.12. *Let T be a C_0 -operator and assume that T has a cyclic vector. Then the set of all cyclic vectors of T forms a G_δ -dense subset of \mathcal{K} .*

III. Weak C_{00} -contractions and Jordan models.

1. We shall need some definitions:

Let T be a bounded linear operator in the (complex separable) Hilbert space K and let Ξ be a subset of K . The (finite or denumerable) cardinal number

$$\mu_T = \inf \left\{ \text{cardinal}(\Xi): K = \bigvee_{n=0}^{\infty} T^n(\Xi) \right\}$$

is called the *multiplicity* of T (see [11]). For example, $\mu_T = 1$ means that T has a cyclic vector.

If T' is a bounded linear operator in K' (another Hilbert space) and there exists a *quasi-invertible* operator $X: K \rightarrow K'$ (i.e., $\ker(X) = \{0\}$ and $\text{range}(X)$ is dense in K') such that $T'X = XT$, we shall write $T < T'$ or $T' > T$.

We have the following properties [11]:

(1) $T' > T$ if and only if $T^* > T'^*$;

(2) $T' > T$ and $T'' > T'$ imply $T'' > T$;

(3) $T' > T$ implies $\mu_T \geq \mu_{T'}$;

(4) if T' and T are C_{00} -contractions such that $T' > T$, and $u \in H^\infty$, then $u(T') > u(T)$;

(5) if T', T are C_0 -contractions and $T' > T$, then $m_T = m_{T'}$.

If $T' > T$ and $T > T'$, the operators T and T' are said to be *quasi-similar*.

Definition E. Let $\{q_n\}_{n=1}^\omega$ be a finite or infinite sequence of nonconstant inner functions satisfying the condition $q_1 < q_2 < \dots < q_n < \dots < 1$. Then the *Jordan operator* corresponding to the sequence $\{q_n\}$ is the C_0 -contraction associated to the invariant subspace

$$\mathfrak{M} = \bigoplus_n (q_n H^2 \phi_n),$$

where $\{\phi_n\}_{n=1}^\omega$ is an ONB of some Hilbert space K' .

This operator will be denoted by

$$J(q_1) \oplus J(q_2) \oplus \dots \oplus J(q_n) \oplus \dots, \quad \text{or} \quad J(q_1, q_2, q_3, \dots, q_n, \dots).$$

The (finite or denumerable) cardinal number ω will be called the *length* of the operator J (see [11]).

It is clear that the invariant subspace \mathfrak{M} is actually an IN-subspace with $\text{mif } q_1$.

It was proved in [11], [13] that if $T \in C_0$ then $\mu_T = \mu_{T^*}$ and if $\mu_T = N < \infty$, then T is quasi-similar to a unique Jordan operator (the *Jordan model* of T !) $J(q_1, \dots, q_N)$ of length N .

We are going to extend the notion of Jordan model to a particular class of C_0 -operators. Some of the proofs of the results of this section can be obtained

from those in [10] and [11] (with perhaps a few formal modifications). In these cases, we shall merely quote the corresponding reference.

Let $(\text{IN})_B = \{U \in (\text{IN}): \text{the mif of } U \text{ is a Blaschke product}\}$ and let $(\det)_B = (\det) \cap (\text{IN})_B$ ($(\text{IN})_B^\sim$ and $(\det)_B^\sim$ are similarly defined).

Lemma 3.1. *Let $\mathfrak{M} = UH_K^2 \in (\text{IN})_B^\sim$ with mif $q = b_\lambda^N$ (i.e., q is the power of order $N \geq 1$ of an elementary Blaschke factor having its zero at the point $\lambda \in D$), and assume that T has a cyclic vector G . Then $\mathfrak{M} \in (\det)_B^\sim$ and $\det U = q$.*

Proof. Observe that $\mathcal{K} = \mathfrak{M}^\perp \subset [b_\lambda^N H_K^2]^\perp$; hence $G = \sum_{n=0}^{N-1} g_\lambda b_\lambda^n \psi_n$, where $g_\lambda(z) = (1 - |\lambda|^2)^{1/2} / (1 - \bar{\lambda}z)$ and $\psi_0, \dots, \psi_{N-1} \in K$.

