TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 173, November 1972

A NEW CLASS OF FUNCTIONS OF BOUNDED INDEX

BY

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ABSTRACT. Entire functions of strongly bounded index have been defined and it is shown that functions of genus zero and having all negative zeros satisfying a one sided growth condition belong to this class.

1. **Introduction.** Let $f(z)$ be an entire function and let

$$\Omega(z) = \Omega_s(z) = \max_{0\le j\le s}\left\{\frac{|f^{(j)}(z)|}{j!}\right\} \quad (f^{(0)}(z) = f(z)). \tag{1.1}$$

Definition 1. An entire function $f(z)$ is said to be of *bounded index* (we shall also say of class B), if for some fixed s, $\{|f^{(n)}(z)|/n!\} \le \Omega_s(z)$ for all n and all z (see [3], [6]).

It is known that given any transcendental entire function f, there exists a transcendental entire function g of unbounded index such that [7]

$$\log M(r, f) \sim \log M(r, g) \qquad (r \to \infty).$$

In particular, given two numbers λ and ρ such that $0 \le \lambda \le \rho \le \infty$, there exists a function g of unbounded index such that g is of order ρ and of lower order λ. A result of this type cannot hold with g of bounded index since a function of bounded index must necessarily be of exponential type [8]. Furthermore, known examples of functions of bounded index and order one are all of regular growth, that is, the order of a function is equal to its lower order ([5], [9]). In this paper we show that there exist functions of bounded index, and of given order ρ and lower order λ provided $0 \le \lambda \le \rho \le 1$ (see also [10]). Our attempts to construct such functions have led us to the remark that a very simple subclass SB, of the class B, displays a particularly useful property. If $f \in SB$ and P is a polynomial then $fP \in SB$.

Definition 2. An entire function $f(z)$ is of *strongly bounded index* (we shall also say of class SB) if there exist quantities χ, $0 < \chi < 1$, r_0, and an integer $s \ge 0$ such that

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Presented to the Society, March 5, 1971 under the title *Entire functions of strongly bounded index* and March 31, 1971 under the title *Entire functions of strongly bounded index*. II; received by the editors October 4, 1971.

AMS (MOS) subject classifications (1970). Primary 30A64; Secondary 30A66.

Key words and phrases. Order and lower order of an entire function, exponential type, bounded index.

(1) Research supported by N.S.F. Grant GP-19533.

(2) Research supported by N.S.F. Grant GP-7507.

$$|f^{(n)}(z)|/n! \le \chi \Omega_s(z), \tag{1.2}$$

for all $n \ge s+1$ and all z with $|z| \ge r_0$. For instance, $f(z) = e^z \in SB$. Here $\chi = \frac{1}{2}$, $s = 1$, $r_0 = 0$. We now state

Theorem 1. *Let $f(z)$ be entire and $f(z) \in SB$. Then*

(i) $f(z) \in B$,

(ii) *if $P(z)$ is a polynomial then $f(z)P(z) \in SB$,*

(iii) $\{f(z)/P(z)\} \in SB$ *provided* $\{f(z)/P(z)\}$ *is entire,*

(iv) *if a is any complex number and*

$$0 < \chi < e^{-2|a|} \tag{1.3}$$

where χ is the constant in (1.2), *then* $e^{az}f(z) \in SB$.

Our main result is

Theorem 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a positive, strictly increasing sequence such that*

$$a_{n+1} - a_n \ge b_n \qquad (n \ge 1), \tag{1.4}$$

where $\{b_n\}_{n=1}^{\infty}$ is positive nondecreasing and

$$\sum_{n=1}^{\infty} \frac{1}{nb_n} < \infty. \tag{1.5}$$

Then

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right) \in SB. \tag{1.6}$$

Theorem 2 has four useful corollaries. Consider the Lindelöf functions

$$f(z) = \prod_{1}^{\infty} \left(1 + \frac{z}{a_n}\right), \tag{1.7}$$

where $a_n = \{n(\log n)^{\alpha}\}^{1/\lambda}$, $0 < \lambda \le 1$ and $\alpha > 1$ if $\lambda = 1$, α an arbitrary real number if $0 < \lambda < 1$.

Corollary 2.1. *All Lindelöf functions defined by* (1.7) *belong to class SB.*

Corollary 2.2. *If α is any nonzero complex number and $f(z)$ satisfies the assumptions of Theorem* 2, *then*

$$f(\alpha z) = F(z) \in SB. \tag{1.8}$$

Corollary 2.3. *Let $\{a_n\}_{n=1}^{\infty}$ satisfy the conditions of Theorem* 2, *and let $\{a_{n_j}\}_{j=1}^{\infty}$ be any one of its infinite subsequences.*

Then

$$\prod_{j=1}^{\infty}\left(1+\frac{z}{a_{n_j}}\right)=g(z)\in SB.$$

From Corollary 2.2, we deduce that $F(z)\in SB\subset B$ and consequently there exists an index s such that

$$\frac{|F^{(n)}(z)|}{n!}\le \max_{0\le j\le s}\left\{\frac{|F^{(j)}(z)|}{j!}\right\},$$

for all n and all z.

