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Π_1^0 CLASSES AND DEGREES OF THEORIES[1]

BY

CARL G. JOCKUSCH, JR. AND ROBERT I. SOARE

ABSTRACT. Using the methods of recursive function theory we derive several results about the degrees of unsolvability of members of certain Π_1^0 classes of functions (i.e. degrees of branches of certain recursive trees). As a special case we obtain information on the degrees of consistent extensions of axiomatizable theories, in particular effectively inseparable theories such as Peano arithmetic, $\mathbf{P}$. For example: THEOREM 1. *If a degree* $\mathbf{a}$ *contains a complete extension of* $\mathbf{P}$, *then every countable partially ordered set can be embedded in the ordering of degrees* $\leq \mathbf{a}$. (This strengthens a result of Scott and Tennenbaum that no such degree $\mathbf{a}$ is a minimal degree.) THEOREM 2. *If* $\mathbf{T}$ *is an axiomatizable, essentially undecidable theory, and if* $\{\mathbf{a}_n\}$ *is a countable sequence of nonzero degrees, then* $\mathbf{T}$ *has continuum many complete extensions whose degrees are pairwise incomparable and incomparable with each* $\mathbf{a}_n$. THEOREM 3. *There is a complete extension* $\mathbf{T}$ *of* $\mathbf{P}$ *such that no nonrecursive arithmetical set is definable in* $\mathbf{T}$. THEOREM 4. *There is an axiomatizable, essentially undecidable theory* $\mathbf{T}$ *such that any two distinct complete extensions of* $\mathbf{T}$ *are Turing incomparable.* THEOREM 5. *The set of degrees of consistent extensions of* $\mathbf{P}$ *is meager and has measure zero.*

1. If $R(x)$ is a recursive predicate of one free number variable, the class of all number-theoretic functions f satisfying $(x)R(\bar{f}(x))$ is called a Π_1^0 *class.* Sets of numbers will constantly be identified with their characteristic functions, and thus a class $\mathcal{S}$ of sets is a Π_1^0 class just if the corresponding class of characteristic functions is a Π_1^0 class. A class $\mathcal{S}$ of functions is called *recursively bounded* (*r.b.*) just if there is a recursive function which bounds every $f \in \mathcal{S}$ on all arguments. In particular, any class of *sets* is r.b. Our purpose is to study r.b. Π_1^0 classes. Each such class may be thought of as the set of (infinite) branches of a special finitely-branching recursive tree, and thus our arguments will combine standard methods from recursion theory with König's lemma for trees.

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In particular, we shall be concerned with the following two special sorts of Π_1^0 classes of sets which were pointed out by Shoenfield in [22, Theorems 3 and 4].

(1.1). The class of sets C which separate a given disjoint pair of r.e. sets (A, B), i.e. contain A and are disjoint from B.

(1.2). The class of consistent (or complete) extensions of a given axiomatizable theory.

In regard to (1.2), we shall especially study Peano arithmetic, and throughout the paper the notation $\mathbf{P}$ will be used for the set of all formulas provable in (first-order) Peano arithmetic. In general, a "theory" is simply a deductively closed set of formulas in a propositional or first-order language, and the terminology of [25] will be used for theories.

Terminology and notation will be given at the close of this section, but first we summarize the rest of the paper.

In § 2 we generalize the Kreisel-Shoenfield basis theorem [22, Theorem 2] by proving that any nonempty r.b. Π_1^0 class has a member f whose degree $\mathbf{f}$ satisfies $\mathbf{f}' = \mathbf{0}'$. A similar argument is then used to show that every such class also has a member f whose degree contains no hyperimmune sets. Finally, we prove that given any axiomatizable, essentially undecidable theory $\mathbf{T}$, and any degree $\mathbf{a} > \mathbf{0}$, there are $2^{\aleph_0}$ degrees, mutually incomparable, and incomparable with $\mathbf{a}$, which are the degrees of complete extensions of $\mathbf{T}$. This extends the result of Scott and Tennenbaum [21] that there is at least one such degree if $\mathbf{T}$ is $\mathbf{P}$ and $\mathbf{a} = \mathbf{0}'$.

In § 3 we study, largely on the basis of earlier results, the sets definable in complete extensions of $\mathbf{P}$. We show that there is a complete extension $\mathbf{T}$ of $\mathbf{P}$ such that no nonrecursive arithmetical set is definable in $\mathbf{T}$ and obtain a similar result for various levels of the arithmetical hierarchy. Also we study the "expansion" of r.e. sets when their definitions are interpreted in various complete extensions of $\mathbf{P}$ and relate this phenomenon to hypersimplicity.

In § 4 we prove that if $\mathbf{a}$ is the degree of a complete extension of $\mathbf{T}$, then every countable partially ordered set is embeddable in the upper semilattice of degrees below $\mathbf{a}$. We accomplish this by first using a priority argument to construct a recursive sequence $\{(A_i, B_i)\}$ of disjoint pairs of r.e. sets such that any sequence of sets $\{C_i\}$, where C_i separates (A_i, B_i), is recursively independent. We also construct an infinite r.b. Π_1^0 class $\mathcal{S}$ all of whose members are mutually Turing incomparable.

In § 5, we prove that the upper cone of degrees generated by any Π_1^0 class $\mathcal{S}$ without recursive members is meager (in the sense of Baire category), while the measure of the set of degrees generated by $\mathcal{S}$ may be either 0 or 1 even if $\mathcal{S}$ is r.b. It follows that the set of degrees of extensions of $\mathbf{P}$ is meager and has measure zero.

In § 6 we make some observations which allow immediate extensions of some of the earlier results. For instance, theorems on complete extensions of **P** are extended to arbitrary consistent extensions of any theory in which the provable and refutable formulas are effectively inseparable. Also existence theorems for r.b. Π_1^0 predicates can be sharpened to existence results for predicates of the special form (1.2) mentioned earlier.

We now consider notation. The set of natural numbers is denoted by N, and a *string* is a partial function from a finite initial segment of N into N. The variables ρ, σ, τ will be reserved for strings, and Σ will stand for the set of all strings whose range is contained in $\{0, 1\}$. If σ is a string, $lh(\sigma)$ is the cardinality of its domain. The notation $\sigma * \tau$ is used for the string obtained when τ is adjoined to the right-hand end of σ. (Here σ and τ are viewed in the obvious way as finite sequences.) If i is a number, $\sigma * i$ is the string obtained by adjoining the term i to the right-hand end of σ.

Let ϕ and ψ be partial functions. We say that ϕ and ψ are *compatible* if they agree on the intersection of their domains and that ϕ *extends* ψ ($\phi \supseteq \psi$) if the graph of ϕ contains that of ψ. This latter notion is used especially when ϕ and ψ are either strings or (characteristic functions of) sets. We write $\operatorname{dom}(\phi)$ for the domain of ϕ.

A set T of strings is called a *tree* if whenever it contains a string σ it also contains all strings extended by σ. We assume that the set of all strings is Gödel-numbered so that we may speak of a *recursive* tree, etc. A tree T is called *recursively bounded* (*r.b.*) if there exists a recursive function f such that, for every string $\sigma \in T$ and every $x \in \operatorname{dom}(\sigma)$, $\sigma(x) \leq f(x)$. If T is a tree, T^* is the set of all total functions f such that every string extended by f is in T. It follows from König's lemma [17, p. 157] that, if T is r.b., then T^* is nonempty iff T is infinite. It is easily seen that a class $\mathcal{S}$ of functions is a [r.b.] Π_1^0 class iff $\mathcal{S} = T^*$ for some [r.b.] recursive tree T. If T is a tree and σ is a string, $T(\sigma)$ is the set of all strings in T which extend σ, and $T^*(\sigma)$ is the set of all functions in T^* which extend σ. Observe that we use the notation T for trees and **T** for theories.

The notation "$\{e\}_s^\sigma(x) = y$" (where σ is a string and e, s, x, y are numbers) has the usual meaning, i.e. roughly that the eth recursive reduction procedure, given input x and oracle information σ, reaches output y within s steps. We write $\{e\}^\sigma(x) = y$ as an abbreviation for $\{e\}_{lh(\sigma)}^\sigma(x) = y$ and so we have (as in [23, p. 540])

(1.3) If $\{e\}^\sigma(x) = y$ and τ extends σ, then $\{e\}^\tau(x) = y$.

(1.4) The predicate "$\{e\}^\sigma(x)$ is defined" is a recursive predicate of e, σ, and x.

Assume that the class of finite subsets of N is Gödel-numbered, and let D_u denote the finite set with index u. If A is a set and f is a recursive function, we say that $D_{f(x)}$ *witnesses* A nonhyperimmune if $D_{f(x)} \cap A$ is nonempty for every x, and $D_{f(x)}$ is disjoint from $D_{f(y)}$ whenever $x \neq y$. We write $\langle u, v\rangle$ for the image of (u, v) under a recursive pairing function from N^2 onto N. If $\mathbf{a}$ is a degree, $\mathbf{a}'$ is the jump of $\mathbf{a}$; $\mathbf{0}$ is the degree of the recursive sets. We write W_e for the eth r.e. set under some standard indexing of the r.e. sets, and $\bar{A}$ for $N - A$. For sets A, B, we define A join B to be $\{2n: n \in A\} \cup \{2n+1: n \in B\}$. For unexplained terminology the reader is referred to [17].