Hence

$$(3.1) \quad \mathcal{K} = \bigvee_{n=0}^{\infty} T^n G = \bigvee_{n=0}^{\infty} (T - \lambda)^n G = \bigvee_{n=0}^{\infty} b_\lambda^n(T)G = \bigvee_{n=0}^{N-1} b_\lambda^n(T)G$$

is a finite dimensional space and therefore $\mathfrak{M} \in (\det)_B^\sim$ and $\det U$ is a Blaschke product of degree less than or equal to N ([4, Chapter III]). Using this result and the relation $\det U < q$ we conclude that $\det U = q$ and $\dim(\mathcal{K}) = N$.

Corollary 3.2. *Let $\mathfrak{M} = UH_K^2 \in (\text{IN})_B^\sim$ with mif $q = \prod_k b_k^{m_k}$, and assume that T admits a cyclic vector. Then $\mathfrak{M} \in (\det)_B^\sim$ and $\det U = q$.*

Proof. By Theorem 1.2, $\sigma(T) = \text{closure } \{\lambda_k: \lambda_k \text{ is the zero of } b_k(z)\}$. Fix k and let $\Gamma_k \subset D \setminus \sigma(T)$ be a (positively oriented) circle separating $\{\lambda_k\}$ from $\sigma(T) \setminus \{\lambda_k\}$; then

$$(3.2) \quad P_k(T) = \frac{1}{2\pi i} \int_{\Gamma_k} (z - T)^{-1} dz$$

is the projection of \mathcal{K} onto the subspace

$$(3.3) \quad \mathcal{K}_k = \{F \in \mathcal{K}: (T - \lambda_k)^{m_k} F = b_k^{m_k}(T)F = 0\}$$

along its complementary space \mathcal{K}_k^\perp , where $\mathcal{K}_k, \mathcal{K}_k^\perp$ are T -invariant subspaces such that

$$(3.4) \quad \mathcal{K} = \mathcal{K}_k \oplus \mathcal{K}_k^\perp, \quad \sigma(T|_{\mathcal{K}_k}) = \{\lambda_k\}, \quad \sigma(T|_{\mathcal{K}_k^\perp}) = \sigma(T) \setminus \{\lambda_k\}$$

(\oplus denotes an algebraic direct sum, which is not orthogonal, in general).

Moreover, $P_k(T)T = TP_k(T)$ (cf. [6, Chapter XI]).

Let F be a cyclic vector for T ; then, from $\mathcal{K} = \bigvee_{n=0}^{\infty} T^n F$, we infer that

$$\mathcal{K}_k = P_k(T)\mathcal{K} = \bigvee_{n=0}^{\infty} P_k(T)T^n F = \bigvee_{n=0}^{\infty} T^n P_k(T)F = \bigvee_{n=0}^{\infty} T_k^n P_k(T)F.$$

Now observe that $\mathfrak{M}_k = \mathcal{K}_k^\perp = B_k H_K^2 \in (\text{IN})_B^\sim$ and its mif is equal to $b_k^{m_k}$. Since $F_k = P_k(T)F$ is a cyclic vector for T_k , we only need to show that $\mathfrak{M}_k \in$

$(\det)_B^\sim$ and $\det B_k = b_k^{m_k}$, from which the result follows (in fact, using results in [4, Chapter III] we obtain $\mathfrak{M} = \bigcap_k \mathfrak{M}_k$ and $\det U = \prod_k (\det B_k) = q$; hence $\mathfrak{M} \in (\det)_B^\sim$).

We have: (1) $K_k = \bigvee_{n=0}^{\infty} T_k^n F_k$, and (2) K_k admits the expression (3.1) of Lemma 3.1. Furthermore, $b_k^{m_k}(T_k) = 0$, and therefore, T_k satisfies a polynomial equation of degree m_k : $(T_k - \lambda_k)^{m_k} = 0$. Hence, $K_k = \bigvee_{n=0}^{m_k-1} T_k^n F_k$ and $\dim K_k \leq m_k$.

