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# CLOSED SUBGROUPS OF LATTICE-ORDERED PERMUTATION GROUPS

BY

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ABSTRACT. Let $G$ be an $l$-subgroup of the lattice-ordered group $A(\Omega)$ of order-preserving permutations of a chain $\Omega$; and in this abstract, assume for convenience that $G$ is transitive. Let $\bar{\Omega}$ denote the completion by Dedekind cuts of $\Omega$. The stabilizer subgroups $G_{\bar{\omega}} = \{g \in G \mid \bar{\omega}g = \bar{\omega}\}$, $\bar{\omega} \in \bar{\Omega}$, will be used to characterize certain subgroups of $G$ which are *closed* (under arbitrary suprema which exist in $G$). If $\Delta$ is an *o-block* of $G$ (a nonempty convex subset such that for any $g \in G$, either $\Delta g = \Delta$ or $\Delta g \cap \Delta$ is empty), and if $\bar{\omega} = \sup \Delta$, $G_\Delta$ will denote $\{g \in G \mid \Delta g = \Delta\} = G_{\bar{\omega}}$; and the *o-block system* $\tilde{\Delta}$ consisting of the translates $\Delta g$ of $\Delta$ will be called *closed* if $G_\Delta$ is closed. When the collection of $o$-block systems is totally ordered (by inclusion, viewing the systems as congruences), there is a smallest closed system $\mathcal{C}$, and all systems above $\mathcal{C}$ are closed. $\mathcal{C}$ is the *trivial system* (of singletons) iff $G$ is *complete* (in $A(\Omega)$). $G_{\bar{\omega}}$ is closed iff $\bar{\omega}$ is a *cut in* $\mathcal{C}$, i.e., $\bar{\omega}$ is not in the interior of any $\Delta \in \mathcal{C}$. Every closed convex $l$-subgroup of $G$ is an intersection of stabilizers of cuts in $\mathcal{C}$. Every closed prime subgroup $\neq G$ is either a stabilizer of a cut in $\mathcal{C}$, or else is minimal and is the intersection of a tower of such stabilizers. $L(\mathcal{C}) = \bigcap \{G_\Delta \mid \Delta \in \mathcal{C}\}$ is the distributive radical of $G$, so that $G$ acts faithfully (and completely) on $\mathcal{C}$ iff $G$ is completely distributive. Every closed $l$-ideal of $G$ is $L(\mathfrak{D})$ for some system $\mathfrak{D}$. A group $G$ in which every nontrivial $o$-block supports some $1 \neq g \in G$ (e.g., a generalized ordered wreath product) fails to be complete iff $G$ has a smallest nontrivial system $\tilde{\Delta}$ and the restriction $G_\Delta|\Delta$ is $o$-2-transitive and lacks elements $\neq 1$ of bounded support.

These results about permutation groups are used to show that if $H$ is an abstract $l$-group having a representing subgroup, its closed $l$-ideals form a tower under inclusion; and that if $\{K_\lambda\}$ is a Holland kernel of a completely distributive abstract $l$-group $H$, then so is the set of closures $\{K_\lambda^*\}$, so that if $H$ has a transitive representation as a permutation group, it has a complete transitive representation.

1. **Introduction.** A permutation $f$ of a chain $\Omega$ is said to *preserve order* if $\beta \leq \gamma$ implies $\beta f \leq \gamma f$ for all $\beta, \gamma \in \Omega$. The group $A(\Omega)$ of all order-preserving permutations becomes a lattice-ordered group ($l$-group) when ordered pointwise. We shall treat $l$-subgroups $G$ of $A(\Omega)$, i.e., subgroups which are also sublattices; these will be known as *l-permutation groups*. $G$ will be assumed transitive only

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in §§4 and 5. A convex $l$-subgroup $P$ of an $l$-group $G$ is *prime* if $g_1 \wedge g_2 \in P$ implies $g_1 \in P$ or $g_2 \in P$. If $(G, \Omega)$ is an $l$-permutation group, each $g \in G$ can be uniquely extended to an order-perserving permutation of $\overline{\Omega}$ which we shall identify with $g$. For the extended group we shall write $(G, \overline{\Omega})$. Stabilizer subgroups $G_{\overline{\omega}}$, $\overline{\omega} \in \overline{\Omega}$, are prime.

Lloyd [7, Theorem 2] showed that in a rather restricted class of (possibly intransitive) $l$-permutation groups, the stabilizers $G_\alpha$, $\alpha \in \Omega$, are closed. In [9], the present author showed that for groups in that class, the stabilizers $G_{\overline{\omega}}$, $\overline{\omega} \in \overline{\Omega}$, constitute all closed primes $\neq G$. In preparation for the description in Theorem 9 of various closed subgroups of $G$, the present paper extends the information in [9], and the discussion in [8] of the conditions (1) All sups in $G$ are pointwise, (2) $G$ is complete, (3) Stabilizers $G_\alpha$ are closed, and (4) $G$ is completely distributive. However, dependence on these papers is kept to a minimum.

Several examples are given at the end of the paper, including one of a transitive group (necessarily with an infinite number of $o$-block systems) with a closed prime which is not a stabilizer.

2. **Prime subgroups of an $l$-group.** If $P$ is a prime subgroup of an $l$-group $H$, then $\{Q \mid P \subseteq Q,\ Q \text{ prime in } H\}$ is totally ordered by inclusion [3, Lemma 3], and this tower contains all convex subgroups of $H$ which contain $P$.

**Lemma 1** (Byrd and Lloyd [2, Lemma 3.3]). *If $P$ and $Q$ are prime subgroups of an $l$-group $H$, with $P$ closed and $P \subseteq Q$, then $Q$ is also closed.*

Another hypothesis that guarantees closure of a prime subgroup is

**Proposition 2.** *Let $P$ be a prime subgroup of an $l$-group $H$ and let $Q$ be a convex $l$-subgroup of $H$, $Q \nsubseteq P$. Then if $P \cap Q$ is closed in $H$, so is $P$.*

**Proof.** Suppose $s = \sup\{s_i \mid i \in I\}$, with $1 \leq s_i \in P$ for all $i$. Pick $1 < g \in Q \setminus P$. Then $q \wedge s = \sup\{q \wedge s_i\}$, and each $q \wedge s_i \in P \cap Q$, so that if $P \cap Q$ is closed, $q \wedge s \in P \cap Q \subseteq P$; and thus $s \in P$ since $P$ is prime and $q \notin P$.