2. In [22, Theorem 2] Shoenfield extended a basis result of Kreisel by proving that any nonempty Π_1^0 class of sets has a member of degree $\mathbf{a} < \mathbf{0}'$. We extend this result further by proving that $\mathbf{a}$ can even be chosen so that $\mathbf{a}' = \mathbf{0}'$, and therefore any consistent axiomatizable theory (in particular Peano arithmetic $\mathbf{P}$) has a complete extension of such a degree. (It is worth noting that the results of this section apply to any Π_1^0 class which contains a recursively bounded function, since each such class has a nonempty r.b. Π_1^0 subclass.)

Theorem 2.1. *If* $\mathcal{S}$ *is a nonempty r.b.* Π_1^0 *class,* $\mathcal{S}$ *contains a function* f *whose degree* $\mathbf{f}$ *satisfies* $\mathbf{f}' = \mathbf{0}'$.

Proof. Let T_0 be a r.b. recursive tree such that $\mathcal{S} = T_0^*$. We shall define a decreasing sequence of infinite recursive trees $\{T_e\}_{e \in N}$ and choose $f \in \bigcap_{e \in N} T_e^*$. The tree T_{e+1} will have the property that $\{e\}^g(e)$ is defined for all or no $g \in T_{e+1}^*$.

Assume that tree T_e has been defined. Let $U_e = \{\sigma \mid \{e\}^\sigma(e)$ is undefined$\}$. It is easy to see from stipulations (1.3) and (1.4) on relative computability that U_e forms a recursive tree.

Case 1. $T_e \cap U_e$ is finite. Let $T_{e+1} = T_e$.

Case 2. $T_e \cap U_e$ is infinite. Let $T_{e+1} = T_e \cap U_e$, which is clearly a recursive tree.

Since $\mathcal{S}$ is nonempty, T_0 is infinite. It follows by induction that T_e is infinite for all $e \in N$, so by König's lemma (i.e. the compactness theorem for finitely branching trees [17, p. 157]), T_e^* is nonempty for all $e \in N$. Therefore, $\bigcap_{e \in N} T_e^*$ is nonempty because it is the intersection of a decreasing sequence of nonempty closed sets (in the compact space $\mathcal{S} = T_0^*$). Choose any function $f \in \bigcap_{e \in N} T_e^*$. (In fact, f is unique.) We claim that $\mathbf{f}' = \mathbf{0}'$, because the entire construction can be carried out recursively in $\mathbf{0}'$. It is easily seen that Case 1 applies in the definition of T_{e+1} just in case

$$(2.1) \qquad (\exists n)(\forall \sigma)[\sigma \in T_e \ \&\ lh(\sigma) = n \Rightarrow \{e\}^\sigma(e) \text{ is defined}].$$

Since T_e is r.b. (by the same function bounding T_0), we can find recursively all $\sigma \in T_e$ of length n uniformly in n. Hence, (2.1) is a predicate of the form $(\exists n) R(n)$, where R is a recursive predicate whose index can be computed effectively from e and an index of T_e. Therefore, we can decide recursively in $\mathbf{0}'$ whether Case 1 holds, and after that we can recursively find a Gödel number for T_{e+1}. Furthermore, $\{e\}^f(e)$ is defined $\Leftrightarrow$ Case 1 applies in defining T_{e+1}, because if Case 1 applies then $\{e\}^\sigma(e)$ is defined for all sufficiently long $\sigma \in T_e$ by (2.1), so $\{e\}^f(e)$ is defined; and if Case 2 applies, $\{e\}^f(e)$ is undefined since $f \in U_e^*$.

Since we can determine recursively in $\mathbf{0}'$ uniformly in e which case applies in defining T_{e+1}, it follows that $\mathbf{f}' \leq \mathbf{0}'$. Since $\mathbf{0}' \leq \mathbf{f}'$ always holds, we have $\mathbf{f}' = \mathbf{0}'$.

The proof of Theorem 2.1 can easily be modified to yield the stronger result that for any nonempty r.b. Π_1^0 class $\mathcal{S}$ with no recursive member and any degree $\mathbf{a} \geq \mathbf{0}'$ there is a function $f \in \mathcal{S}$ such that $\mathbf{f}' = \mathbf{f} \cup \mathbf{0}' = \mathbf{a}$. In the proof of Theorem 2.1, first choose effectively from T_{e+1} two incompatible strings $\sigma_{e+1}^0, \sigma_{e+1}^1$. (These exist since T_{e+1}^* is nonempty and has no recursive member.) Then replace T_{e+1} by the subtree of those strings $\sigma \in T_{e+1}$ which are compatible with $\sigma_{e+1}^{A(e)}$, where A is a fixed set of degree $\mathbf{a}$. The proof that $\mathbf{f}' = \mathbf{f} \cup \mathbf{0}' = \mathbf{a}$ for the function f thus obtained is very similar to that of the Friedberg completeness criterion [3] or [17, p. 265], of which the present result is a generalization.

Corollary 2.2. *Any consistent axiomatizable theory* (*in particular* **P**) *has a complete extension of degree whose jump is* $\mathbf{0}'$.

Proof. By (1.2).

Corollary 2.3. *Let* **T** *be a consistent axiomatizable theory. Then* **T** *has a model in which the domain is a set of natural numbers, and the predicates are of a degree whose jump is* $\mathbf{0}'$.

Proof. As in [22, Theorem 5].

A degree $\mathbf{a}$ is called *hyperimmune-free* if no hyperimmune set has degree $\mathbf{a}$. The existence of nonzero hyperimmune-free degrees was proved by Miller and Martin [13, Theorem 2.1]. The following extension of their result is proved by combining their methods with those in the proof of Theorem 2.1.

Theorem 2.4. *If* $\mathcal{S}$ *is a nonempty r.b.* Π_1^0 *class,* $\mathcal{S}$ *contains a function* f *whose degree* $\mathbf{f}$ *is hyperimmune-free.*

Proof. We find $f \in \mathcal{S}$ such that for every function g recursive in f there is a recursive function h such that $h(x) \geq g(x)$ for all x. Once this is done it follows immediately from [16, Theorem 21] that the degree $\mathbf{f}$ is hyperimmune-free.

Let $\mathcal{S} = T_0^*$, where T_0 is a r.b. recursive tree. As in Theorem 2.1, we construct a descending sequence of recursive trees $\{T_e\}_{e\in N}$, and choose $f \in \bigcap_{e\in N} T_e^*$. We now arrange that $\{e\}^g$ is total for all or no $g \in T_{e+1}^*$. Assume that T_e has been defined. Let

$$U_e^x = \{\sigma | \{e\}^\sigma(x) \text{ is undefined}\}.$$

Case 1. For all x, $T_e \cap U_e^x$ is finite. Let $T_{e+1} = T_e$.

Case 2. For some x, $T_e \cap U_e^x$ is infinite. Let x_e be the least such x and define $T_{e+1} = T_e \cap U_e^{x_e}$, which is a recursive tree just as in Theorem 2.1.

As in Theorem 2.1, choose (the unique) function $f \in \bigcap_{e\in N} T_e^*$. Fix a number e and consider the function $\{e\}^f$. If Case 2 applies in the definition of T_{e+1}, then $\{e\}^f(x_e)$ is undefined, so $\{e\}^f$ is not total. If Case 1 applies, we define a recursive function $h_e(x)$ which bounds $\{e\}^f$. To compute $h_e(x)$, first find a number k_x such that $\{e\}^\sigma(x)$ is defined for every $\sigma \in T_e$ of length k_x. The number k_x exists because Case 1 applies, and k_x can be found effectively uniformly in x because T_e is r.b. Therefore the following function is both total and recursive:

$$h_e(x) = \max\{\{e\}^\sigma(x) \colon \sigma \in T_e \ \& \ lh(\sigma) = k_x\}.$$

Clearly $h_e(x) \geq \{e\}^f(x)$ because $\{e\}^f(x) = \{e\}^\sigma(x)$ for some $\sigma \in T_e$ of length k_x. Therefore, by the remarks at the beginning of the proof, the degree of f is hyperimmune-free. (It follows from our proof that each degree $\mathbf{a} < \mathbf{f}$ is also hyperimmune-free but this is automatically true by [13, Theorem 1.1]. The degree $\mathbf{f}$ also satisfies $\mathbf{f}'' = \mathbf{0}''$.)

Theorem 2.5. *Given any nonempty r.b.* Π_1^0 *class* $\mathcal{S}$ *which has no recursive members, and any countable sequences of nonrecursive degrees* $\{\mathbf{a}_i\}$, $\mathcal{S}$ *has* $2^{\aleph_0}$ *members* f, *mutually Turing incomparable, such that the degree* $\mathbf{f}$ *is incomparable with each* $\mathbf{a}_i$.

Our proof combines the standard technique[2] for constructing $2^{\aleph_0}$ mutually incomparable degrees with the following Lemmas 2.6 and 2.7 which enable the construction to be carried out within $\mathcal{S}$. Given an infinite recursive tree T, define the (nonrecursive) tree of extendible finite branches of T, denoted T^{ext}, to consist of all $\sigma \in T$ for which $T^*(\sigma)$ is nonempty. If T is r.b., then by König's lemma, $\sigma \in T^{\text{ext}}$ just if $T(\sigma)$ is infinite.

Lemma 2.6 (Splitting Lemma). *For every infinite recursively bounded recursive tree* T *and index* e *either*

(2) The construction of $2^{\aleph_0}$ mutually incomparable degrees was first given by Sacks [18, p. 13]. We use a modification due to T. G. McLaughlin (unpublished).