Now we can apply a result from [4, Chapter III-2]. If $\dim K_k < m_k$, then the mif of \mathfrak{M}_k is a Blaschke product of degree $N < m_k$, which contradicts our previous result. Hence, $\dim K_k = m_k$, and we conclude that $B_k \in (\det)_B$ and $m_k = \text{degree}(\det B_k) = \dim K_k \geq \text{degree}(\text{mif of } B_k) = m_k$; i.e., $\det B_k = \text{mif of } B_k = b_k^{m_k}$.

The following theorem is contained in [10, Theorem 2]:

Theorem 3.3. *Let $T \in C_0(N)$ (i.e., T can be represented in the form (0.1) where K is a Hilbert space of dimension $N < \infty$); then the following conditions are equivalent:*

- (i) *there exists a cyclic vector for T ;*
- (ii) *if $\mathfrak{M} = UH_K^2$ (U an inner function-operator), then $\det U(z) = q(z)$, $q = \text{mif of } U$;*
- (iii) *for each inner factor p of q there exists a unique T -invariant subspace K_p such that $m_T|_{K_p} = p$, and this subspace has the expression*

$$K_p = \{F \in K: p(T)F = 0\};$$

- (iv) *there is no proper T -invariant subspace \mathfrak{L} such that $m_T|_{\mathfrak{L}} = q$.*

Theorem 3.3'. *Let $T \in C_0$ be the contraction associated to the IN-subspace \mathfrak{M} , and let (i), ..., (iv) be as in Theorem 3.3. Then:*

- (1) *in the general case (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i);*
- (2) *moreover, if $\mathfrak{M} \in (\text{IN})_B^\sim$, then the four properties are equivalent.*

Sketch of the proof. (1) (ii) \Rightarrow (iii) and (iv) \Rightarrow (i) follow from Proposition 1.7, Theorem 2.1, and the corresponding argument in [10, Theorem 2]. For the proof that (iii) \Rightarrow (iv), see [10, Theorem 2].

(2) (iv) \Rightarrow (i) is the statement of Corollary 3.2. Hence if $\mathfrak{M} \in (\text{IN})_B^\sim$, then (i), (ii), (iii) and (iv) are equivalent.

Remark. A closer analysis of the subspaces \mathfrak{M}_k'' of Theorem 2.1, (iii) shows that, if $\mathfrak{M} \in (\text{IN})_B^\sim$ and T has a cyclic vector, then

$$\mathfrak{E}(\mathfrak{M}) = \bigcup_k \{g_{\lambda_k} \phi_k\}^\perp \quad \text{and} \quad K_\sigma = \bigcup_k [K \cap \{g_{\lambda_k} \phi_k\}^\perp],$$

for some unitary vectors $\phi_k \in K$; i.e., $\mathfrak{E}(\mathfrak{M})$ and K_σ are countable unions of subspaces of codimension one.

Corollary 3.4. *If $\mathfrak{M} \in (\text{IN})_B^\sim$, then the restriction of T to any cyclic T -invariant subspace is a weak contraction.*

Corollary 3.5. *If $\mathfrak{M} = \mathfrak{M}_b \cap \mathfrak{M}_d \cap \mathfrak{M}_s$ is the decomposition of the IN-subspace \mathfrak{M} with $\text{mif } q = bds$, given by Proposition 1.5, and T admits a cyclic vector, then $\mathfrak{M}_b = BH_K^2 \in (\det)_B^\sim$ and $\det B = b = \text{mif of } B$.*

2. **Theorem 3.6.** *Let $\mathfrak{M} = BH_K^2 \in (\text{IN})_B^\sim$ with $\text{mif } q = \prod_k b_k^{m_k}$ and let $K = \bigvee_k K_k$ be the decomposition of $K = \mathfrak{M}^\perp$ associated to the zeroes of q (i.e., the one given by (3.2) to (3.4)). Let T be defined as usual; then*