Assume now α real and greater than one so that (1.8) and (1.1) imply

$$\frac{\alpha^n|f^{(n)}(\alpha z)|}{n!}\le \alpha^s\Omega_s(\alpha z).$$

Replacing αz by ζ, we obtain

$$|f^{(n)}(\zeta)|/n!\le \Omega_s(\zeta)/\alpha^{n-s}\qquad (n=s+1,\ s+2,\cdots)$$

for all ζ.

We thus see that the functions in Theorem 2 belong to SB in a very special sense: the constant χ in (1.2) may be chosen arbitrarily small (a diminution of χ will of course increase, in general, the value of the index s).

In particular, if a is given, we can choose χ so as to satisfy (1.3). Consequently assertion (iv) of Theorem 1 leads to

Corollary 2.4. *If $f(z)$ satisfies the conditions of Theorem 2 then $e^{az+b}f(z)$ $\in SB$.*

In Corollary 2.3 we can choose a subsequence $\{a_{n_j}\}$ by omitting from the given sequence $\{a_n\}$ long sections of consecutive terms. The entire function $h(z)$ corresponding to this subsequence belongs the the class SB and it is obvious that we may, by suitable choices of the gaps, obtain irregularities in the growth of $h(z)$. We are thus led to the following result which we state without proof.

Theorem 3. *Let $f(z)$ be given by* (1.6) *and let a sequence $\{a_n\}_{n=1}^{\infty}$ satisfy the conditions of Theorem* 2. *Let*

$$R_1,\ R_2,\cdots\qquad (R_m<R_{m+1},\ m=1,2,\cdots,\ R_m\to\infty)\tag{1.9}$$

be a given sequence.

It is always possible to select a subsequence $\{c_j\}_{j=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ and two subsequences $\{R_k'\}_{k=1}^{\infty}$, $\{R_k''\}_{k=1}^{\infty}$ of (1.9) *such that*

$$h(z)=\prod_{j=1}^{\infty}\left(1+\frac{z}{c_j}\right)\in SB,\tag{1.10}$$

and such that for all $k = 1, 2, \cdots$

$$\log M(R'_k, h) > (1 - 1/(k+1)) \log M(R'_k, f),$$

and

$$\frac{\log \log M(R''_k, h)}{\log R''_k} < \frac{1}{k+1}.$$

By an appropriate choice of gaps we can also construct a function h belonging to B and of given order ρ and of given lower order λ where $0 \le \lambda \le \rho \le 1$. We omit the details of construction.

In §2 we give the proof of Theorem 1. §3 contains necessary lemmas and §4 gives the proof of Theorem 2.

The authors wish to thank Professor Albert Edrei who suggested the consideration of functions of the class SB and conjectured Theorems 1 and 2.

2. **Proof of Theorem 1. Proof of assertion (i).** By Definition 2, there exist fixed quantities χ, $0 < \chi < 1$, r_0 and $s \ge 0$ such that (1.2) holds for all $n \ge s+1$ and all z with $|z| \ge r_0$.

We examine $f(z)$ and its successive derivatives in the closed disk

$$|z| \le r_0. \tag{2.1}$$

Since the number of zeros of f in (2.1) is $n(r_0, 1/f)$, it is obvious that one of the quantities $f(z), f'(z), \cdots, f^{(N)}(z)$ $(N = n(r_0, 1/f))$ is different from zero.

Let

$$\Omega_N(z) = \max_{0 \le j \le N} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\}. \tag{2.2}$$

It is clear that $\Omega_N(z)$ is continuous and does not vanish in (2.1). Hence for some $\alpha > 0$,

$$\Omega_N(z) \ge \alpha \qquad (|z| \le r_0). \tag{2.3}$$

Assume $|z| \le r_0$. By Cauchy's formula, for the nth derivative,

$$\frac{|f^{(n)}(z)|}{n!} \le \frac{1}{2^n} M(r_0 + 2, f). \tag{2.4}$$

If n is sufficiently large, say $n \ge n_0 \ge s+1$, (2.3) and (2.4) imply

$$\frac{|f^{(n)}(z)|}{n!} \le \frac{M(r_0+2, f)}{2^n} \le \chi\alpha \le \chi\Omega_N(z) \tag{2.5}$$

for all z such that $|z| \le r_0$ and for all $n \ge n_0$. Let $p = \max(n_0, N)$. Then (2.5) and (2.2) imply

$$\frac{|f^{(n)}(z)|}{n!} \le \chi \max_{0\le j\le p} \left\{\frac{|f^{(j)}(z)|}{j!}\right\} \qquad (n \ge p+1), \tag{2.6}$$

provided $|z| \le r_0$. On the other hand, since $n_0 \ge s+1$, and $f \in SB$, (2.6) holds for $n \ge p+1$ and $|z| \ge r_0$. Hence we can drop the restriction on the size of $|z|$ and this completes the proof.