**Corollary 3.** *Let $\mathcal{P}$ be a meet-irreducible set of prime subgroups of an $l$-group $H$ (i.e., $\bigcap \mathcal{P} \subset \bigcap \mathcal{P}\setminus\{P\}$) for every $P \in \mathcal{P}$). If $\bigcap \mathcal{P}$ is closed, so is every $P \in \mathcal{P}$.*

We mention that the special case $\bigcap \mathcal{P} = \{1\}$ is a trivial consequence of a much stronger statement due to Lloyd [6, Theorem 2.3].

A convex subset $\Delta \neq \Box$ of $\Omega$ is an *o-block* of $(G, \Omega)$ if, for every $g \in G$, either $\Delta g = \Delta$ or $\Delta g \cap \Delta = \Box$. The *convexification* $\operatorname{Conv}(\overline{\Lambda})$ of $\overline{\Lambda} \subseteq \overline{\Omega}$ means $\{\omega \in \Omega \mid \overline{\lambda}_1 \leq \omega \leq \overline{\lambda}_2 \text{ for some } \overline{\lambda}_1, \overline{\lambda}_2 \in \overline{\Lambda}\} \subseteq \Omega$. An $o$-block $\Delta$ of $G$ will be called *extensive* if $\operatorname{Conv}(\overline{\delta} G_\Delta) = \Delta$ for one (hence every) $\overline{\delta} \in \overline{\Delta}$. The special case of the following theorem in which $G$ is transitive (so that all $o$-blocks are extensive) and $\bar{\pi} \in \Omega$ was established by the author in [10, Theorem 11].

**Theorem 4.** *Let $(G, \Omega)$ be an $l$-permutation group, and let $\bar{\pi} \in \bar{\Omega}$. An o-correspondence between the tower of nontrivial extensive o-blocks $\Delta$ of $(G, \Omega)$ for which $\bar{\pi} \in \bar{\Delta}$ and the tower of prime subgroups $C$ of $G$ for which $G_{\bar{\pi}} \subset C$ is given by $\Delta \to G_\Delta$ and $C \to \mathrm{Conv}(\bar{\pi}C)$.*

**Proof.** [5, Theorem 3] states that nondisjoint $o$-blocks of transitive groups are comparable under inclusion, and the same proof works in the intransitive case for extensive $o$-blocks, so the lattice of $o$-blocks $\Delta$ for which $\bar{\pi} \in \bar{\Delta}$ is indeed totally ordered. If $\bar{\pi} \in \bar{\Delta}$, $\Delta$ an extensive $o$-block, then $G_\Delta$ is clearly a prime subgroup of $G$ containing $G_{\bar{\pi}}$, and $\mathrm{Conv}(\bar{\pi}G) = \Delta$ since $\Delta$ is extensive. If $G_{\bar{\pi}} \subset C$, $C$ prime, then $\Delta = \mathrm{Conv}(\bar{\pi}C)$ is an $o$-block of $G$ and $G_\Delta = C$. For if $\bar{\pi}c_1 \le \bar{\pi}c_2 g \le \bar{\pi}c_3$, then $c_2 g \le c_2 g \vee c_3 \in C$ since $(c_2 g \vee c_3)c_3^{-1} \in G_{\bar{\pi}} \subseteq C$, and similarly $c_2 g \ge c_2 g \wedge c_1 \in C$. Since $C$ is convex, $c_2 g \in C$, so that $g \in C$ and $\Delta g = \Delta$, as required. Clearly $\Delta$ is extensive.

3. **Closed subgroups of an $l$-permutation group.** The author showed in [8, Theorem 7] that for a *transitive* $l$-subgroup $G$ of $A(\Omega)$, the following are equivalent:

(1) *Sups in $G$ are pointwise,* i.e., if in $G$, $g = \bigvee_{i \in I} g_i$, then, for each $\beta \in \Omega$, $\beta g$ is the sup in $\Omega$ of $\{\beta g_i \mid i \in I\}$.

(2) $G$ is a *complete* subgroup of $A(\Omega)$, i.e., if in $G$, $g = \bigvee_{i \in I} g_i$, then $g = \bigvee_{i \in I} g_i$, also in $A(\Omega)$.

(3) $G_\alpha$ is a *closed* subgroup of $G$ for all $\alpha \in \Omega$. (Of course, if any $G_{\bar{\omega}}$, $\bar{\omega} \in \bar{\Omega}$, is closed, so is $G_{\bar{\sigma}}$ for every $\bar{\sigma} \in \bar{\omega}G$.)

Moreover, it was shown in [8, Corollary 15] that in the presence of these conditions, we have

(4) $G$ is a *completely distributive* $l$-group, i.e., $\bigwedge_{i \in I} \bigvee_{k \in K} g_{ik} = \bigvee_{f \in K^I} \bigwedge_{i \in I} g_{if(i)}$ for any collection $\{g_{ik} \mid i \in I,\ k \in K\}$ of elements of $G$ for which the indicated sups and infs exist.

Here we discuss these conditions without assuming transitivity.

**Lemma 5.** *If $s = \sup\{s_i\}$ and $\sup\{\bar{\omega}s_i\} < \bar{\omega}s$ for some $\bar{\omega} \in \bar{\Omega}$, this sup also fails to be pointwise for every $\bar{\tau}$ in the interval $((\sup\{\bar{\omega}s_i\})s^{-1}, \bar{\omega}]$ of $\bar{\Omega}$.*

**Proof.** $\sup\{\bar{\tau}s_i\} \le \sup\{\bar{\omega}s_i\} < \bar{\tau}s$.

The *support* of $g \in A(\Omega)$ means $\{\omega \in \Omega \mid \omega g \ne \omega\}$, and any $\Delta \subseteq \Omega$ containing this set is said to support $g$. We let $\bar{\Omega}_c = \{\bar{\omega} \in \bar{\Omega} \mid G_{\bar{\omega}} \text{ closed}\}$.