(i) *if $\dim K_k = t_k < \infty$, for all k , then T admits a (unique) Jordan model; i.e., there exists a Jordan operator J such that $T \succ J \succ T$. Moreover, if $J = J(q_1, q_2, \dots, q_n, \dots)$, then $q_1 = q$ and length $J = \mu_T$.*

(ii) *Moreover, if $\mathfrak{M} \in (\det)_B^\sim$, then the IN-subspace \mathfrak{M}' associated to J belongs to $(\det)_B^\sim$ and*

$$(3.5) \quad b = \det B = \prod_k b_k^{t_k} = \prod_k q_n.$$

(iii) *Conversely, if $T \succ J$ (or $J \succ T$), where J is a Jordan operator such that $\prod_n q_n(z)$ converges (uniformly on compact subsets of D) to a function $b(z) \neq 0$, then J is the Jordan model of T , $\mathfrak{M} \in (\det)_B^\sim$ and the equalities (3.5) are satisfied.*

Observe that the proof of [11, Proposition 3] can be easily modified to show that

Proposition 3.7. *Let*

$$J = J(q_1, q_2, \dots, q_n, \dots) \text{ and } J' = J'(q'_1, q'_2, \dots, q'_1, \dots)$$

be two Jordan operators of lengths ω and ω' resp., and assume that $J \succ J'$. Then $\omega = \omega'$, $q'_n = q_n$ (for all n), and both operators coincide (up to unitary equivalence).

Proof of Theorem 3.6. Let $K = \bigvee_k K_k$ and let $T_k = P_k(T)T = TP_k(T)$ as in Corollary 3.2. Then $\mu_{T_k} \leq \mu_T$, for all k . Since T_k satisfies a polynomial equation of degree m_k , it is clear that $\mu_{T_k} < \infty$ if and only if $\dim K_k < \infty$.

(i) Let $\dim K_k = t_k < \infty$, for all k . Then (see [11, Theorem 2]) there exists a Jordan operator of finite length

$$J'_k = J(b_k^{m_k}, \dots, b_k^{m_k}, b_k^{m_k-1}, \dots, b_k^{m_k-1}, \dots, b_k, \dots, b_k),$$

where r_{m_k-1} terms are equal to $b_k^{m_k}$, $r_{m_k-2} - r_{m_k-1}$ terms are equal to $b_k^{m_k-1}$, \dots , $r_0 - r_1$ terms are equal to b_k ,

$$t_k = \sum_{h=0}^{m_k-1} (b+1)(r_n - r_{b+1}) \quad (r_{m_k} = 0)$$

and J'_k is similar to T_k .

Define the operator J as follows: $J = J(q_1, q_2, \dots, q_n, \dots)$, where $q_n = \prod_k (n\text{th term of } J'_k)$ (the product is taken over all those k such that the length of J'_k is $\geq n$).

Claim. J is a Jordan model for T . Let K' be the space on which J acts and let $K' = \bigvee_k K'_k$, $K' = K'_k \oplus K'_k \hat{}$, $P'_k(J)$ and $J_k = JP'_k(J) = P'_k(J)J$ be defined by the formulas (3.2) to (3.4) of Corollary 3.2. Then J_k is unitarily equivalent to J'_k ; hence it is similar to T_k . Let $X_k: K'_k \rightarrow K_k$ be an invertible map such that $T_k X_k = X_k J_k$ ($k = 1, 2, \dots$) and define the bounded linear map $X: K' \rightarrow K$ by means of $X = \sum_k c_k X_k P'_k(J)$ ($c_k > 0$, for all k), where the constants c_k are chosen so that $\|X\| \leq 1$ (e.g., we can take $c_k = 2^{-k} \|X_k\|^{-1}$, in which case X is a compact operator). We want to show that X is actually a quasi-invertible operator from K' to K , and

$$(3.6) \quad TX = XJ.$$

Let $\mathfrak{L}_N = \bigoplus_{k=1}^N K_k$, $\mathfrak{R}_N = \bigvee_{k=N+1}^\infty K_k$, $P_{\mathfrak{L}_N} = \sum_{k=1}^N P_k(T)$ and let $\mathfrak{L}'_N, \mathfrak{R}'_N$ and $P_{\mathfrak{L}'_N}$ be similarly defined.