Proof of assertion (ii). By hypothesis (1.2) holds for all $n \ge s+1$ and all z such that $|z| \ge r_0$. Let

$$g(z) = (z - z_0)f(z), \tag{2.7}$$

$$\Omega(z) = \max_{0\le j\le s} \left\{\frac{|f^{(j)}(z)|}{j!}\right\}, \tag{2.8}$$

$$\Omega^*(z) = \max_{0\le j\le s+1} \left\{\frac{|g^{(j)}(z)|}{j!}\right\}. \tag{2.9}$$

Since

$$\frac{g^{(n)}(z)}{n!} = (z - z_0)\frac{f^{(n)}(z)}{n!} + \frac{f^{(n-1)}(z)}{(n-1)!} \tag{2.10}$$

we have, for $n \ge s+2$ and $|z| \ge r_0$,

$$\frac{|g^{(n)}(z)|}{n!} \le \chi\Omega(z)\{1 + |z - z_0|\}. \tag{2.11}$$

From (2.7) we obtain, for $z \ne z_0$,

$$\frac{f^{(n)}(z)}{n!} = \frac{g^{(n)}(z)}{n!}\frac{1}{(z-z_0)} + \frac{g^{(n-1)}(z)}{1!(n-1)!}\frac{(-1)1!}{(z-z_0)^2} + \cdots + \frac{g(z)}{0!}\frac{(-1)^n n!}{n!(z-z_0)^{n+1}}. \tag{2.12}$$

(2.9) and (2.12) yield, for $z \ne z_0$,

$$\frac{|f^{(n)}(z)|}{n!} \le \Omega^*(z)\left\{\frac{1}{|z-z_0|} + \frac{1}{|z-z_0|^2} + \cdots + \frac{1}{|z-z_0|^{n+1}}\right\} \qquad (0 \le n \le s+1).$$

Consequently we have from (2.11) and (2.8), for $n \ge s+2$ and $|z| \ge r_0$, $z \ne z_0$,

$$\frac{|g^{(n)}(z)|}{n!} \le \chi\Omega^*(z)\{1 + |z-z_0|\}\left\{\frac{1}{|z-z_0|} + \cdots + \frac{1}{|z-z_0|^{n+1}}\right\}.$$

If $|z|$ is sufficiently large, say $|z| \ge R_1$, then

$$\chi\{1 + |z-z_0|\}\left\{\frac{1}{|z-z_0|} + \cdots + \frac{1}{|z-z_0|^{n+1}}\right\} < \chi'$$

where $0 < \chi' < 1$. This shows that $g(z) \in SB$. Now if $P(z) = (z - z_0)\cdots(z - z_p)$

and $Q(z) = AP(z)$ (A a constant) then the above argument applied $(p+1)$ times shows that $fP \in SB$, $fQ \in SB$. This completes the proof.

Proof of assertion (iii). Let

$$g(z) = f(z)/(z - z_0) \tag{2.13}$$

and let $\Omega(z)$ and $\Omega^*(z)$ have the same meaning as in (2.8) and (2.9). Then for $n \geq s+1$ and $|z| \geq r_0$, $z \neq z_0$,

$$\frac{|g^{(n)}(z)|}{n!} \leq \Omega(z) \left\{ \frac{\chi}{|z - z_0|} + \frac{1}{|z - z_0|^2} + \cdots + \frac{1}{|z - z_0|^{n+1}} \right\}. \tag{2.14}$$