(1) *there exists an infinite recursive tree* $T_1 \subseteq T$ *such that, for every* $f \in T_1^*$, $\{e\}^f$ *is recursive or not total*; *or*

(2) *for every* $\sigma \in T^{\text{ext}}$, *there exist* $\tau_1, \tau_2 \in T^{\text{ext}}(\sigma)$ *such that, for some* x, $\{e\}^{\tau_1}(x)$ *and* $\{e\}^{\tau_2}(x)$ *are defined and unequal.*

Proof of Lemma 2.6. Fix T and e, and assume that (1) and (2) fail. Fix $\sigma \in T^{\text{ext}}$ which is a counterexample to (2). We will get a contradiction by defining a recursive function $g(x)$ such that

$$(f)[f \in T^*(\sigma) \Rightarrow \{e\}^f \text{ is total } \& \ \{e\}^f = g].$$

Choose any argument x_0. Since (1) fails, there are only finitely many $\tau \in T$ for which $\{e\}^\tau(x_0)$ is undefined. (An infinite set of such τ would constitute an infinite subtree $S \subseteq T$, and S is clearly a recursive tree.) Hence, there exists n_0 such that, for all $\tau \in T(\sigma)$ of length n_0, $\{e\}^\tau(x_0)$ is defined. But since (2) fails for x_0, there are no two *extendible* strings $\rho, \tau \in T(\sigma)$ of length n_0 such that $\{e\}^\rho(x_0) \neq \{e\}^\tau(x_0)$. Hence, by König's lemma there exist $m_0 \geq n_0$ and s_0 such that, for all $\rho, \tau \in T(\sigma)$ of length m_0, $\{e\}^\rho_{s_0}(x)$ and $\{e\}^\tau_{s_0}(x_0)$ are defined and equal. Since s_0 and m_0 exist they can be computed recursively. (We use the fact that $T(\sigma)$ is r.b. in order to compute recursively all $\tau \in T(\sigma)$ of length n, uniformly in n.) Having computed s_0 and m_0, we choose any $\tau \in T(\sigma)$ of length m_0 and define $g(x_0) = \{e\}^\tau_{s_0}(x_0)$. Now for any $f \in T^*(\sigma)$, $\{e\}^f(x_0)$ is defined and equal to $g(x_0)$ by the continuity of the functional $\{e\}$.

Lemma 2.7. *If* T *is a r.b. recursive tree, and* T^* *has members but no recursive members, then for every* $\sigma \in T^{\text{ext}}$, *there exist two incompatible strings* $\tau_1, \tau_2 \in T^{\text{ext}}(\sigma)$.

Proof of Lemma 2.7. This lemma is obvious but can be thought of as the special case of Lemma 2.6 in which $\{e\}$ is the identity operator.

Proof of Theorem 2.5. Choose a r.b. recursive tree T_0 such that $T_0^* = \mathcal{S}$, the given r.b. Π_1^0 class. To simplify the details we will construct the functions f incomparable with a *single* nonrecursive degree $\mathbf{a}$, but it will be clear how to generalize to countably many nonrecursive degrees $\{\mathbf{a}_i\}$ simply by considering $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ at stage n in place of just $\mathbf{a}$. Let $h \in 2^N$ be some function of degree $\mathbf{a}$.

A *rooted tree* is an ordered pair (σ, S) where σ is a string, S is an infinite recursive tree, and $S = S(\sigma)$. (If S is r.b. then clearly $\sigma \in S^{\text{ext}}$ by König's lemma since $S(\sigma)$ is infinite.) If (σ_0, S_0), (σ_1, S_1) are rooted trees we say that (σ_1, S_1) *extends* (σ_0, S_0) if σ_1 extends σ_0 and $S_1 \subseteq S_0$; and that (σ_0, S_0) and (σ_1, S_1) are *incompatible* if σ_0 and σ_1 are incompatible. We say that f *belongs to* (σ, S) if $f \in S^*$.

We must define a sequence $\{\mathcal{R}_n\}_{n\in N}$ such that each $\mathcal{R}_n$ is a set of 2^n pairwise incompatible rooted trees and each rooted tree in $\mathcal{R}_n$ has exactly two incompatible extensions in $\mathcal{R}_{n+1}$. Furthermore, $\mathcal{R}_0 = \{(\emptyset, T_0)\}$, where $\emptyset$ is the empty string and T_0 is the given recursive tree, and $\mathcal{R}_{n+1}$ satisfies the following conditions for all functions f, g:

(2.2) If f belongs to a rooted tree in $\mathcal{R}_{n+1}$ then $\{n\}^f \neq h$.

(2.3) If f and g belong to different rooted trees of $\mathcal{R}_{n+1}$ and $e \leq n$, then $\{e\}^f \neq g$.

It is sufficient to define such a sequence $\mathcal{R}_n$ because once this is achieved, we may define $\mathcal{C}$ to be the set of functions which belong to some rooted tree in $\mathcal{R}_n$ for each n and which are not recursive in h. Then $\mathcal{C}$ is contained in the given class $\mathcal{S}$ by definition of the $\mathcal{R}_n$, and $\mathcal{C}$ contains a continuum of functions by the condition that each member of $\mathcal{R}_n$ has two incompatible extensions in $\mathcal{R}_{n+1}$, and because only countably many functions are recursive in h. Finally, it follows from (2.2) that h is not recursive in any member of $\mathcal{C}$, and from (2.3) that any two distinct members of $\mathcal{C}$ are Turing incomparable.

We now define the sequence $\{\mathcal{R}_n\}$ by induction. Let $\mathcal{R}_0 = \{(\emptyset, T_0)\}$, and assume that $\mathcal{R}_n$ has been defined. To insure that each member of $\mathcal{R}_n$ has two incompatible extensions in $\mathcal{R}_{n+1}$ we first choose, for each member of $\mathcal{R}_n$, *any* two incompatible extensions. This may be done by Lemma 2.7. Let $\mathcal{C}_{n+1}$ denote this set of 2^{n+1} rooted trees. These may be thought of as "candidates" for $\mathcal{R}_{n+1}$. We must now keep extending each candidate in successive substages until we have insured that (2.2) and (2.3) hold for the extended collection. Thus, each member of $\mathcal{R}_{n+1}$ will be an extension of one of the original candidates. We shall not carry out this extension procedure in complete detail but shall indicate how each condition is attacked.

To satisfy condition (2.2), consider any rooted tree (σ, S) in $\mathcal{C}_{n+1}$ and apply Lemma 2.6 to S with $e = n$. If (1) of Lemma 2.6 holds, extend (σ, S) to (σ, S_1) where $S_1 \subseteq S$ is given by Lemma 2.6. If (2) of Lemma 2.6 holds, choose $\tau_1, \tau_2 \in S^{\text{ext}}$ and x such that $\{n\}^{\tau_1}(x)$ and $\{n\}^{\tau_2}(x)$ are defined and unequal. At least one of these, say $\{n\}^{\tau_1}(x)$, must differ from $h(x)$, in which case we extend (σ, S) to $(\tau_1, S(\tau_1))$. Whichever case of Lemma 2.6 applies, our construction clearly guarantees that for all f in the extension of (σ, S) we have $\{n\}^f \neq h$. (In the first case, the nonrecursiveness of h is used.) By applying this procedure to every rooted tree in $\mathcal{C}_{n+1}$, we get 2^{n+1} rooted trees each of which extends some member of $\mathcal{C}_{n+1}$ and which satisfy (2.2). Denote this collection by $\mathcal{C}^{\rho}_{n+1}$.

To satisfy (2.3), we choose any two distinct members of $\mathcal{C}'_{n+1}$, say (σ_0, S_0) and (σ_1, S_1), and any Gödel number $e \leq n$. Apply Lemma 2.6 to (σ_0, S_0). If (1) holds, extend (σ_0, S_0) exactly as before, and leave (σ_1, S_1) unchanged. If (2) holds, let τ_1, τ_2, x be such that $\{e\}^{\tau_1}(x)$ and $\{e\}^{\tau_2}(x)$ are defined and unequal, and $\tau_1, \tau_2 \in S_0^{\text{ext}}$. Extend (if necessary) σ_1 to any $\sigma_2 \in S_2^{\text{ext}}$ such that $\sigma_2(x)$ is defined. Choose $i \in \{0, 1\}$ such that $\{e\}^{\tau_i}(x) \neq \sigma_2(x)$. Extend (σ_0, S_0) to $(\tau_i, S(\tau_i))$ and extend (σ_1, S_1) to $(\sigma_2, S_1(\sigma_2))$. Clearly, in either case, if f and g belong to the extensions of (σ_0, S_0) and (σ_1, S_1) respectively, then $\{e\}^f \neq g$.

To satisfy (2.3), we repeat the above procedure for the extensions corresponding to every pair of distinct members of $\mathcal{C}'_{n+1}$ and every Gödel number $e \leq n$. (Of course, if (σ_0, S_0) and (σ_1, S_1) in $\mathcal{C}'_{n+1}$ extend *different* elements in $\mathcal{R}_n$ and if f_i belongs to (σ_i, S_i) for $i = 0$ and 1, then f_0 and f_1 are already incomparable with respect to all Gödel numbers $e < n$ by definition of $\mathcal{R}_n$. In this case one need consider only Gödel number $e = n$.)