Let $F' = F'_1 + F'_2 + \dots + F'_N \in \mathfrak{L}'_N$ ($F'_k \in K'_k$). Then

$$\begin{aligned} XJF' &= \sum_{k=1}^N X(JF'_k) = \sum_{k=1}^N c_k X_k JF'_k = \sum_{k=1}^N c_k TX_k F'_k \\ &= T \left(\sum_{k=1}^N c_k X_k F'_k \right) = TXF'. \end{aligned}$$

Since $\bigcup_N \mathfrak{L}'_N$ is dense in K' and X is continuous, we obtain (3.6). Moreover, it is not hard to infer from the definition of X that $X(\mathfrak{L}'_N) = \mathfrak{L}_N$ and $X|_{\mathfrak{L}'_N}: \mathfrak{L}'_N \rightarrow \mathfrak{L}_N$ is an invertible map, for all N . Therefore,

$$\overline{X(K')} = \overline{X\left(\bigcup_N \mathfrak{L}'_N\right)} = \bigcup_N \overline{X(\mathfrak{L}'_N)} = \bigcup_N \mathfrak{L}_N = K.$$

It only remains to show that $\ker(X) = \{0\}$.

Observe that \mathfrak{L}_N and \mathfrak{R}_N form a positive angle in K , and $K = \mathfrak{L}_N \oplus \mathfrak{R}_N$; moreover, \mathfrak{L}'_N and \mathfrak{R}'_N have similar properties in K' .

Let $G' \in \ker(X)$. Then G' has a unique decomposition $G' = G'_N + F'_N$, where $G'_N \in \mathfrak{L}'_N$ and $F'_N \in \mathfrak{R}'_N$.

Then $G_N = X(G'_N) \in \mathfrak{L}_N$ and $F_N = X(F'_N) \in \mathfrak{R}_N$. Since $X(G') = 0$, we have: $G_N = -F_N \in \mathfrak{L}_N \cap \mathfrak{R}_N = \{0\}$.

On the other hand, the fact that $X|_{\mathfrak{L}'_N}$ is invertible implies that $G'_N = 0$, for all N . Hence

$$\ker(X) \subset \bigcap_k \mathfrak{R}'_k = \{0\}.$$

Therefore, X is quasi-invertible. Thus we have: $T \succ J$.

The same argument proves that $Y = \sum_k c'_k X_k^{-1} P_k(T)$ is a quasi-invertible and $JY = YT$; i.e., $J \succ T \succ J$. Hence J is a Jordan model for T , and $J^* \succ T^* \succ J^*$ (i.e., J^* is a Jordan model for T^*).

Now, the uniqueness of J (and J^*) follows from Proposition 3.4. Thus we have $J^* = J(q_1^-, q_2^-, \dots, q_n^-, \dots)$, where $q^-(z) = \overline{q(\bar{z})}$; in particular, $\mu_{T^*} = \mu_T = \text{length of } J = \text{length of } J^*$.

(ii) If $K_k^\perp = \mathfrak{M}_k = B_k H_K^2$, $K_k'^\perp = \mathfrak{M}'_k = B'_k H_K^2$, and $\mathfrak{M} \in (\det)_B^\sim$, then it follows from the constructions of (i) that

$$b = \det B = \prod_k \det B_k = \prod_k \det B'_k = \prod_k b_k^{t_k} = \prod_n q_n.$$

(iii) If $T \succ J$, then $\dim K_k = \dim K'_k = t_k$, for all k .

If t_k is not finite for some k , then b_k^m divides $\prod_n q_n$ for all values of m , and this implies $b(z) \equiv 0$, contradicting our assumption. Hence t_k is always finite.

It is clear that $b_k^{t_k}$ divides b for all k ; thus $\prod_k b_k^{t_k} < b = \prod_n q_n$ ($\prod_k b_k^{t_k}$ converges and it defines an inner function) and therefore $\mathfrak{M} \in (\det)_B^\sim$.