From (2.13) we have, for $1 \leq n \leq s$,

$$\frac{|f^{(n)}(z)|}{n!} \leq \frac{|g^{(n)}(z)|}{n!} |z - z_0| + \frac{|g^{(n-1)}(z)|}{(n-1)!} \leq \Omega^*(z)\{1 + |z - z_0|\} \tag{2.15}$$

and

$$|f(z)| = |g(z)|\,|z - z_0| \leq \Omega^*(z)\{1 + |z - z_0|\}.$$

Hence (2.15) holds for $0 \leq n \leq s$ and

$$\Omega(z) \leq \Omega^*(z)\{1 + |z - z_0|\}. \tag{2.16}$$

The inequalities (2.14) and (2.16) imply

$$\frac{|g^{(n)}(z)|}{n!} \leq \Omega^*(z)\{1 + |z - z_0|\} \left\{ \frac{\chi}{|z - z_0|} + \cdots + \frac{1}{|z - z_0|^{n+1}} \right\} \tag{2.17}$$

for $n \geq s+1$ and $|z| \geq r_0$, $z \neq z_0$. Hence for $|z|$ sufficiently large, say $|z| \geq R_2$, we have $|g^{(n)}(z)|/n! \leq \chi'' \, \Omega^*(z)$, where $0 < \chi'' < 1$, for $n \geq s+1$ and $|z| \geq R_2$. This means that $g(z) \in SB$.

If $P(z) = \Pi_{j=0}^{p}(z - z_j)$ and $Q(z) = AP(z)$, then the above argument shows that $f/P \in SB$, $f/Q \in SB$. This completes the proof.

Proof of assertion (iv). Let

$$g(z) = e^{az} f(z). \tag{2.18}$$

Then

$$\frac{g^{(n)}(z)}{n!} = e^{az} \left\{ \frac{f^{(n)}(z)}{n!} + a\frac{f^{(n-1)}(z)}{(n-1)!} + \cdots + \frac{a^n}{n!} f(z) \right\}. \tag{2.19}$$

There is a similar relation where f and g are exchanged and a is replaced by $-a$. From this latter formula we deduce

$$\frac{|f^{(n)}(z)|}{n!} \leq |e^{-az}| \max_{0\leq j\leq n} \left\{\frac{|g^{(j)}(z)|}{j!}\right\} e^{|a|}.$$

In particular if

$$\Omega^{**}(z) = \max_{0\leq j\leq s} \left\{\frac{|g^{(j)}(z)|}{j!}\right\}, \tag{2.20}$$

we have, in view of (1.1),

$$\Omega(z) \leq |e^{-az}| e^{|a|} \Omega^{**}(z). \tag{2.21}$$

By assumption, (1.2) is satisfied for some fixed $\chi < 1$ and consequently (2.19) yields for all $n \geq s+1$ and all z such that $|z| \geq r_0$

$$\frac{|g^{(n)}(z)|}{n!} \leq |e^{az}| \left\{\left(1 + \frac{|a|}{1!} + \cdots + \frac{|a|^{n-s-1}}{(n-s-1)!}\right)\chi\Omega(z) + \left(\frac{|a|^{n-s}}{(n-s)!} + \cdots + \frac{|a|^n}{n!}\right)\Omega(z)\right\}$$

$$\leq |e^{az}| e^{|a|} \left\{\chi + \frac{|a|^{n-s}}{(n-s)!}\right\}\Omega(z).$$

Using (2.21) we find, for $n \geq s+1$ and $|z| \geq r_0$

$$\frac{|g^{(n)}(z)|}{n!} \leq e^{2|a|}\left(\chi + \frac{|a|^{n-s}}{(n-s)!}\right)\Omega^{**}(z).$$

Since s is fixed $|a|^{n-s}/(n-s)! \to 0$ as $n \to \infty$, and so we may select, in view of (1.3), an integer $s_0 \geq s$, so that

$$e^{2|a|}\left(\chi + \frac{|a|^{n-s}}{(n-s)!}\right) < \chi' < 1$$

as soon as $n \geq s_0 + 1$. Hence for all $n \geq s_0 + 1$ and all z with $|z| \geq r_0$,

$$\frac{|g^{(n)}(z)|}{n!} \leq \chi'\Omega^{**}(z) \leq \chi' \max_{0\leq j\leq s_0}\left\{\frac{|g^{(j)}(z)|}{j!}\right\}.$$

The proof of Theorem 1 is now complete.

3. **Lemmas.** We require several lemmas. The first two lemmas contain known results.

Lemma A [2, Example B. 18]. *If $\{b_n\}_1^\infty$ is positive nondecreasing and $\Sigma(nb_n)^{-1} < \infty$, then*

$$\lim_{n\to\infty} \frac{\log n}{b_n} = 0. \tag{3.1}$$

Lemma B [4, p. 261]. *If $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$, then*

(3.2) $$\alpha_1\beta_{j_1} + \alpha_2\beta_{j_2} + \cdots + \alpha_n\beta_{j_n} \le \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$$

for every permutation $j_1, \cdots, j_n$ *of* $1, 2, \cdots, n$.