The rooted trees resulting from all these extensions form $\mathcal{R}_{n+1}$, and satisfy our requirements. Theorem 2.5 now follows as previously indicated.

The following corollary generalizes a result of Scott and Tennenbaum [20].

Corollary 2.8. *Given any axiomatizable, essentially undecidable theory* $\mathbf{T}$, *and any degree* $\mathbf{a} > \mathbf{0}$, *there are* $2^{\aleph_0}$ *degrees, mutually incomparable, and incomparable with* $\mathbf{a}$, *which are the degrees of complete extensions of* $\mathbf{T}$.

Proof. This follows from Theorem 2.5 immediately by (1.2).

The following corollaries do not use the full strength of Theorem 2.5 and accordingly they have direct proofs which are somewhat simpler than that for Theorem 2.5.

Corollary 2.9. *If* $\mathcal{S}$ *is any nonempty r.b.* Π_1^0 *class, then* $\mathcal{S}$ *contains functions* f *and* g *whose degrees* $\mathbf{f}$ *and* $\mathbf{g}$ *have greatest lower bound* $\mathbf{0}$.

Proof. We may assume that $\mathcal{S}$ has no recursive members since otherwise the result is immediate. Let f be any member of $\mathcal{S}$. Apply Theorem 2.5 to obtain a function $g \in \mathcal{S}$ of a degree incomparable with each of the (countably many) nonzero degrees $\leq \mathbf{f}$.

The next corollary, due to A. H. Lachlan, strengthens Corollary 6.6 of [8] and was stated without proof at the close of [8].

Corollary 2.10 (A. H. Lachlan). *If* A *is an r.e. set whose complement* $\bar{A}$ *is introreducible (i.e. recursive in each of its infinite subsets), then* A *is either recursive or hypersimple.*

Proof. Assume the corollary is false for A and that f is a recursive function such that $\{D_{f(x)}\}$ witnesses that $\bar{A}$ is nonhyperimmune. Define $\mathcal{S}$ to be the class of all subsets of $\bar{A}$ which intersect every $D_{f(x)}$. Then $\mathcal{S}$ is a nonempty Π_1^0 class of sets and A is recursive in every member of $\mathcal{S}$. It now follows from either Theorem 2.5 or Corollary 2.9 that A must be recursive.

The next corollary is a consequence of the proof of Theorem 2.5 and will be useful in § 3. For each n, let $0^{(n)}$ be a set of degree $\mathbf{0}^{(n)}$.

Corollary 2.11. *Let $\{A_i\}$ be any countable sequence of sets, and let C be a set such that A_i is recursive in C, uniformly in i. Let $\mathcal{S}$ be any nonempty r.b. Π_1^0 class. Then $\mathcal{S}$ has a member f, recursive in C join $0^{(2)}$, such that no nonrecursive A_i is recursive in f.*

Proof. The proof of Theorem 2.5 yields a "binary tree" B of functions $f \in \mathcal{S}$ such that no nonrecursive A_i is recursive in f. (More precisely, B is the set of all strings extended by any string in $\bigcup_{n \in N} \mathcal{R}_n$.) The effectiveness of the proof allows B to be made recursive in C join $0^{(2)}$. Thus B has a branch f recursive in C join $0^{(2)}$ because every string in B extends to an (infinite) branch of B, i.e. $B = B^{\text{ext}}$.

3. If A is a set and $\mathbf{T}$ is a theory, we say that A is *definable* in $\mathbf{T}$ if there is a formula $\mathfrak{A}(x)$ of one free variable in the language $\mathfrak{L}$ of $\mathbf{T}$ such that $A = \{k: \mathfrak{A}(\mathbf{k}) \in \mathbf{T}\}$. (We assume that $\mathfrak{L}$ has a numeral $\mathbf{k}$ corresponding to each $k \in N$.) The results of § 2 easily yield complete extensions of Peano arithmetic in which the definable sets are rather pathological. Throughout this section we use $\mathcal{P}$ to denote the r.b. Π_1^0 class of complete extensions of Peano arithmetic.

Corollary 3.1. *There exists a theory $\mathbf{T} \in \mathcal{P}$ such that every set definable in $\mathbf{T}$ is either recursive or nonarithmetical.*

Proof. By Theorem 2.5, there is a theory $\mathbf{T} \in \mathcal{P}$ whose degree is incomparable with each of the (countably many) nonzero Turing degrees of arithmetical sets. Since each set definable in $\mathbf{T}$ is recursive in $\mathbf{T}$, the corollary follows.

Corollary 3.2. *For each $n \geq 2$ there exists a theory $\mathbf{T} \in \mathcal{P}$ such that every nonrecursive set definable in $\mathbf{T}$ is recursive in $0^{(n)}$ but not r.e. in $0^{(n-1)}$.*

Proof. Let $n \geq 2$ be given, let $C = 0^{(n)}$, and let A_i be the ith set r.e. in $0^{(n-1)}$, under some standard indexing. It follows from Corollary 2.11 that there is a theory $\mathbf{T} \in \mathcal{P}$, recursive in C join $0^{(2)}$ and thus in $0^{(n)}$, such that no nonrecursive A_i is recursive in $\mathbf{T}$. As before, the corollary now follows from the fact that each set definable in $\mathbf{T}$ is recursive in $\mathbf{T}$.

We would like very much to know whether the previous corollary holds for $n = 1$.

We now narrow our attention to definitions for r.e. sets. For each e, let $\mathfrak{A}_e(x)$ be the formula of Peano arithmetic which expresses "$(\exists y)T(e, x, y)$", where T is Kleene's T-predicate [12, p. 281]. We assume that the r.e. sets $\{W_e\}$ are indexed so that $\mathfrak{A}_e$ defines W_e in Peano arithmetic. If $\mathbf{T}$ is a theory, let

$$W_e^{\mathbf{T}} = \{k: \mathfrak{A}_e(\mathbf{k}) \in \mathbf{T}\}.$$

If $\mathbf{T}$ extends Peano arithmetic, then $W_e \subseteq W_e^{\mathbf{T}}$. We say that the theory $\mathbf{T}$ *blows up* W_e if $W_e^{\mathbf{T}}$ is cofinite. (Note that this definition really depends on the index e, not just the set W_e.) The remaining results of this section show that the hypersimple and cofinite sets are precisely the r.e. sets which can be blown up with respect to all of their indices, while every r.e. set can be blown up with respect to at least one index.

Corollary 3.3. *There is a theory* $\mathbf{T} \in \mathcal{P}$ *which blows up every hypersimple set* W_e.

Proof. By Theorem 2.4 and the remarks at the close of its proof, there is a theory $\mathbf{T} \in \mathcal{P}$ such that no hyperimmune set is recursive in $\mathbf{T}$. If W_e is hypersimple, then $\overline{W_e^{\mathbf{T}}}$ is contained in $\overline{W}_e$ and so must be hyperimmune or finite. The former case is impossible since $\overline{W_e^{\mathbf{T}}}$ is recursive in $\mathbf{T}$.

Proposition 3.4. *Let* A *be r.e. but neither hypersimple nor cofinite. Then there exists* e *such that* $A = W_e$ *and no consistent extension* $\mathbf{T}$ *of Peano arithmetic blows up* W_e.

Proof. *Let* f be a recursive function such that $D_{f(x)}$ witnesses that $\overline{A}$ is nonhyperimmune. We may assume without loss of generality that $\bigcup_x D_{f(x)} = N$. Choose some recursive enumeration of A and let A^s be the finite subset of A obtained after s steps of this enumeration. Let $\mathfrak{A}(x)$ be the formula

$$(\exists s)(\exists y)[x \in A^s \cap D_{f(y)} \ \&\ D_{f(y)} \not\subseteq A^s].$$

By the normal form theorem, $\mathfrak{A}(x)$ can be expressed as $\mathfrak{A}_e(x)$ for some e. Then $A = W_e$ since all the $D_{f(x)}$ intersect $\overline{A}$ and $\bigcup_x D_{f(x)} = N$. If $\mathbf{T}$ is a consistent extension of Peano arithmetic, all the $D_{f(x)}$ intersect $\overline{W_e^{\mathbf{T}}}$, so $W_e^{\mathbf{T}}$ is coinfinite.

Proposition 3.5. *For any r.e. set* A *there exist a number* e *and a theory* $\mathbf{T} \in \mathcal{P}$ *such that* $A = W_e$ *and* $W_e^{\mathbf{T}} = N$.

Proof. Suppose the assertion is false for some r.e. set A. Let K be any r.e. nonrecursive set, and let f be a recursive function such that

$$W_{f(e)} = \begin{cases} N & \text{if } e \in K, \\ A & \text{otherwise.} \end{cases}$$

We claim that, for all e,

$$e \in \overline{K} \Leftrightarrow [\neg\, (\forall x)\mathfrak{A}_{f(e)}(x)] \in \mathbf{P}.$$

Recall here that $\mathbf{P}$ is Peano arithmetic and thus r.e.; hence the claim, once established, yields the desired contradiction by implying that $\overline{K}$ is r.e.