Now (iii) follows from (i) and (ii).

As in [11, Corollaries 1 and 2], we have

Corollary 3.8. Let $\mathfrak{M} = BH_K^2 \in (\text{IN})_B^\sim$, $\mathfrak{M}' = B'H_K^2 \in \mathcal{F}^\sim$ and assume that $K = \mathfrak{M}^\perp = \bigvee_k K_k$, where $\sigma(T|K_k) = \{\lambda_k\}$ and $\dim K_k = t_k < \infty$. Then

(i) If $T \succ T'$ or $T' \succ T$, then T and T' have the same Jordan model. Conversely, if T and T' have the same Jordan model, then they are quasi-similar. If, in particular, $\mathfrak{M} \in (\det)_B^\sim$, then $\mathfrak{M}' \in (\det)_B^\sim$ and $\det B' = \det B$.

(ii) T and its restriction T'' to a proper T -invariant subspace cannot have the same Jordan model.

Proof. If $T \succ T'$ or $T' \succ T$, then $\mathfrak{M}' \in (\text{IN})_B^\sim$ and it has the same mif as \mathfrak{M} ([9, Proposition VII, 2.1]). Let $K = \bigvee_k K_k$, $K' = \bigvee_k K'_k$ be the decomposition (3.2) to (3.4). It follows from the proof of Theorem 3.6 that $\dim K'_k = \dim K_k = t_k$, for all k , and the proof proceeds along the lines of Theorem 3.6.

3. The sufficient condition for the existence of a Jordan model for the contraction T (where $\mathfrak{M} \in (\text{IN})_B^\sim$) is clearly not necessary; however, we believe that the only interesting cases are those considered in Theorem 3.6. In fact, the operators satisfying the condition (i) of Theorem 3.6 can be redefined as those C_0 -contractions such that $I - T^*T$ is compact and the associated invariant subspace \mathfrak{M} belongs to the subclass $(\text{IN})_B^\sim$; this definition (and the one corresponding to the determinant class, or some intermediate class) can be naturally extended.

We conjecture that Theorems 3.3 and 3.6 (and related results) remain true without the assumption that the mif of the IN-subspace \mathfrak{M} is a Blaschke product; moreover, this seems to be the most general situation in which all results of this type are valid.

For example, if $\mathfrak{M} = zH_K^2$, where K is an infinite dimensional Hilbert space, then $\mathcal{K} = K$ and $T = 0$; then T and its restriction to any infinite dimensional subspace have the same Jordan model; namely, $J = J(z, z, z, \dots)$. This proves that the hypothesis of Corollary 3.8, (ii) cannot be relaxed.

A final remark. It was proved in [11] that if $T \in C_0(N)$, then it has a Jordan model of length $\omega \leq N$.

Assume that $\mathfrak{M} = UH_K^2 \in (\det)^\sim$ with mif q and $\det U = p$ and assume that T admits a Jordan model of finite length $J(q_1, q_2, \dots, q_\omega)$, where $q_1 = q$ and $p = \prod_{n=1}^\omega q_n$; then $q^\omega < \prod_{n=1}^\omega q_n = p$, and therefore,

$$\omega \geq \inf\{k: q^k < p\} = \omega_0.$$

Let $b = \prod_{k=1}^\infty b_k$, $r = \prod_{k=1}^\infty b_k$ be two Blaschke products with the same zeroes (all the zeroes of r have order one), and let

$$J = J\left(\prod_{k=1}^\infty b_k, \prod_{k=2}^\infty b_k, \dots, \prod_{k=n}^\infty b_k, \dots\right) \text{ and } J' = J(r) \oplus J.$$

In the first case, there is no finite ω_0 satisfying the above condition; in the second one, $\omega_0 = 2$. However, J' has also infinite length.

Note. After this paper was written, the author received the preprint "Boundedness from measure theory", by Henry Helson. In this article, H. Helson gives (among other results) new proofs of the *constant directions theorem* of [8] and several results related to §2.

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