Lemma 1. *Let* $a_{n+1} - a_n \ge b_n$ $(n \ge 1)$ *where* $\{b_n\}_1^\infty$ *is positive nondecreasing and* $\Sigma(nb_n)^{-1} < \infty$. *Given* $K \ge 1$ *and* $\epsilon > 0$, *it is possible to find an integer* $N \ge 1$ *and a positive nondecreasing sequence* $\{c_n\}_{n=1}^\infty$ *such that the following conditions hold simultaneously*:

(3.3) $$c_n \ge 8 \quad (n \ge N),$$

(3.4) $$\sum_{j=N}^\infty \frac{1}{jc_j} < \frac{\epsilon}{4},$$

(3.5) $$a_{n+1} - a_n \ge Kc_n + 8 \quad (n \ge N),$$

(3.6) $$\sum_{j=1}^\infty \frac{1}{a_j} < +\infty,$$

(3.7) $$a_{N+2m+1} - a_N > a_{N+2m} - a_N \ge (K+1)mc_{N+m} \quad (m \ge 1),$$

(3.8) $$\frac{1}{c_{n+1}} + \frac{1}{c_{n+1} + c_{n+2}} + \frac{1}{c_{n+1} + c_{n+2} + c_{n+3}} + \cdots \le \epsilon \quad (n \ge N),$$

(3.9) $$\frac{1}{c_{n-1}} + \frac{1}{2c_{n-2}} + \frac{1}{3c_{n-3}} + \cdots + \frac{1}{(n-N)c_N} < \epsilon \quad (n > N).$$

Proof. (i) Let $c_n = b_n/(K+1)$. Then $\{c_n\}_{n=1}^\infty$ is the required sequence such that $\Sigma(nc_n)^{-1} < +\infty$. By Lemma A and the convergence of this series we can choose N so large that (3.3) and (3.4) are satisfied.

(ii) $$a_{n+1} - a_n \ge b_n = (K+1)c_n \ge Kc_n + 8 \quad (n \ge N).$$

This proves (3.5).

(iii) Since

$$a_{p+2m} - a_p = \sum_{j=0}^{2m-1} (a_{p+j+1} - a_{p+j}) \ge (K+1) \sum_{j=0}^{2m-1} c_{p+j},$$

we have on taking $p = 2$, $p + 2m = 2n$,

$$a_{2n} > (K+1)(c_n + \cdots + c_{2n-1}) \ge (K+1)nc_n.$$

Hence

$$\frac{1}{a_{2n}} + \frac{1}{a_{2n+1}} < \frac{2}{K+1}\frac{1}{nc_n},$$

and the convergence of the series in (3.6) follows from (3.4).

(iv) Taking $p = N$ we get

$$a_{N+2m} - a_N \geq (K+1)m\, c_{N+m}.$$

Since $\{a_n\}\uparrow$, (3.7) follows.

(v) We have

$$c_{n+1} + \cdots + c_{n+2m-1} \geq c_{n+m} + \cdots + c_{n+2m-1} \geq m\, c_{n+m} \qquad (m \geq 1)$$

and so

$$\begin{aligned}\Sigma_n &= \frac{1}{c_{n+1}} + \frac{1}{c_{n+1} + c_{n+2}} + \cdots \\ &\leq 2\sum_{j=1}^{\infty} \frac{1}{j\, c_{n+j}} < \frac{2}{c_n}\sum_{j=1}^{n}\frac{1}{j} + 2\sum_{j=n+1}^{\infty}\frac{1}{j\, c_{n+j}} \\ &< \frac{2}{c_n}(1 + \log n) + 2\sum_{j=n+1}^{\infty}\frac{1}{j\, c_{n+j}}.\end{aligned}$$

By Lemma A

$$2(1+\log n)/c_n < \epsilon/2 \quad \text{if } n \geq N_1$$

and

$$2\sum_{j=n+1}^{\infty}\frac{1}{j\, c_{n+j}} < \frac{\epsilon}{2} \quad \text{if } n \geq N_2.$$

Let $N = \max(N_1, N_2)$. Then for $n \geq N$, $\Sigma_n < \epsilon$ and (3.8) is proved.

(vi) Let $n > N$ and $r = 1/c_{n-1} + 1/2c_{n-2} + \cdots + 1/(n-N)c_N$. By (i),

(a) $$\frac{1}{c_{n-1}} \leq \frac{1}{c_{n-2}} \leq \cdots \leq \frac{1}{c_N},$$

and

(b) $$\frac{1}{n-N} < \frac{1}{n-N-1} < \cdots < \frac{1}{2} < \frac{1}{1}.$$

By applying Lemma B to (a) and (b) we have

$$\begin{aligned} r &\leq \frac{1}{c_N} + \frac{1}{2c_{N+1}} + \cdots + \frac{1}{(n-N)c_{n-1}} \\ &\leq \frac{1}{c_N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) + \frac{1}{(N+1)c_{N+N}} + \cdots + \frac{1}{(n-N)c_{n-1}} \\ &\leq \frac{1+\log N}{c_N} + \sum_{j=1}^{\infty}\frac{1}{(N+j)c_{N+j}} < \epsilon \end{aligned}$$

for N sufficiently large. This proves (3.9) and also completes the proof of Lemma 1.