**Lemma 6.** *If $\{s_i\} \subseteq G_{\bar{\omega}}$, $\bar{\omega} \in \bar{\Omega}$, and if $\sup\{s_i\} = s \notin G_{\bar{\omega}}$, then $G_{\bar{\tau}}$ is not closed for any $\bar{\tau} \in [\bar{\omega}, \bar{\omega}s)$, and this interval does not support any $1 \ne g \in G$. Hence $\bar{\Omega}_c$ is closed (in the order topology on $\bar{\Omega}$).*

**Proof.** We may assume without loss of generality that the $s_i$'s are positive. Suppose by way of contradiction that $G_{\bar{\tau}}$ is closed for some $\bar{\tau} \in [\bar{\omega}, \bar{\omega}s)$, and let $\bar{\sigma} = \bar{\tau}s^{-1} < \bar{\omega}$. Then $G_{\bar{\sigma}}$ is closed, and $\bar{\sigma}s < \bar{\sigma}s^2$, so that $\bar{\sigma} < \bar{\sigma}s$. Thus $\bar{\sigma} < \bar{\sigma}s_j < \bar{\omega}$ for some $s_j$, and letting $\bar{\rho} = \bar{\sigma}s_j$, $G_{\bar{\rho}}$ is closed. But $s_i s^{-1} s_j \vee 1 \in G_{\bar{\rho}}$ for each $i$ since $s_i \in G_{\bar{\omega}}$ guarantees that $\bar{\rho}s_i s^{-1} s_j < \bar{\omega}s^{-1}s_j \leq \bar{\tau}s^{-1}s_j = \bar{\sigma}s_j = \bar{\rho}$; whereas $\sup\{s_i s^{-1} s_j \vee 1\} = s_j \notin G_{\bar{\rho}}$, a contradiction. Hence $G_{\bar{\tau}}$ is not closed for any $\bar{\tau} \in [\bar{\omega}, \bar{\omega}s)$. But then $G_{\bar{\tau}}$ is not closed for any $\bar{\tau} \in [\bar{\omega}, \bar{\omega}s)s^{-1} = [\bar{\omega}s^{-1}, \bar{\omega})$. Therefore, $\bar{\Omega}\backslash\bar{\Omega}_c$ is open in the order topology on $\bar{\Omega}$. Also, if $[\bar{\omega}, \bar{\omega}s)$ supports some $1 \neq g \in G$ (which can be assumed positive), then $s_i \leq sg^{-1} < s$ for each $i$ (since $\bar{\omega}s_i = \bar{\omega}$), a contradiction.

**Theorem 7.** *For any l-permutation group, each condition* (1)–(4) *implies the next. Moreover,*

(a) *If sups in $G$ are pointwise* (*at points of* $\Omega$), *they are also pointwise at cuts of* $\Omega$.

(b) *If $G$ is complete in $A(\Omega)$ and transitive on $\Omega$, then $G$ is also complete in $A(\bar{\Omega})$.*

(c) *If stabilizers of points are closed, so are stabilizers of cuts.*

**Proof.** The proof of the first statement is exactly as in [8]. (None of the implications can be reversed; a counterexample for the last one can be found in §7.) (a) follows from Lemma 5, and (c) from Lemma 6. When $(G, \Omega)$ is transitive, (2) for $(G, \Omega)$ implies (1) for $(G, \Omega)$, which by (a) implies (1) for $(G, \bar{\Omega})$, which in turn implies (2) for $(G, \bar{\Omega})$, yielding (b); and (b) does in fact fail without transitivity.

A $G$-congruence $\mathfrak{D}$ is said to be *convex* if its congruence classes are convex. The corresponding partition of $\Omega$ into o-blocks of $G$ will also be denoted by $\mathfrak{D}$. The set of o-blocks associated with a given $\mathfrak{D}$ inherits a natural total order from $\Omega$, namely $\Delta_1 \leq \Delta_2$ iff $\delta_1 \leq \delta_2$ for some $\delta_1 \in \Delta_1$, $\delta_2 \in \Delta_2$. If $G$ is transitive, the o-blocks associated with $\mathfrak{D}$ are all translates of each other. In this case the partition $\mathfrak{D}$ will be called an *o-block system* of $G$, and if $\Delta$ is an o-block, the system it determines will be denoted by $\tilde{\Delta}$.

We shall say that $\bar{\omega} \in \bar{\Omega}$ is a *cut in* $\mathfrak{D}$ if $\bar{\omega}$ does not lie in the interior of $\bar{\Delta}$ for any $\Delta \in \mathfrak{D}$. If $G$ is transitive and $\mathfrak{D}$ nontrivial, then $\Delta \in \mathfrak{D}$ lacks end points, so the same is true of $\bar{\Delta}$, and thus the condition reals "$\bar{\omega} \notin \bar{\Delta}$ for any $\Delta \in \mathfrak{D}$".

We shall say that a convex $G$-congruence $\mathfrak{D}$ is *closed* if for each $\mathfrak{D}$-class $\Delta$, $G_\Delta$ $(= G_{\sup\Delta} = G_{\inf\Delta})$ is closed. If $\bar{\omega}$ is a cut in a closed $\mathfrak{D}$, then $G_{\bar{\omega}}$ is closed (by definition if $\bar{\omega}$ is an end point in $\bar{\Omega}$ of some $\Delta \in \mathfrak{D}$, and then in general by Lemma 6). If $G$ is transitive, and $\Delta$ is an o-block of $G$, then if $G_\Delta$ is closed, $\tilde{\Delta}$ is closed, for if $\Gamma \in \tilde{\Delta}$, $G_\Gamma$ is a conjugate of $G_\Delta$.

We partially ordered the collection of convex $G$-congruences by inclusion. In the transitive case, this gives a total order [5, Theorem 3].