To prove the claim, first assume that $\neg\,(\forall x)\mathfrak{A}_{f(e)}(x) \in \mathbf{P}$. Then it follows from the ω-consistency of $\mathbf{P}$ that $\mathfrak{A}_{f(e)}(\mathbf{k}) \notin \mathbf{P}$ for some $k \in N$. Thus $W_{f(e)} \neq N$ and so $e \in \overline{K}$. For the converse, assume that $\neg\,(\forall x)\mathfrak{A}_{f(e)}(x) \notin \mathbf{P}$. Then by Lindenbaum's lemma [6, p. 162] there is a theory $\mathbf{T} \in \mathcal{P}$ containing the formula $(\forall x)\mathfrak{A}_{f(e)}(x)$. Since theories are deductively closed, $W^{T}_{f(e)} = N$ for this theory $\mathbf{T}$, and hence $A \neq W_{f(e)}$ by our original assumption. Thus we may conclude that $e \in K$. This completes the proof of the claim, and the proposition follows.

4. Scott and Tennenbaum announced [20] that if **a** is the degree of a complete extension of Peano arithmetic $\mathbf{P}$ then **a** is not a minimal degree. The main purpose of this section is to considerably generalize this result by proving that if **a** is such a degree then any countable partially ordered set is embeddable in the upper semilattice of degrees below **a**. We accomplish this by first combining a priority argument with techniques used in constructing an infinite set with no subset of higher degree [24] in order to prove Theorem 4.1.

We define the (recursive) join of a sequence of sets $\{A_n\}_{n\in N}$ to be the set $\{\langle n, x\rangle: x \in A_n\}$. We say that the sequence of sets $\{A_n\}_{n\in N}$ is *recursively independent* if for each $n \geq 0$, A_n is not recursive in the join of the sequence $A_0, A_1, \cdots, A_{n-1}, \emptyset, A_{n+1}, \cdots$. If $\{(A_n, B_n)\}_{n\in N}$ is a sequence of disjoint pairs of sets, then a *sequence of separating sets* is a sequence of sets $\{C_n\}_{n\in N}$ such that C_n separates (A_n, B_n) for all $n \in N$.

Theorem 4.1. *There is a recursive sequence of disjoint pairs of r.e. sets $\{(A_n, B_n)\}_{n\in N}$, such that any sequence of separating sets $\{C_n\}_{n\in N}$ is recursively independent.*

Proof. We will define a partial recursive function ψ, taking only values 0 and 1, such that the sequence of disjoint r.e. sets $\{(A_n, B_n)\}_{n\in N}$ satisfies the theorem where we let $A_n = \{x: \psi(\langle n, x\rangle) = 1\}$, and $B_n = \{x: \psi(\langle n, x\rangle) = 0\}$. We will define ψ as the limit of a recursive sequence of finite functions ψ_s, where $\psi_{s+1} \supseteq \psi_s$. Each ψ_s naturally determines a recursive tree

$$T_s = \{\sigma: \sigma \in \Sigma \;\&\; \sigma \text{ compatible with } \psi_s\}.$$

Since $\psi_{s+1} \supseteq \psi_s$, clearly $T_{s+1} \subseteq T_s$. If we define $\mathcal{S} = \bigcap_{s\in N} T^*_s$, then $\mathcal{S}$ is clearly a r.b. Π^0_1 class and

$$\mathcal{S} = \{f: f \text{ is compatible with } \psi\}.$$

For any $f \in 2^N$, we define the functions $\pi_i f = \lambda x[f(\langle i, x\rangle)]$, and $\Delta_i f$, where

$$\Delta_i f(\langle j, x\rangle) = \begin{cases} f(\langle j, x\rangle) & \text{if } j \neq i, \\ 0 & \text{if } j \neq i. \end{cases}$$

(Think of π_i as a projection operator and Δ_i as a deletion operator.) For partial functions such as strings $\sigma \in \Sigma$, the partial functions $\pi_i \sigma$ and $\Delta_i \sigma$ are defined similarly on $\operatorname{dom}\sigma$ and are undefined off $\operatorname{dom}\sigma$.

For each $i, e \in N$, we define a *requirement* denoted R_{ie}, which asserts that

$$(f)[f \in \mathcal{S} \Rightarrow \pi_i f \neq \{e\}^{\Delta_i f}].$$

We say that R_{ie} has *higher priority* than R_{jk} just if $\langle i, e\rangle < \langle j, k\rangle$.

To prove the theorem it clearly suffices to construct ψ such that the resulting class $\mathcal{S}$ satisfies R_{ie} for all $i, e \in N$, because as f ranges through $\mathcal{S}$, $\pi_i f$ ranges through all separating sets of the pair (A_i, B_i) and $\Delta_i f$ ranges through all recursive joins of separating sets of (A_j, B_j) for $j \neq i$. (Recall that we identify sets with their characteristic functions.)

In order to reveal the intuition behind the construction, we use Rogers' terminology [17], and begin by designating the set $\{\langle i, x\rangle : x \in N\}$ as the i-list. On the i-list we place an infinite sequence of "markers" $\{\Lambda_{ie}\}_{e \in N}$ arranged in ascending order according to subscript e. The integer occupied by Λ_{ie} at stage s (denoted $x_s(i, e)$ from now on) will not be in $\operatorname{dom}\psi_s$ but may later enter $\operatorname{dom}\psi_t$, $t > s$, in order to satisfy R_{ie} or some requirement of higher priority, in which case Λ_{ie} is moved to some $y \notin \operatorname{dom}\psi_t$. As in Yates' maximal set construction (see Rogers [17, p. 235]), every element not covered by a marker at s is enumerated in $\operatorname{dom}\psi_s$. Hence, Λ_{ie} comes to rest on the $(e+1)$th element of the i-list which is not in $(A_i \cup B_i)$.

Since R_{ie} can move all markers except those Λ_{jk} for $\langle j, k\rangle < \langle i, e\rangle$, it is appropriate to think of R_{ie} as the conjunction of $2^{\langle i, e\rangle}$ "subrequirements" R_{ie}^D for sets $D \subseteq D_{ie}$, where

$$D_{ie} = \{\Lambda_{jk} : \langle j, k\rangle < \langle i, e\rangle\}.$$

For $D \subseteq D_{ie}$, the *subrequirement* R_{ie}^D is defined as the assertion that requirement R_{ie} holds for those $f \in \mathcal{S}$ such that $f(x) = 1$ if x is occupied by a marker in D, and $f(x) = 0$ if x is occupied by a marker in $D_{ie} - D$.

Fix a requirement R_{ie}, a set $D \subseteq D_{ie}$, and a stage s. Let w be the finite function

$$w(x_s(j, k)) = \begin{cases} 1 & \text{if } \Lambda_{jk} \in D, \\ 0 & \text{if } \Lambda_{jk} \in D_{ie} - D, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(Note that w depends upon s as well as upon D, i, and e.)

We say that R_{ie}^D is *satisfied at stage* s if there is a string $\sigma \in T_s$ such that

(4.1) σ extends w;

(4.2) $(x)\,[x \in (\mathrm{dom}\,\sigma - \mathrm{dom}\,w) \Longrightarrow x \in \mathrm{dom}\,\psi_s]$; and

(4.3) $(\exists x)\,[\{e\}_s^{\Delta_i\sigma}(x)$ and $\pi_i\sigma(x)$ are defined and unequal].

(Condition (4.2) insures that any $f \in T_s^*$ which extends w also extends ψ_s, while (4.3) insures that any such f satisfies R_{ie}. Notice also that if for some s, every subrequirement of R_{ie} is satisfied then R_{ie} holds for all $f \in T_s^*$.)

We say that subrequirement R_{ie}^D *requires attention at stage* s if R_{ie}^D is not satisfied at stage s, and there exists a string $\sigma \in T_s$ which extends w and such that

(4.4) $\{e\}_s^{\Delta_i\sigma}(x_s(i, e))$ is defined.

In this case we also say that requirement R_{ie} *requires attention at stage* s.

We now define the finite functions ψ_s, $s \in N$. Define $\psi_0 = \emptyset$, and place marker Λ_{ie} on integer $\langle i, e\rangle$. Thus $x_0(i, e) = \langle i, e\rangle$.

Stage $s \geq 0$. Choose the requirement of highest priority, say R_{ie}, which requires attention at s. If no such exists, set $\psi_{s+1} = \psi_s$, and $x_{s+1}(j, k) = x_s(j, k)$. Otherwise, choose D so that R_{ie}^D requires attention at s; let w be the corresponding finite function; let $\sigma \in T_s$ extend w and satisfy (4.4). Define

$$\psi_{s+1}(y) = \begin{cases} 1 \dot{-} \{e\}_s^{\Delta_i\sigma}(y) & \text{if } y = x_s(i, e), \\ \sigma(y) & \text{if } y \in \mathrm{dom}\,\sigma - (\mathrm{dom}\,w \cup \mathrm{dom}\,\psi_s)\ \&\ y \neq x_s(i, e), \\ \psi_s(y) & \text{otherwise.} \end{cases}$$

(Note that subrequirement R_{ie}^D is now satisfied at $s + 1$ because we have insured that there is a $\tau \in T_{s+1}$ which agrees with σ except perhaps on $x_s(i, e)$, so that $\Delta_i\sigma = \Delta_i\tau$, and such that τ satisfies (4.1), (4.2) and (4.3).)