In what follows in this section and in the next section we shall take $\epsilon = 1/100$ and $N \geq 1$ such that (3.3)–(3.9) hold and also

$$\frac{1}{c_N} + \sum_{j=1}^{\infty} \frac{1}{j c_{N+j}} < \frac{K}{100}.$$

Lemma 2. *Let* $\Gamma_n = \{z: |z + a_n| < 4\}$, $n \geq N$, *and suppose* $z \notin \bigcup_{n=N}^{\infty} \Gamma_n$. *Then*

$$\sum_{j=N}^{\infty} \frac{1}{|z + a_j|} < \chi < 1 \tag{3.10}$$

where χ *is some fixed number.*

Proof. Let $z = x + iy$. Then either

(i) $x \geq - a_N$, or

(ii) $- a_{n+1} \leq x < - a_n$ for some well-determined $n \geq N$.

Suppose first that (i) holds. By assumption

$$1/|z + a_N| \leq 1/4 \tag{iii}$$

and

$$|z + a_{N+1}| \geq |x + a_{N+1}| \geq a_{N+1} - a_N \geq (K + 1) c_N.$$

Hence

$$1/|z + a_{N+1}| \leq 1/(K + 1) c_N < 1/100. \tag{iv}$$

For $j \geq N + 2$,

$$|z + a_j| \geq |x + a_j| \geq a_j - a_N.$$

Hence by (3.7) we have for $j = N + 2m$ or $N + 2m + 1$, $m \geq 1$,

$$|z + a_j| \geq (K + 1)m c_{N+m}.$$

Consequently

$$\sum_{j=N+2}^{\infty} \frac{1}{|z + a_j|} \leq \frac{1}{K + 1} \sum_{m=1}^{\infty} \frac{1}{m c_{N+m}} < \frac{1}{100} \tag{v}$$

and (3.10) follows from (iii)–(v). Suppose now (ii) holds. Then

$$1/|z + a_n| + 1/|z + a_{n+1}| < 1/2. \tag{vi}$$

For $j \geq n + 2$,

$$|z + a_j| \geq |x + a_j| \geq a_j - a_{n+1} \geq (K + 1)(c_{j-1} + c_{j-2} + \cdots + c_{n+1})$$

and so, by (3.8),

(vii) $$\sum_{j=n+2}^{\infty} \frac{1}{|z+a_j|} < \frac{1}{K+1}\left\{\frac{1}{c_{n+1}} + \frac{1}{c_{n+1}+c_{n+2}} + \cdots\right\} < \frac{1}{100}.$$

If $n > N$ we have for $N \le j \le (n-1)$

$$|z+a_j| \ge |x+a_j| \ge |a_n - a_j| \ge (K+1)(n-j)\,c_j$$

and (3.9) yields

(viii) $$\sum_{j=N}^{n-1} \frac{1}{|z+a_j|} \le \frac{1}{K+1}\left\{\frac{1}{c_{n-1}} + \frac{1}{2c_{n-2}} + \cdots + \frac{1}{(n-N)\,c_N}\right\} < \frac{1}{100}.$$

From (vi)–(viii) we get (3.10) in this case also. The proof of Lemma 2 is complete.

Lemma 3. *Let*

$$f(z) = \prod_{j=N}^{\infty}\left(1 + \frac{z}{a_j}\right)$$

and let $\{-d_j\}_{j=N}^{\infty}$ *be the zeros of* $f'(z)$. *Then for all* j *and* k $(j \ge N,\ k \ge N)$,

(3.11) $$|d_j - a_k| > 8.$$

Proof. We need to show that if $|z + a_k| \le 8$ for some k, then

$$\frac{f'(z)}{f(z)} = \sum_{j=N}^{\infty} \frac{1}{z+a_j} \ne 0.$$

For $j \ge k+1$, we have by (3.5),

$$|z+a_j| \ge |a_j - a_k| - |z+a_k| \ge (a_j - a_k) - 8 \ge K(c_{j-1} + c_{j-2} + \cdots + c_k).$$

Hence by (3.8)

$$\sum_{j=k+1}^{\infty} \frac{1}{|z+a_j|} < \frac{1}{K}\left\{\frac{1}{c_k} + \frac{1}{c_k + c_{k+1}} + \cdots\right\} < \frac{1}{50}.$$