Now we define an extremely important convex $G$-congruence $\mathcal{C}$ as follows: $\sigma\mathcal{C}\tau$ iff either $\sigma = \tau$ or no $\bar{\delta} \in \bar{\Omega}$ lying between $\sigma$ and $\tau$ (inclusive) has a closed stabilizer $G_{\bar{\delta}}$. Since $\bar{\Omega}_c = \{\bar{\omega} \in \bar{\Omega} \mid G_{\bar{\omega}}$ is closed$\}$ satisfies $\bar{\Omega}_c g = \bar{\Omega}_c$ for every $g \in G$, $\mathcal{C}$ is indeed a (convex) $G$-congruence.

**Theorem 8.** *Let $(G, \Omega)$ be an $l$-permutation group. Then*

(1) *$\mathcal{C}$ is the smallest closed convex $G$-congruence, and a convex $G$-congruence $\mathcal{D}$ is closed iff $\mathcal{D} \geq \mathcal{C}$.*

(2) *$\mathcal{C}$ is the trivial system (of singletons) iff all stabilizers of points in $\Omega$ are closed.*

(3) *The $\mathcal{C}$-classes are extensive o-blocks of $G$.*

(4) *$G_{\bar{\omega}}$ is closed iff $\bar{\omega}$ is a cut in $\mathcal{C}$, i.e., $\bar{\Omega}_c$ is the set of cuts in $\mathcal{C}$.*

(5) *If $\bar{\omega} \in \bar{\Delta}$, $\Delta \in \mathcal{C}$, then $G^*_{\bar{\omega}} = G_\Delta$.*

(6) *If $\bar{\omega}$ is a cut in $\mathcal{C}$ and $G_{\bar{\omega}}$ fixes $\bar{\tau}$, then $\bar{\tau}$ is also a cut in $\mathcal{C}$.*

**Proof.** If $\Delta$ is a $\mathcal{C}$-class, then every interval $[\sup \Delta, \omega)$ meets $\bar{\Omega}_c$, so by Lemma 6, $G_{\sup \Delta}$ is closed. Hence $\mathcal{C}$ is closed. (4) now follows immediately from the construction of $\mathcal{C}$. Hence if $\mathcal{D} \geq \mathcal{C}$, $\mathcal{D}$ is closed; and conversely, if $\mathcal{D}$ is closed, then for any $\Delta \in \mathcal{D}$, $G_{\sup \Delta}$ and $G_{\inf \Delta}$ are closed, so that $\sup \Delta$ and $\inf \Delta$ are cuts in $\mathcal{C}$, and thus $\mathcal{D} \geq \mathcal{C}$. This proves (1).

$\mathcal{C}$ is trivial iff $\bar{\Omega}_c$ is dense in $\bar{\Omega}$ iff $\bar{\Omega}_c = \bar{\Omega}$ (since $\bar{\Omega}_c$ is closed by Lemma 6) iff all point stabilizers are closed (by (a) of Theorem 7). If $\delta \in \Delta \in \mathcal{C}$, then $\mathrm{Conv}(\delta G_\Delta)$ has the same (closed) stabilizer as $\Delta$, so its end points cannot lie in $\bar{\Delta}$, proving that $\Delta$ is extensive.

If $\bar{\omega}$ is not a cut in $\mathcal{C}$, so that $\bar{\omega} \in \bar{\Delta}$, $\Delta \in \mathcal{C}$, then by Theorem 4, the tower of primes properly containing $G_{\bar{\omega}}$ is $\{G_\Gamma \mid \bar{\omega} \in \bar{\Gamma}$, $\Gamma$ an extensive o-block of $G\}$, and by (4), $G_\Delta$ is the smallest closed member of this tower, so that $G^*_{\bar{\omega}} = G_\Delta$. For (6), if $G_{\bar{\omega}}$ fixes $\bar{\tau}$, then $G_{\bar{\omega}} \subseteq G_{\bar{\tau}}$. Now if $\bar{\omega}$ is a cut in $\mathcal{C}$, so that $G_{\bar{\omega}}$ is closed, then $G_{\bar{\tau}}$ is also closed by Lemma 1, so that $\bar{\tau}$ is also a cut in $\mathcal{C}$.

Our main theorem extends a line of thought begun in [9]. For transitive groups, it will be sharpened somewhat in Theorem 12. The closure of a convex $l$-subgroup $C$ will be denoted by $C^*$, and $\{\bar{\omega} \in \bar{\Omega} \mid \bar{\omega}C = \bar{\omega}\}$ by $\bar{F}xC$ (written without the bar in [9]). If $\bar{\Gamma} \subseteq \bar{\Omega}$, $G(\bar{\Gamma})$ will denote $\{g \in G \mid \bar{\gamma}g = \bar{\gamma}$ for each $\bar{\gamma} \in \bar{\Gamma}\} = \bigcap \{G_{\bar{\gamma}} \mid \bar{\gamma} \in \bar{\Gamma}\}$. If $\mathcal{D}$ is a convex $G$-congruence, we let $L(\mathcal{D})$ be the $l$-ideal $\{g \in G \mid \Delta g = \Delta$ for all $\Delta \in \mathcal{D}\} = \bigcap \{G_{\bar{\omega}} \mid \bar{\omega}$ is a cut in $\mathcal{D}\}$. If $\mathcal{D}$ is closed, $L(\mathcal{D})$ is closed. If $\mathcal{D} < \mathcal{E}$, then $L(\mathcal{D}) \subseteq L(\mathcal{E})$. We let $G$ act on $\mathcal{D}$ by defining $\Delta g \in \mathcal{D}$ to be $\Delta g \subseteq \Omega$. This action is faithful iff $L(\mathcal{D}) = \{1\}$. The *distributive radical* $D(G)$ of $G$ is the intersection of all the closed primes of $G$, and is $\{1\}$ iff $G$ is completely distributive [2, Theorem 3.4 and Corollary 3.8].

**Main Theorem 9.** *Let* $(G, \Omega)$ *be an l-permutation group.*

(1) *If* $C$ *is any convex l-subgroup of* $G$, *then* $C^* \supseteq G(\overline{F}xC)$.