Beginning with $k = 0$, any marker Λ_{jk} which now lies on some $y \in \mathrm{dom}\,\psi_{s+1}$ is moved to a new integer in the j-list, $L_j = \{\langle j, k\rangle : k \in N\}$. Namely, we define for each $j \in N$,

$$x_{s+1}(j, 0) = \mu y[y \in L_j\ \&\ y \notin \mathrm{dom}\,\psi_{s+1}],$$

$$x_{s+1}(j, k + 1) = \mu y[y \in L_j\ \&\ y > x_{s+1}(j, k)\ \&\ y \notin \mathrm{dom}\,\psi_{s+1}].$$

(Notice that if $y \in \mathrm{dom}\,\sigma - \mathrm{dom}\,w$ is occupied by a marker Λ_{jk} at s, then $\langle j, k\rangle \geq \langle i, e\rangle$ by definition of w. This insures that R_{ie} moves only those markers Λ_{jk} for which $\langle j, k\rangle \geq \langle i, e\rangle$.)

This completes the construction at stage s. To complete the construction, define $\psi = \lim_s \psi_s$, which exists because $\psi_{s+1} \supseteq \psi_s$. Furthermore, ψ is partial recursive because our procedure at stage s is recursive. Hence, $\mathcal{S}$ is a r.b. Π_1^0 class, where we define $\mathcal{S} = \bigcap_{s \in N} T_s^*$.

Lemma 4.2. *For all $i, e \in N$, requirement R_{ie} requires attention only finitely often.*

Proof. Fix R_{ie}, and assume by induction that all requirements of higher priority have required attention for the last time before some stage, say s_0. Then for each j and k, where $\langle j, k\rangle < \langle i, e\rangle$, we know that $x_s(j, k) = x_{s_0}(j, k)$ for all $s > s_0$. Hence, once a subrequirement R_{ie}^D has been satisfied at some $s > s_0$, it remains satisfied at all $t \geq s$. Hence, R_{ie} requires attention at most $2^{\langle i,e\rangle}$ times after s_0.

Lemma 4.3. *For all $i, e \in N$, requirement R_{ie} holds for $\mathcal{S}$.*

Proof. Fix R_{ie}. By Lemma 4.2, we can choose s_0 so that every requirement of priority R_{ie} or higher has required attention for the last time before s_0. Hence, for all $s \geq s_0$, and all $\langle j, k\rangle < \langle i, e\rangle$, we have $x_s(j, k) = x_{s_0}(j, k)$ which can thus be denoted simply by $x(j, k)$.

To show that R_{ie} holds, assume by contradiction that there exists $f \in \mathcal{S}$ such that $\pi_i f = \{e\}^{\Delta_i f}$. Let w be the finite subfunction of f with domain $\{x(j, k):$ all $\langle j, k\rangle < \langle i, e\rangle\}$, and let $D \subseteq D_{ie}$ correspond to w as explained earlier. Then there exists some initial segment σ of f, σ extending w, and some $s > s_0$ such that $\{e\}_s^{\Delta_i\sigma}(x_s(i, e))$ is defined. But then R_{ie}^D requires attention at s contrary to the assumption on s_0. This completes the proof of Lemma 4.3 and hence that of Theorem 4.1.

Corollary 4.4. *If* **a** *is the degree of a complete extension of Peano arithmetic then any countable partially ordered set is embeddable in the upper semilattice of degrees below* **a**.

Proof. Our proof uses Scott's theorem [19] which implies that given any nonempty Π_1^0 class $\mathcal{S}$ of sets and any complete extension **T** of **P**, $\mathcal{S}$ contains a member which is recursive in **T**. Let **T** be a complete extension of **P** of degree **a**. Let $\{(A_n, B_n)\}_{n\in N}$ be given by Theorem 4.1, and let $\mathcal{S}$ be the class of sets determined by this sequence as in the proof of Theorem 4.1. Namely,

$$\mathcal{S} = \{C: (n)[\pi_n C \text{ separates } (A_n, B_n)]\},$$

where the projection operator π_n is extended to sets via their characteristic functions. Since $\mathcal{S}$ is a nonempty Π_1^0 class of sets, Scott's theorem implies that $\mathcal{S}$ has a member, say C, whose degree **c** satisfies $\mathbf{c} \leq \mathbf{a}$ (in fact $\mathbf{c} < \mathbf{a}$). But the sequence $\{\pi_n C\}_{n\in N}$ is recursively independent. Therefore, Sacks' technique [18, p. 53] can be directly applied to the sequence $\{\pi_n C\}_{n\in N}$ to embed any countable partially ordered set in the upper semilattice of degrees below **c**.

Corollary 4.5. *For any degree* **a**, *there is a nonempty r.b.* Π_1^0 *class which has no members of degree* **a** *or* **0**.

Proof. Given $\{(A_n, B_n)\}_{n\in N}$ by Theorem 4.1, define two r.b. Π_1^0 classes

$$\mathcal{S}_n = \{C: C \text{ separates } (A_n, B_n)\}, \qquad n = 0, 1.$$

Now every member of $\mathcal{S}_0$ is Turing incomparable with every member of $\mathcal{S}_1$. Thus either $\mathcal{S}_0$ or $\mathcal{S}_1$ has no member of degree **a**.

In [10] we generalize this corollary by constructing disjoint pairs of r.e. sets (A_0, B_0) and (A_1, B_1) such that if C separates (A_0, B_0) and D separates (A_1, B_1) then C and D form a *minimal pair* (i.e. C and D are nonrecursive and any set recursive in both C and D is recursive). Corollary 4.5 is thus generalized to the assertion that, for any nonrecursive degree **a**, there is a nonempty r.b. Π_1^0 class which has no members of degree $\geq$ **a** or of degree **0**. This answers a question suggested by G. E. Sacks which arose as follows. There is a certain analogy between r.b. Π_1^0 classes and arbitrary Π_1^0 classes when one changes "recursive" to "hyperarithmetic," "degree" to "hyperdegree," etc. H. Friedman has shown [4] that the analogy of the generalization of Corollary 4.5 holds precisely for those hyperdegrees not above the hyperdegree of Kleene's $\mathcal{O}$.

The techniques in the proof of Theorem 4.1 can be easily modified to prove

Theorem 4.6. *There exist disjoint r.e. sets A and B such that $A \cup B$ is coinfinite, but for any two sets C, D which each separate (A, B), C and D are Turing incomparable unless their symmetric difference $C \,\Delta\, D$ is finite.*

We omit the proof which is an easy variation of the former method. Naturally in Theorem 4.6 the separating set C cannot be made incomparable with *all* other separating sets D since there are clearly such sets D whose symmetric difference with C is finite. If we are willing to consider more general r.b. Π_1^0 classes, however, we can strengthen the conclusion of Theorem 4.6 so that *all* members are incomparable.

Theorem 4.7. *There is an infinite r.b. Π_1^0 class $\mathcal{S}$ such that if $f, g \in \mathcal{S}$ and $f \neq g$, then f and g are Turing incomparable.*

Proof. Recall from the introduction that Σ is the set of all strings with range $\subseteq \{0, 1\}$ appropriately Gödel numbered. We will define a recursive sequence $\{\psi_s\}_{s\in N}$ of total recursive functions from Σ to Σ such that, for all $\sigma \in \Sigma$,

(4.5) $\psi_s(\sigma * 1)$ and $\psi_s(\sigma * 0)$ are incompatible extensions of $\psi_s(\sigma)$ for all $s \in N$;

(4.6) $\mathrm{range}(\psi_{s+1}) \subseteq \mathrm{range}(\psi_s)$ for all $s \in N$;

(4.7) $\lim_s \psi_s(\sigma)$ exists.

Each ψ_s determines a recursive tree

$$T_s = \{\tau: (\exists\sigma)[\psi_s(\sigma) \text{ extends } \tau]\}.$$

(These functions ψ_s are similar to those used by Shoenfield in [23].) Condition

(4.6) guarantees that $T_{s+1} \subseteq T_s$. If we define $\mathcal{S} = \bigcap_{s \in N} T_s^*$, then $\mathcal{S}$ is clearly a nonempty r.b. Π_1^0 class. The theorem thus follows if we define the functions ψ_s such that the class $\mathcal{S}$ they determine satisfies for all $e \in N$ the requirement R_e which asserts that

$$(f)[[f \in \mathcal{S} \ \& \ \{e\}^f \text{ total } \& \ \{e\}^f \neq f] \Rightarrow \{e\}^f \notin \mathcal{S}].$$

For any $\sigma \in \Sigma$, let $\mathcal{S}(\sigma)$ denote the set of those $f \in \mathcal{S}$ (if any) which extend σ. We think of requirement R_e as the conjunction of 2^{e+1} "subrequirements." For each $\sigma \in \Sigma$ of length $e+1$, we define a *subrequirement*, denoted R_e^σ, which asserts that

$$(f)[[f \in \mathcal{S}(\psi(\sigma)) \ \& \ \{e\}^f \text{ total } \& \ \{e\}^f \neq f] \Rightarrow \{e\}^f \notin \mathcal{S}].$$

If $\{e\}_s^\tau(y)$ is defined for all $y < x$, we define $\overline{\{e\}}_s^\tau(x)$ to be the string $\{e\}_s^\tau(0), \ldots, \{e\}_s^\tau(x-1)$.

We say that subrequirement R_e^σ is *satisfied at stage* s if there is an x such that $\overline{\{e\}}_s^\tau(x)$ is defined and $\notin T_s$, where τ denotes $\psi_s(\sigma)$. Note that if all subrequirements R_e^σ of a given requirement R_e are satisfied at some stage s, then R_e clearly holds for T_s^* and hence for $\mathcal{S}$, because any $f \in T_s^*$ extends $\psi_s(\sigma)$ for some σ of length $e+1$.