If $k > N$ we have for $N \le j \le k-1$ (see (3.5)),

$$|z+a_j| \ge (a_k - a_j) - |z+a_k| \ge a_k - a_j - 8 \ge K(k-j)c_j.$$

This gives, by (3.9),

(ii) $$\sum_{j=N}^{k-1} \frac{1}{|z+a_j|} \le \frac{1}{K}\left(\frac{1}{c_{k-1}} + \frac{1}{2c_{k-2}} + \cdots + \frac{1}{(k-N)c_N}\right) < \frac{1}{100}.$$

Since $|z + a_k| \le 8$, (i) and (ii) imply

$$\frac{f'(z)}{f(z)} \ge \frac{1}{8} - \frac{1}{100} - \frac{1}{50} > 0.$$

This completes the proof of Lemma 3.

Lemma 4. *If* $z \in \Gamma_n = \{z: |z + a_n| < 4\}$ *for some* $n \geq N$, *then*

$$\sum_{j=N}^{\infty} \frac{1}{|z + d_j|} < \chi < 1 \tag{3.12}$$

where χ *is some fixed number.*

Proof. (a) We have, by Laguerre's theorem [1, p. 23], $a_N < d_N < a_{N+1} < \cdots$. Suppose first

(i) $$|z + a_N| < 4.$$

By (3.11) and (i)

(ii) $$|z + d_N| \geq (d_N - a_N) - |z + a_N| > 8 - 4 = 4.$$

For $j \geq N + 1$ we have by (3.5)

$$|z + d_j| \geq (d_j - a_N) - |z + a_N| \geq a_j - a_N - 4 \geq K\{c_{j-1} + c_{j-2} + \cdots + c_N\}.$$

Hence by (3.8),

(iii) $$\sum_{j=N+1}^{\infty} \frac{1}{|z + d_j|} < \frac{1}{K}\left\{\frac{1}{c_N} + \frac{1}{c_N + c_{N+1}} + \cdots\right\} < \frac{1}{50}.$$

These two inequalities (ii) and (iii) give (3.12).

(b) Suppose now $|z + a_n| < 4$ for some $n \geq N + 1$. Then

$$|z + d_n| \geq (d_n - a_n) - |z + a_n| \geq 8 - 4 = 4.$$

Similarly $|z + d_{n-1}| > 4$ and so

(iv) $$1/|z + d_{n-1}| + 1/|z + d_n| < \tfrac{1}{2}.$$

For $j \geq n + 1$, we have by (3.5)

$$|z + d_j| \geq (d_j - a_n) - |z + a_n| > (a_j - a_n) - 4 \geq K(c_{j-1} + c_{j-2} + \cdots + c_n).$$

This gives (see (3.8))

(v) $$\sum_{j=n+1}^{\infty} \frac{1}{|z + d_j|} < \frac{1}{K}\left\{\frac{1}{c_n} + \frac{1}{c_n + c_{n+1}} + \cdots\right\} < \frac{1}{100}.$$

If $N \leq n - 2$ we have for $N \leq j \leq n - 2$,

$$|z + d_j| \geq |d_j - a_n| - |z + a_n| \geq a_n - a_{j+1} - 4 \geq K(n - j - 1)\, c_{j+1}.$$

Hence (3.9) yields

(vi) $$\sum_{j=N}^{n-2} \frac{1}{|z+d_j|} < \frac{1}{K}\left\{\frac{1}{c_{n-1}} + \frac{1}{2c_n} + \cdots + \frac{1}{(n-N-1)c_{N+1}}\right\} < \frac{1}{100}.$$

The inequalities (iv)–(vi) imply (3.12). The proof of Lemma 4 is complete.

Lemma 5. *Let*

$$f(z) = \prod_{j=N}^{\infty}\left(1 + \frac{z}{a_j}\right).$$

Then $f(z)$ is an entire function of genus zero and

$$f'(z) = f'(0)\prod_{j=N}^{\infty}\left(1 + \frac{z}{d_j}\right)$$

where $f'(0) = \Sigma_{j=N}^{\infty} a_j^{-1}$. If for some z at least one of the two inequalities

(3.13)

(i) $$\sum_{j=N}^{\infty} \frac{1}{|z+a_j|} < \chi < 1,$$

(ii) $$\sum_{j=N}^{\infty} \frac{1}{|z+d_j|} < \chi < 1,$$

where χ is a constant, holds, then for this z

(3.14) $$\frac{|f^{(n+1)}(z)|}{(n+1)!} \le \max\left\{\chi^{n+1}|f(z)|, \frac{\chi^n}{(n+1)}|f'(z)|\right\} < \chi^n \max\{|f(z)|, |f'(z)|\}, \qquad n = 1, 2, \cdots.$$