(2) *If* $C$ *is any closed convex l-subgroup of* $G$, $C = G(\overline{F}xC) = \bigcap \{G_{\overline{\delta}} \mid C \subseteq G_{\overline{\delta}}\}$; *and if* $C$ *is the intersection of a finite collection of stabilizers, all stabilizers containing* $C$ *are closed.*

(3) *If the stabilizers of points of* $\Omega$ *are closed, the closed convex l-subgroups of* $G$ *are precisely the* $G(\overline{\Delta})$*'s,* $\overline{\Delta} \subseteq \overline{\Omega}$, *and thus are intersections of closed primes.*

(4) *If* $P$ *is any prime subgroup of* $G$, $\overline{F}xP^* = \overline{F}xP \cap \overline{\Omega}_c$ *and* $P^* = G(\overline{F}xP \cap \overline{\Omega}_c)$.

(5) *If* $P$ *is any closed prime subgroup of* $G$, *the tower of primes* $Q$ *such that* $P \subset Q \subset G$ *consists entirely of closed stabilizers* $G_{\overline{\omega}}$, *and* $P$ *is a minimal closed prime unless* $G_{\overline{\tau}} \subset P$ *for some closed* $G_{\overline{\tau}}$.

(6) *In (5), if* $P \neq G$ *is not itself a stabilizer, then* $P$ *has no cover in the tower of primes which contain it, and thus is an intersection of closed stabilizers; and* $P$ *is a minimal closed prime.*

(7) *The distributive radical* $D(G)$ *is* $L(\mathcal{C})$.

(8) *Every closed l-ideal of* $G$ *is* $L(\mathfrak{D})$ *for some* $\mathfrak{D}$.

**Proof.** The proof of [9, Theorem 3] proves (1), which gives the first part of (2), which, in view of Theorem 7, gives (3). For the second part of (2), if $C$ is the intersection of a finite collection of stabilizers, it is the intersection of a (finite) meet-irreducible subcollection $\{G_{\overline{\gamma}} \mid \overline{\gamma} \in \overline{\Gamma}\}$, and the $G_{\overline{\gamma}}$'s are closed by Corollary 3. Now let $C \subseteq G_{\overline{\delta}}$. If $G_{\overline{\delta}}$ is not closed, then for each $\overline{\gamma}$, $G_{\overline{\gamma}} \not\subseteq G_{\overline{\delta}}$ by Lemma 1, so there exists $1 < g_{\overline{\gamma}} \in G_{\overline{\gamma}}$ such that $\overline{\delta}g > \overline{\delta}$. Now $\bigwedge g_{\overline{\gamma}} \in \bigcap G_{\overline{\gamma}} = C$, but $\bigwedge g_{\overline{\gamma}} \notin G_{\overline{\gamma}}$, a contradiction.

If $P$ is any prime subgroup of $G$, then $P^* \subseteq \bigcap \{G_{\overline{\omega}} \mid P \subseteq G_{\overline{\omega}}, \overline{\omega} \in \overline{\Omega}_c\}$ since the latter is closed, so that $\overline{F}xP^* \supseteq \overline{F}xP \cap \overline{\Omega}_c$. But of course $\overline{F}xP^* \subseteq \overline{F}xP$, and since $P^* \subseteq G_{\overline{\omega}}$ and $P^*$ closed implies $G_{\overline{\omega}}$ closed by Lemma 1, we have $\overline{F}xP^* \subseteq \overline{\Omega}_c$, so that $\overline{F}xP^* = \overline{F}xP \cap \overline{\Omega}_c$. By (1), $P^* = G(\overline{F}xP^*) = G(\overline{F}xP \cap \overline{\Omega}_c)$.

Now let $P \neq G$ be a closed prime and let $\mathfrak{T}$ be the tower of primes $Q$ (automatically closed by Lemma 1) such that $P \subset Q \subset G$. Let $C \in \mathfrak{T}$. If $P$ is not a stabilizer, then since $P$ is the intersection of the stabilizers which contain it (by (2)), $\mathfrak{T}$ has no smallest member (proving part of (6)) and the subset of $\mathfrak{T}$ consisting of stabilizers is coinitial with $\mathfrak{T}$. Thus in any case, $C$ contains a stabilizer $G_{\overline{\pi}}$. By Theorem 4, $\Delta = \mathrm{Conv}(\overline{\pi}C)$ is an o-block of $G$ and $G_\Delta = C$. Hence $C = G_{\sup\Delta}$. This establishes the first part of (5), which, together with the part of (6) already proved, yields the rest of (5) and (6).

The distributive radical $D(G)$ is the intersection of the closed primes of $G$, and by (6), every closed prime is an intersection of closed stabilizers, so that $D(G) = \bigcap \{G_{\overline{\alpha}} \mid \overline{\omega} \in \overline{\Omega}_c\} = L(\mathcal{C})$.

Finally, let $C$ be any closed $l$-ideal of $G$. Since $C$ is normal in $G$, $\overline{F}xC$ is a union of orbits of $(G, \overline{\Omega})$. Hence we obtain a convex $G$-congruence $\mathfrak{D}$ by setting $\sigma\mathfrak{D}\tau$ iff either $\sigma = \tau$ or no $\overline{\delta}$ between $\sigma$ and $\tau$ (inclusive) lies in $\overline{F}xC$. By (2), $C = G(\overline{F}xC) = L(\mathfrak{D})$. This concludes the proof.