We define the functions ψ_s as follows:

Stage $s = 0$. Define $\psi_0(\sigma) = \sigma$ for all $\sigma \in \Sigma$.

Stage $s+1$. We say that subrequirement R_e^σ *requires attention at* $s+1$ (and thus requirement R_e *requires attention at* $s+1$) if R_e^σ is not satisfied at $s+1$, and there exists σ' extending σ such that if τ denotes $\psi_s(\sigma')$, then there exists an x such that

(4.8) $\overline{\{e\}}_s^\tau(x)$ is defined and incompatible with τ; and

(4.9) $\overline{\{e\}}_s^\tau(x)$ extends $\psi_s(\rho * i)$ for some ρ of length $e+1$, and some $i \in \{0, 1\}$.

If no subrequirement requires attention at $s+1$, let $\psi_{s+1} = \psi_s$. Otherwise, let R_e^σ be the first subrequirement (in lexicographic ordering first on e and then on the well ordering of $\sigma \in \Sigma$) which requires attention at $s+1$, and let σ', ρ, and i satisfy (4.8) and (4.9).

Define

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\sigma' * \nu') & \text{if } \nu = \sigma * \nu', \\ \psi_s(\rho * (i \dot{-} 1) * \nu') & \text{if } \nu = \rho * \nu', \\ \psi_s(\nu) & \text{if } \nu \text{ does not extend either } \sigma \text{ or } \rho. \end{cases}$$

(Notice that for any $f \in T_{s+1}^*$, the first clause insures that if f extends $\psi_s(\sigma)$ then f extends $\psi_s(\sigma') = \tau$, and hence $\{e\}^f$ extends $\overline{\{e\}}^\tau(x)$. However,

$\{\bar{e}\}^{\tau}(x) \notin T_{s+1}$ because $\{\bar{e}\}^{\tau}(x)$ extends $\psi_s(\rho)$, but if $\rho = \sigma$, $\{\bar{e}\}^{\tau}(x)$ extends $\psi_s(\sigma)$ and is incompatible with $\psi_{s+1}(\sigma) = \tau$; if $\rho \neq \sigma$ then $\{\bar{e}\}^{\tau}(x)$ extends $\psi_s(\rho * i)$ which is incompatible with $\psi_{s+1}(\rho) = \psi_s(\rho * (i \dot{-} 1))$. In either case subrequirement R_e^{σ} is satisfied at $s + 1$.)

This completes the construction at stage $s + 1$. We complete the construction by defining $\mathcal{S} = \bigcap_{s \in N} T_s^*$. The functions ψ_s clearly satisfy (4.5) and (4.6). To see that (4.7) holds, fix $\sigma \in \Sigma$ of length $e + 1$ and assume by induction on the length of strings that $\lim_s \psi_s(\tau)$ exists for all τ of length e or less. Let all these limits be obtained by s_0, and let $\sigma = \tau * i$, where τ has length e. (Possibly $e = 0$ so that $\tau = \emptyset$.) For $t \geq s > s_0$, $\psi_t(\tau) = \psi_s(\tau)$, and hence $\psi_t(\sigma)$ extends $\psi_s(\sigma)$. But then once the subrequirement R_e^{σ} is satisfied at $s > s_0$, it is satisfied at all $t \geq s$. Hence, requirement R_e requires attention at most 2^{e+1} times after s_0. But if $lh(\sigma) = e + 1$, then $\psi_{s+1}(\sigma) \neq \psi_s(\sigma)$ for $s > s_0$ only when R_e requires attention at $s + 1$. Thus, $\lim_s \psi_s(\sigma)$ exists for all σ of length $e + 1$, and R_e requires attention only finitely often.

Finally, for all $e \in N$, $\mathcal{S}$ meets requirement R_e, because if $f \in \mathcal{S}$ were a counterexample to R_e, then $f \in \mathcal{S}(\psi(\sigma))$ for some σ of length $e + 1$, so that R_e^{σ} would require attention at some arbitrarily large s contrary to the above proof that R_e requires attention finitely often.

5. If $\mathcal{S}$ is a class of functions, let $\mathfrak{D}(\mathcal{S})$ be the collection of all sets Turing equivalent to some function in $\mathcal{S}$, and let $\mathfrak{U}(\mathcal{S})$ be the collection of all sets to which some function in $\mathcal{S}$ is Turing reducible. In this section we investigate measure and category (as defined in [18, p. 153]) for $\mathfrak{D}(\mathcal{S})$ and $\mathfrak{U}(\mathcal{S})$ when $\mathcal{S}$ is a Π_1^0 class.

Theorem 5.1. *If $\mathcal{S}$ is any Π_1^0 class of functions (not necessarily r.b.) which has no recursive member, then $\mathfrak{U}(\mathcal{S})$ (and hence $\mathfrak{D}(\mathcal{S})$) is meager.*

Proof. Assume that $\mathcal{S}$ is a Π_1^0 class and $\mathfrak{U}(\mathcal{S})$ is not meager. Then for some number e, $\mathcal{S}_e$ is not meager, where $\mathcal{S}_e = \{C: \{e\}^C$ is total and $\{e\}^C \in \mathcal{S}\}$. Hence for some string $\sigma \in \Sigma$, every string $\tau \in \Sigma$ extending σ can be extended to a set $A \in \mathcal{S}_e$, since otherwise $\mathcal{S}_e$ is nowhere dense. Fix such e and σ, and let T be a recursive tree such that $\mathcal{S} = T^*$.

Note that if τ extends σ and $\{e\}^{\tau}(0), \cdots, \{e\}^{\tau}(j)$ are all defined, then the string $\{e\}^{\tau}(0), \cdots, \{e\}^{\tau}(j)$ is in the tree T since otherwise τ could not be extended to a set A with $\{e\}^A \in T^*$. Given such a τ, there exists τ' extending τ such that $\{e\}^{\tau'}(j + 1)$ is defined, since otherwise τ could not be extended to a set A such that $\{e\}^A$ is total. By iterating these observations ω times, one easily constructs a recursive function $f \in T^* = \mathcal{S}$. This establishes the theorem.

The above theorem shows in particular that the degrees of theories extending Peano arithmetic form a meager set.

Corollary 5.2. *Suppose* $\mathcal{S}$ *is a class of functions having no recursive member and either*

(i) $\mathcal{S}$ *is an* F_σ *set in* N^N, *or*

(ii) $\mathcal{S}$ *is a* Σ_3^0 *set of functions.*

Then $\mathfrak{U}(\mathcal{S})$ *is meager.*

Proof. (i) It clearly suffices to treat the case where $\mathcal{S}$ is closed. In this case, $\mathcal{S} = T^*$ for some tree T, and the proof of Theorem 5.1 applies since the recursiveness of T was never used in the proof.

(ii) It suffices to treat the case where $\mathcal{S}$ is Π_2^0. Then by Theorem 3.1 of [9], there is a Π_1^0 class $\mathcal{S}'$ such that $\mathfrak{D}(\mathcal{S}) = \mathfrak{D}(\mathcal{S}')$ (and hence $\mathfrak{U}(\mathcal{S}) = \mathfrak{U}(\mathcal{S}')$). But $\mathfrak{U}(\mathcal{S}')$ is meager by the theorem.

It follows from (i) of the corollary that $\mathfrak{U}(\{A\})$ is meager for A nonrecursive, as was first proved by Sacks [18, p. 158]. If $\mathcal{S}$ is the class of all recursive functions, then part (ii) yields Shoenfield's result [21] that $\mathcal{S}$ is not a Π_3^0 class. On the other hand, $N^N - \mathcal{S}$ is both G_δ and Π_3^0 so that Corollary 5.2 is, in a sense, best possible.

We now turn to measure where, as the next two theorems show, the situation is slightly less straightforward than that for category. If $\mathcal{F}$ is a measurable subset of 2^N, $\mu(\mathcal{F})$ is the measure of $\mathcal{F}$. If $\sigma \in \Sigma$, the "conditional probability" $\mu_\sigma(\mathcal{F})$ is $2^n \cdot \mu(\mathcal{F} \cap \mathcal{G}(\sigma))$, where $n = lh(\sigma)$ and $\mathcal{G}(\sigma)$ is the collection of all sets which extend σ. For each $\sigma \in \Sigma$, μ_σ is a measure on 2^N.

Theorem 5.3. *If* A *and* B *are disjoint recursively inseparable sets and* $\mathcal{S}$ *is the collection of all sets which separate* A *and* B, *then* $\mu(\mathfrak{U}(\mathcal{S})) = 0$.

Proof. Assume that $\mu(\mathfrak{U}(\mathcal{S})) > 0$. Then for some number e, $\mu(\mathcal{S}_e) > 0$, where $\mathcal{S}_e = \{C: \{e\}^C$ is total and $\{e\}^C \in \mathcal{S}\}$. Thus for some string $\sigma \in \Sigma$, $\mu_\sigma(\mathcal{S}_e) > 2/3$. (This is easily seen from the fact that each Borel set in 2^N can be approximated by a finite union of basic open sets $\mathcal{G}(\sigma)$ to within arbitrarily small positive measure.) We fix such e and σ. Now, for $i = 0$ or 1, let

$$C_i = \{n: \mu_\sigma(\{C: \{e\}^C(n) = i\}) > 1/3\}.$$

Since

$$\mu_\sigma(\{C: \{e\}^C(n) \text{ defined and } \{e\}^C(n) \in \{0, 1\}\}) > 2/3 \quad \text{for each } n,$$

$C_0 \cup C_1 = N$. Thus by the reduction principle applied to the r.e. sets C_0, C_1, there are disjoint r.e. sets E_0, E_1 such that $E_0 \subseteq C_0$, $E_1 \subseteq C_1$, and $E_0 \cup E_1 = N$. We claim that E_1 is a recursive set separating A and B. E_1 is clearly recursive. If $n \in A$, then $\mu_\sigma(\{C: \{e\}^C(n) = 1\}) > 2/3$ by choice of σ, and hence $n \notin C_0$ because $\mu_\sigma(2^N) = 1$. Thus $A \subseteq \overline{C}_0 \subseteq \overline{E}_0 = E_1$. Similarly, $B \subseteq \overline{E}_1$. This contradiction to the assumption of recursive inseparability completes the proof.