Proof. Let

$$p(z) = \sum_{j=N}^{\infty} \frac{1}{(z+a_j)}$$

and suppose (i) of (3.13) holds. Then

(iii)
$$|p(z)| < \chi < 1,$$
$$p^{(n)}(z) = (-1)^n n! \sum_{j=N}^{\infty} \frac{1}{(z+a_j)^{n+1}}, \qquad n = 1, 2, \cdots.$$

Hence

(iv) $$\frac{|p^{(n)}(z)|}{n!} \le \sum_{j=N}^{\infty} \frac{1}{|z+a_j|^{n+1}} \le \left(\sum_{j=N}^{\infty} \frac{1}{|z+a_j|}\right)^{n+1} < \chi^{n+1}.$$

Since $f' = fp$ we have

(v) $$|f'(z)| \le \chi|f(z)|$$

and

$$\text{(vi)}\quad \begin{aligned}\frac{|f^{(n+1)}(z)|}{(n+1)!} &= \frac{1}{n+1}\left|\left\{\frac{f^{(n)}(z)}{n!}\frac{p(z)}{0!} + \frac{f^{(n-1)}(z)}{(n-1)!}\frac{p'(z)}{1!} + \cdots + \frac{f(z)}{0!}\frac{p^{(n)}(z)}{n!}\right\}\right| \\ &\le \frac{1}{n+1}\left\{\frac{|f^{(n)}(z)|}{n!}\chi + \frac{|f^{(n-1)}(z)|}{(n-1)!}\chi^2 + \cdots + |f(z)|\chi^{n+1}\right\}.\end{aligned}$$

We now use an induction argument to show that

$$\text{(vii)}\qquad \frac{|f^{(n)}(z)|}{n!} \le \chi^n |f(z)|, \qquad n = 1, 2, \cdots.$$

For the inequality holds by (v) for $n = 1$. Suppose it is true for $n = 1, 2, \cdots, m$. Then by (vi),

$$\text{(viii)}\quad \begin{aligned}\frac{|f^{(m+1)}(z)|}{(m+1)!} &\le \frac{1}{m+1}\left\{\frac{f^{(m)}(z)}{m!}\chi + \cdots + |f(z)|\chi^{m+1}\right\} \\ &< \frac{1}{m+1}\left\{\chi^m\chi + \chi^{m-1}\chi^2 + \cdots + \chi^{m+1}\right\}|f(z)| = \chi^{m+1}|f(z)|.\end{aligned}$$

This proves (vii). Suppose now (ii) of (3.13) holds. We have then

$$f''(z) = f'(z)\sum_{j=N}^{\infty}\frac{1}{(z+d_j)}$$

and the above reasoning yields

$$\text{(ix)}\qquad \frac{|f^{(n+2)}(z)|}{(n+1)!} < \chi^{n+1}|f'(z)| \qquad (n = 0, 1, 2, \cdots).$$

From (vii) and (ix) we have

$$\begin{aligned}\frac{|f^{(n+1)}(z)|}{(n+1)!} &\le \max\left\{\chi^{n+1}|f(z)|, \frac{\chi^n}{n+1}|f'(z)|\right\} \\ &< \chi^n \max\{|f(z)|, |f'(z)|\} \qquad (n = 1, 2, \cdots).\end{aligned}$$

This completes the proof of Lemma 5.

4. **Proof of Theorem 2.** We have

$$f(z) = \prod_{j=1}^{\infty}\left(1 + \frac{z}{a_j}\right) = \prod_{1}^{N-1}\left(1 + \frac{z}{a_j}\right)\prod_{j=N}^{\infty}\left(1 + \frac{z}{a_j}\right) = P(z)f_N(z),$$

where $P(z)$ is a polynomial of degree $(N-1)$ and $f_N(z) = \Pi_{j=N}^{\infty}(1 + z/a_j)$ and N is determined as in the remark following Lemma 1.

Let z be given. Then either $z \in \Gamma_n$ for some $n \ge N$ or $z \notin \bigcup_{n=N}^{\infty}\Gamma_n$. If $z \in \Gamma_n$ for some $n \ge N$, then by Lemma 4, (ii) of (3.13) holds and hence, by Lemma 5, (3.14) holds with f replaced by f_N.

If $z \notin \bigcup_{n=N}^{\infty}\Gamma_n$, then by Lemma 2, (i) of (3.13) holds and we have, again by Lemma 5, (3.14) with f replaced by f_N. Hence $f_N \in SB$ and so by Theorem 1,

$f(z) = P(z)/f_N(z)$ belongs to SB. This completes the proof of Theorem 2.

The corollaries are almost obvious and we leave the proofs to the reader. Note that if $f(\alpha z) \in SB$ then $f(|\alpha| z) \in SB$ and conversely..

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