**Theorem 10.** *Let $(G, \Omega)$ be an $l$-permutation group. The following are equivalent:*

(1) *$G$ is completely distributive.*

(2) *$G$ acts faithfully on $\mathcal{C}$.*

(3) *$C^* = G(\overline{F}xC \cap \overline{\Omega}_c)$ for every convex $l$-subgroup $C$ of $G$.*

(4) *Every closed convex $l$-subgroup $C$ of $G$ is the intersection of the closed stabilizers which contain it.*

**Proof.** $G$ is completely distributive iff $D(G) = \{1\}$, and $D(G) = L(\mathcal{C})$ by Theorem 9, so (1) and (2) are equivalent. In any completely distributive $l$-group, every closed convex $l$-subgroup is the intersection of the closed primes which contain it [9, Corollary 5], and hence, by (6) of Theorem 9, of the closed stabilizers which contain it. Thus (1) implies (4). Now suppose (4) holds, and let $C$ be a convex $l$-subgroup of $G$. Then if $G_{\overline{\omega}}$ is closed, $C^* \subseteq G_{\overline{\omega}}$ iff $C \subseteq G_{\overline{\omega}}$, so by (4), $C^* = \bigcap \{G_{\overline{\omega}} \mid C^* \subseteq G_{\overline{\omega}}$ and $G_{\overline{\omega}}$ closed$\} = \bigcap \{G_{\overline{\omega}} \mid \overline{\omega} \in \overline{F}xC \cap \overline{\Omega}_c\} = G(\overline{F}xC \cap \overline{\Omega}_c)$, proving (3). Finally, if (3) holds, then since $\overline{F}x\{1\} = \overline{\Omega}$, we have $\{1\} = \{1\}^* = G(\overline{\Omega}_c) = L(\mathcal{C})$, which is (2).

A *representation* of an $l$-group $H$ on a chain $\Sigma$ is an $l$-isomorphism $\theta$ of $H$ into $A(\Sigma)$. $\theta$ is *complete* if it preserves arbitrary sups that exist in $H$, or equivalently, if $H\theta$ is a complete $l$-subgroup of $A(\Sigma)$. $H$ has a complete representation iff $H$ is completely distributive [2, Theorem 3.10].

By the *normal kernel* of a convex $l$-subgroup $C$ of an $l$-group $H$ we mean the intersection of all the conjugates of $H$, which is the largest $l$-ideal of $G$ contained in $C$. A collection $\{K_\lambda \mid \lambda \in \Lambda\}$ of primes of $H$ whose intersection has trivial normal kernel is called a *Holland kernel* of $H$. These kernels were used by Holland to obtain representations of $H$ as $l$-permutation groups [3, Theorem 2].

**Corollary 11.** *Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be a Holland kernel of an abstract $l$-group $H$, so that the normal kernel of $\bigcap K_\lambda$ is $\{1\}$. Then the normal kernel of $\bigcap K_\lambda^*$ is $D(H)$. Hence if $H$ is completely distributive, $\{K_\lambda^*\}$ is also a Holland kernel.*

**Proof.** Use the Holland representation to represent $H$ as an $l$-permutation group on a chain $\Omega$ so that each $K_\lambda$ is the stabilizer of some point $\omega_\lambda$ and the union of the orbits $\omega_\lambda H$ is $\Omega$. Then by Theorem 9, the $K_\lambda^*$'s are stabilizers of $o$-blocks in the smallest convex $H$-congruence $\mathcal{C}$. Hence the collection of all conjugates of $K_\lambda^*$'s is precisely the collection of stabilizers of all $o$-blocks in $\mathcal{C}$, and the intersection of these is $L(\mathcal{C})$, which by Theorem 9 is $D(H)$.

4. **Transitive $l$-permutation groups.** Recall that in a transitive group $(G, \Omega)$, all the $o$-blocks of a given convex $G$-congruence are translates of each other, and that the $o$-block systems form a tower, so that the concept of the smallest closed system $\mathcal{C}$ is particularly nice. Further, for any $\omega \in \Omega$, the map $\Delta \to \mathfrak{X}$ is an $o$-isomorphism from the tower of $o$-blocks $\Delta$ containing $\omega$ onto the tower of $o$-block systems.

**Theorem 12.** *Let $(G, \Omega)$ be a transitive $l$-permutation group. Then*

(1) *$\mathcal{C}$ is the trivial system iff $G$ is a complete subgroup of $A(\Omega)$.*

(2) *The closed $l$-ideals of $G$ form a tower under inclusion.*

(3) *If $G$ has only a finite number of closed $o$-block systems, every closed prime $\neq G$ is $G_{\bar{\omega}}$ for some cut $\bar{\omega}$ in $\mathcal{C}$ (i.e., (6) of Theorem 9 cannot occur).*

(4) *If $G$ is completely distributive, the closed $l$-ideals of $G$ are precisely the $L(\mathfrak{D})$'s.*

**Proof.** $\mathcal{C}$ is the trivial system iff stabilizers of points are closed, and by [8, Theorem 7] this holds iff $G$ is complete in $A(\Omega)$. (2) follows from (8) of Theorem 9 since the $o$-block systems form a tower.

For (3), let $q$ be the number of closed $o$-block systems, and suppose that $P$ is as in (6) of Theorem 9, and thus has no prime cover. Then there exists above $P$ a stabilizer $G_{\bar{\omega}}$ above which there lies a tower of $q+1$ distinct closed stabilizers. Theorem 4 supplies a tower of $q+1$ distinct $o$-blocks whose systems are closed, a contradiction.

By Theorem 9, every closed $l$-ideal is $L(\mathfrak{D})$ for some $\mathfrak{D}$; and if $\mathfrak{D} \geq \mathcal{C}$, then $\mathfrak{D}$ and hence $L(\mathfrak{D})$ are closed. If $\mathfrak{D} < \mathcal{C}$, then $L(\mathfrak{D}) = L(\mathcal{C}) = \{1\}$ if $G$ is completely distributive, completing the proof.

**Question.** Does the conclusion of (4) hold even without complete distributivity? Nontrivial $L(\mathfrak{D})$'s *can* be closed even when $\mathfrak{D}$ is not, and perhaps they always are.

A representation $\theta$ of an $l$-group $H$ on a chain $\Sigma$ is *transitive* if $H\theta$ is transitive on $\Sigma$. A *representing subgroup* of $H$ is a one element Holland kernel, i.e., a prime subgroup which contains no nontrivial $l$-ideal of $H$. If $H$ has a representing subgroup $K$, then the Holland representation obtained from $K$ is transitive and has $K$ as the stabilizer of a point; and conversely, if $(G, \Omega)$ is transitive, the stabilizers of points are representing subgroups of $G$. Thus if $H$ has a complete transitive representation, $H$ is completely distributive and has a representing subgroup. Conversely, we have

**Corollary 13.** *Let $H$ be a completely distrubutive abstract $l$-group. Then*

(1) *If $\theta$ is a transitive representation of $H$ on a chain $\Sigma$, then $(H\theta, \Sigma)$ can be faithfully "reduced" to the complete transitive group $(H\theta, \mathcal{C})$, where $\mathcal{C}$ is the smallest closed system of $(H\theta, \Sigma)$.*

(2) *If* $K$ *is a representing subgroup of* $H$, *so is* $K^*$. *Hence maximal representing subgroups are closed.*

**Proof.** Since $H\theta$ is completely distributive, its action on $\mathcal{C}$ is faithful, and the stabilizers in this action are closed in $G$, so $G$ is complete in $A(\mathcal{C})$, proving (1). Since a representing subgroup is precisely a one element Holland kernel, (2) follows from Corollary 11.

A transitive group $(G, \Omega)$ is said to be *weakly o-primitive* [4] if $L(\mathfrak{D}) = \{1\}$ only when $\mathfrak{D}$ is the trivial system, so that $(G, \Omega)$ has no proper faithful reductions.

**Corollary 14.** *Conditions* (1), (2), (3), *and* (4) *of Theorem* 7 *are equivalent for weakly o-primitive groups.*

**Proof.** We need only show that (4) implies (3). But by Theorem 10, complete distributivity implies $L(\mathcal{C}) = \{1\}$, so that by weak o-primitivity, $\mathcal{C}$ must be the trivial system, i.e., (3) holds.

**Corollary 15.** *Let* $H$ *be an abstract l-group having a representing subgroup. Then the closed l-ideals of* $H$ *form a tower under inclusion.*

**Proof.** Since $H$ has a transitive representation, we may use part (8) of Theorem 9.

6. **The support property.** If an o-block $\Delta$ of a transitive group $G$ supports some $1 \neq g \in G$, so does every $\Gamma \in \mathfrak{X}$, and by Lemma 6, $\mathfrak{X}$ must be closed. We shall say that $G$ has the *support property* if each of its nontrivial o-blocks supports some $1 \neq g \in G$. Clearly groups having the support property are weakly o-primitive.

A transitive group $(G, \Omega)$ is called *locally o-primitive* if it has a smallest nontrivial o-block system $\mathfrak{D}$. In this case the o-blocks of $\mathfrak{D}$ are called *primitive segments*. The $(G_\Delta \mid \Delta, \Delta)$'s for the various primitive segments $\Delta$ are all isomorphic as $l$-permutation groups, and are o-primitive; and a property enjoyed by them is said to be enjoyed *locally* by $(G, \Omega)$. If $G$ is not locally o-primitive, then $\{\alpha\} = \bigcap \{\Delta \mid \alpha \in \Delta,\ \Delta \text{ a nontrivial } o\text{-block}\}$, so that the support property implies that every nondegenerate interval of $\Omega$ supports some $1 \neq g \in G$. In view of the next lemma, this implication also holds if $G$ is locally nonpathologically o-2-transitive. (An o-2-transitive group is *pathological* if it lacks elements $\neq 1$ of bounded support.)

**Lemma 16.** *Let* $(G, \Omega)$ *be a locally o-primitive group having the support property. Let* $\Delta$ *be a primitive segment, let* $K = G_\Delta \mid \Delta$, *and let* $L = \{g \mid \Delta : \Delta \text{ supports } g\}$. *Then if* $G$ *is locally o-2-transitive,* $L$ *is an o-2-transitive l-ideal of* $K$ *containing all elements of bounded support; and otherwise* $L = K$.

**Proof.** It is easily checked that $L$ is an $l$-ideal of $K$, and the support property states that $L \neq \{1\}$. In pathologically $o$-2-transitive groups, all nontrivial $l$-ideals are (pathologically) $o$-2-transitive [11, Theorem 6], and in nonpathologically $o$-2-transitive groups, the $o$-2-transitive $l$-ideal consisting of the elements of bounded support is contained in every nontrivial $l$-ideal (by the proof of [3, Theorem 6]). $O$-primitive groups which are not $o$-2-transitive have no proper $l$-ideals [10, Corollary 46]. The lemma follows.

The following theorem generalizes [12, Theorem 4], which states that an $o$-primitive group fails to be complete iff it is pathologically $o$-2-transitive.

**Theorem 17.** *Let $(G, \Omega)$ be a transitive l-permutation group having the support property. Then $(G, \Omega)$ fails to be complete if and only if it is locally pathologically o-2-transitive; and in this case, the smallest closed o-block system $\mathcal{C}$ is the smallest nontrivial system.*

**Proof.** Since by Lemma 6 the support by an $o$-block $\Delta$ of some $1 \neq g \in G$ implies the closure of $\overline{\Delta}$, the support property guarantees that either $\mathcal{C}$ is the trivial system (i.e., $G$ is complete), or $G$ is locally $o$-primitive and $\mathcal{C}$ is the smallest nontrivial system. It remains to show that when $G$ is locally $o$-primitive, the stabilizers $G_\alpha$ fail to be closed iff $G$ is locally pathologically $o$-2-transitive.

Let $\alpha \in \Omega$, let $\Delta$ be the primitive segment containing $\alpha$, and let $K$ and $L$ be as in the lemma. Suppose $s = \sup\{s_i\}$, with each $s_i \in G_\alpha$ and $s \in G \backslash G_\alpha$. Since $G_\Delta$ is closed, $\alpha s \in \Delta$, $(K, \Delta)$ is $o$-primitive, and if it is not pathologically $o$-2-transitive, $K_\alpha$ is closed in $K$ [12, Theorem 4]. Thus, letting $\hat{s} = s \mid \Delta$ and $\hat{s}_i = s_i \mid \Delta$, there exists $1 < \hat{t} \in K$ such that, for each $i$, $\hat{s}_i \leq \hat{s}\hat{t}^{-1} < \hat{s}$. By the lemma, $\hat{t} \in L$. (If $(K, \Delta)$ is nonpathologically $o$-2-transitive, $\hat{t}$ exceeds some $1 < k \in K$ having bounded support, so we may assume that $\hat{t}$ has bounded support.) Hence $\Delta$ supports some $t \in G$ such that $t \mid \Delta = \hat{t}$. $st^{-1}$ agrees with $\hat{s}\hat{t}^{-1}$ on $\Delta$ and with $s$ outside $\Delta$, so for each $i$, $s_i \leq st^{-1} < s$, a contradiction. Therefore $G_\alpha$ is closed in $G$.

Conversely, suppose that $G$ is locally pathologically $o$-2-transitive. By the lemma, $L$ is (pathologically) $o$-2-transitive, so there exists $\{\hat{s}_i\} \subseteq L_\alpha$ having in $L$ a sup $\hat{s} \notin L_\alpha$. $\Delta$ supports $s, s_i \in G$ such that $s \mid \Delta = \hat{s}$ and $s_i \mid \Delta = \hat{s}_i$. Then $s = \sup\{s_i\}$ in $G$, for if $s_i \leq t < s$, then $\Delta$ supports $t$, so that $t \mid \Delta$ yields a contradiction. Therefore $G_\alpha$ is not closed in $G$.

If an $l$-permutation group $(G, \Omega)$ has the support property and is locally $o$-primitive, but not locally pathologically $o$-2-transitive, the other transitive representations of $G$ can be very precisely described. (The description is due to Holland [4, Theorem 7]. In [11, Theorem 7], the hypotheses are relaxed to those just mentioned, and it is shown that pathologically $o$-2-transitive groups need not satisfy the description.) Theorem 17 says that the groups $(G, \Omega)$ satisfying these hypotheses are precisely those locally o-primitive groups having the support property which are complete.

**Corollary 18.** *Let* $\{(G_\gamma, \Omega_\gamma) \mid \gamma \in \Gamma\}$ *be a collection of transitive l-permutation groups indexed by a chain* $\Gamma$, *and let* $(W, \Omega)$ *be its ordered wreath product* [5, § 3]. *The transitive l-permutation group* $(W, \Omega)$ *fails to be complete if and only if* $\Gamma$ *has a least element* 0 *and* $(G_0, \Omega_0)$ *is pathologically o-2-transitive.*

7. **Examples.** Our first example shows that completely distributive transitive groups need not have closed stabilizers (cf. Theorem 7 and Corollary 14). Holland [4, p. 433] pointed out that if $\Omega$ is the chain of real numbers, the stabilizer subgroups of $A(\Omega)$ are not minimal representing subgroups. Pick a representing subgroup $K$ properly contained in a stabilizer. By part (3) of Theorem 12, $K$ is not closed. Thus in the Holland representation of $A(\Omega)$ on $R(K)$, $K$ is the stabilizer of a point and is not closed, even though $A(\Omega)$ is completely distributive.

Next, we exhibit a transitive group $(G, \Omega)$ with $G$ totally ordered, so that $\{1\}$ is a closed prime, but with no stabilizer $G_{\bar{\omega}} = \{1\}$ (cf. Theorem 9). Let $I$ be the o-group of integers. Let $J$ be the o-group

$$\overleftarrow{\cdots \oplus I \oplus I \oplus \cdots},$$

the small direct sum (indexed by $I$) of copies of $I$, ordered lexicographically from the right. Let $f$ be the o-group automorphism of $J$ which shifts each sequence one place to the right. Map $1 \in I$ onto $f$ and extend the map to a group homomorphism of $I$ into the group of o-group automorphisms of $J$. Take $G$ to be the semi-direct sum $\overleftarrow{J \oplus I}$ (i.e., $\langle j_1, i_1\rangle + \langle j_2, i_2\rangle = \langle j_1 f^{i_2} + j_2, i_1 + i_2\rangle$), ordered lexicographically from the right to give an o-group. Let $K = \{\langle j, 0\rangle \mid j(n) = 0$ for all $n \geq 0\}$. $K$ is a representing subgroup of $G$, for if $\langle 0, 0\rangle \neq \langle j, 0\rangle \in K$, and if $n < 0$ is the greatest integer such that $j(n) \neq 0$, then $\langle 0, n\rangle + \langle j, 0\rangle + \langle 0, -n\rangle \notin K$. Let $G$ act on $\Omega = R(K)$ via the Holland representation. Letting $\alpha = K$, we have $K + \langle j, i\rangle \in FxG_\alpha$ iff $i \geq 0$, so that $FxG_\alpha$ is a cofinal segment of $\Omega$ extending properly below $\alpha$. For any $\bar{\omega} \in \bar{\Omega}$, we can pick $\beta < \bar{\omega}$, and then $\{1\} \neq G_\beta \subseteq G_{\bar{\omega}}$. Therefore $\{1\}$ is not a stabilizer $G_{\bar{\omega}}$.

The argument that $K$ is a representing subgroup also shows that $K$ is properly contained in certain of its conjugates. (In [10, § 3], groups exhibiting this pathology were called *imbalanced*, but the present example, which had been pointed out to the author by Charles Holland, was omitted for the sake of brevity.) Therefore, $K$ ($= G_\alpha$) is not a maximal representing subgroup, so that $(G, \Omega)$ is not weakly o-primitive. However, $G_\alpha$ is closed in $G$ since $G$ is an o-group, so that $(G, \Omega)$ is complete. Thus among completely distributive transitive groups, weak o-primitivity is properly stronger than completeness (cf. Corollary 14).

Closed primes which are not stabilizers can occur even in groups which are complete and weakly and locally $o$-primitive: Form an ordered wreath product $(W, \Sigma)$ of two factors, with the previous group $(G, \Omega)$ as top factor, and $(A(\Lambda),\Lambda)$, $\Lambda$ the reals, as bottom factor. $W$ is complete by Corollary 18. But $L(\mathfrak{D})$, $\mathfrak{D}$ the smallest nontrivial $o$-block system, is a closed prime (since $G$ is an $o$-group) which is not a stabilizer.

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