The above proof is analogous to that for Sacks' theorem [18, p. 154] that $\mu(\mathfrak{U}(\{A\})) = 0$ for A nonrecursive. Indeed, our theorem coincides with Sacks' in the case where $B = \overline{A}$.

Corollary 5.4. *If $\mathcal{S}$ is the collection of all consistent extensions of Peano arithmetic, then $\mu(\mathfrak{U}(\mathcal{S})) = 0$.*

Proof. Apply the theorem with A as the set of theorems and B as the set of refutables of Peano arithmetic.

Theorem 5.3 shows that certain r.b. Π_1^0 classes $\mathcal{S}$ (namely classes of separating sets for disjoint r.e. sets) must, if they have no recursive member, satisfy $\mu(\mathfrak{D}(\mathcal{S})) = 0$. The next theorem demonstrates that this is not true of r.b. Π_1^0 predicates in general.

Theorem 5.5. (i) *There exists a simple set A such that $\mu(\mathfrak{D}(\mathcal{S})) = 1$, where $\mathcal{S}$ is the collection of all subsets of $\overline{A}$.*

(ii) *There exists a recursively bounded Π_1^0 class $\mathcal{S}_1$ without recursive members such that $\mu(\mathfrak{D}(\mathcal{S}_1)) = 1$.*

Proof. (i) We construct such a set $A \subset \Sigma$. Of course, A can be thought of as a set of natural numbers via a Gödel numbering of Σ. The construction of A follows Post's simple set construction [11, § 5]. We write W_e for the eth r.e. subset of Σ. For each e, enumerate W_e until the first time (if ever) a string σ_e appears in W_e such that $lh(\sigma_e) > e$. Let A be the set of all strings σ_e thus obtained. If W_e is infinite, σ_e exists and so $W_e \cap A \neq \emptyset$. Also A contains at most e strings of length e for each e, so $\overline{A}$ is infinite and thus A is simple.

For each set X and number k, let $S_k(X)$ be the set of strings in Σ of length $\geq k$ extended by X. Clearly, $S_k(X)$ always has the same degree as X. We now obtain an upper bound on the measure m_k of $\{X: S_k(X) \cap A \neq \emptyset\}$. For each n, let $m_{k,n}$ be the measure of $\{X: S_k(X) \cap A_n \neq \emptyset\}$, where A_n is the set of strings in A of length n. Since A contains at most n of the 2^n strings of length n, $m_{k,n} \leq n/2^n$. Also, $m_{k,n} = 0$ for $n < k$. Thus

$$m_k \leq \sum_{n=0}^{\infty} m_{k,n} \leq \sum_{n=k}^{\infty} n/2^n.$$

It follows that $\lim_{k\to\infty} m_k = 0$, and so for almost every set X, $S_k(X) \subseteq \overline{A}$ for some k. Therefore, the measure of $\mathfrak{D}(\mathcal{S})$ is 1, where $\mathcal{S}$ is the collection of all subsets of $\overline{A}$.

(ii) Choose k so that $\{X: S_k(X) \subseteq \overline{A}\}$ has positive measure, where $S_k(X)$ and A are as defined in part (i). Let $\mathcal{S}_1$ be the class of all sets B such that $B \subseteq \overline{A}$ and B contains exactly one sequence of length n for each $n \geq k$. Then

$\mathcal{S}_1$ is a Π_1^0 class of sets and has no recursive member because A is simple. If $S_k(X) \subset \overline{A}$, then $S_k(X) \in \mathcal{S}_1$, so $\mathfrak{D}(\mathcal{S}_1)$ has positive measure by choice of k. Hence it follows from the 0-1 law [2, p. 122] that $\mu(\mathfrak{D}(\mathcal{S}_1)) = 1$ because $\mathcal{S}_1$ is closed under finite symmetric differences.

6. In this final section we note some relationships between r.b. Π_1^0 predicates in general and the two special types of such predicates mentioned in § 1. These relationships are then used to improve some earlier results such as Corollary 4.4. We also point out that many important theories coincide with respect to the degrees of their complete extensions. The notations $\mathcal{P}$, $\mathfrak{D}$, and $\mathfrak{U}$ from the first paragraphs of §§ 3 and 4 will remain in force. We will also use $\mathcal{C}$ to denote the class of all sets which separate some effectively inseparable pair of disjoint r.e. sets. Since the provable and refutable formulas of **P** are effectively inseparable, $\mathcal{P} \subseteq \mathcal{C}$. Of course, the reverse inclusion does not hold, but the following proposition is a weakened form of the reverse inclusion.

Proposition 6.1. $\mathfrak{U}(\mathcal{C}) \subseteq \mathfrak{U}(\mathcal{P})$.

Proof. Let C be any set in $\mathcal{C}$. It is sufficient to construct a complete extension **T** of Peano arithmetic which is recursive in C. This is done by the well-known method of Lindenbaum's lemma [6, p. 162]. It is possible to make **T** recursive in C because, by the universality property of effectively inseparable pairs, there is an effective procedure which, when given indices of any two disjoint r.e. sets A, B, yields an index of a recursive function f such that $f^{-1}(C)$ separates A and B. We omit the details.

Corollary 6.2. *If* **a** *is the degree of any set in* $\mathcal{C}$, *then every countable partially ordered set can be embedded in the degrees* $\leq$ **a**.

Proof. Immediate from Corollary 4.4 and Proposition 6.1.

In particular, the conclusion of Corollary 6.2 holds for any degree **a** which contains a consistent extension (complete or not) of any effectively inseparable theory, i.e. any axiomatizable theory in which the provable and refutable formulas are effectively inseparable. Of course, many important theories, besides **P**, such as ZF set theory and the theories Q, N, and R of [25], are effectively inseparable. The next proposition gives further information on such theories.

Definition 6.3. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of N^N, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be *degree-isomorphic* if there is a degree preserving 1-1 map of $\mathcal{S}_1$ onto $\mathcal{S}_2$.

Proposition 6.4. *If* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are any two effectively inseparable theories and* $\mathcal{S}_1$, $\mathcal{S}_2$ *are the classes of all complete extensions of* $\mathbf{T}_1$, $\mathbf{T}_2$, *respectively, then* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are degree-isomorphic.*

Proof. By a theorem of Pour-El and Kripke [15], there is a "deduction preserving'' recursive permutation f which takes $\mathbf{T}_1$ onto $\mathbf{T}_2$ and preserves both negation and implication (and hence modus ponens and tautologies). Thus complete extensions of $\mathbf{T}_1$ are carried to complete extensions of $\mathbf{T}_2$, and every complete extension of $\mathbf{T}_2$ is the isomorphic image of a complete extension of $\mathbf{T}_1$.

We do not know whether the degrees of theories in $\mathcal{P}$ form a "final segment" of the degrees, i.e., whether $\mathfrak{D}(\mathcal{P}) = \mathfrak{U}(\mathcal{P})$. However, it is rather easy to show, with the aid of Proposition 6.1, that $\mathfrak{D}(\mathcal{P}) = \mathfrak{U}(\mathcal{P})$ iff $\mathfrak{D}(\mathcal{P}) = \mathfrak{D}(\mathcal{C})$, and we suspect this latter formulation of the problem may be more tractable.

The following proposition combines the method of Ehrenfeucht [1, p. 19] with results by Janiczak [7].

Proposition 6.5. *If $\mathcal{S}$ is any nonempty r.b. Π_1^0 class, then there exists an axiomatizable theory* $\mathbf{T}$ *having a single nonlogical constant such that $\mathcal{S}$ is degree-isomorphic to the class of all complete extensions of* $\mathbf{T}$.

Proof. Let $\mathcal{S}$ be any nonempty r.b. Π_1^0 class, and let T be a r.b. recursive tree such that $\mathcal{S} = T^*$. Our theory, $\mathbf{T}$, contains one nonlogical constant, a binary relation symbol R, and contains axioms asserting that R is an equivalence relation. Let $\Phi(n, k)$ be the statement that there are exactly k equivalence classes of R consisting of exactly n members. For string σ, let P_σ denote the conjunction $\bigwedge \{\Phi(n, \sigma(n) + 1): n < lh(\sigma)\}$. As further axioms of $\mathbf{T}$, adjoin for all $n \geq 0$ the disjunction $\bigvee \{P_\sigma : \sigma \in T \ \& \ lh(\sigma) = n\}$.

Clearly $\mathbf{T}$ is consistent and axiomatizable. It follows from [23] that every sentence Ψ of $\mathbf{T}$ is equivalent in $\mathbf{T}$ to a boolean combination of the $\Phi(n, k)$, and this boolean combination is computable from Ψ. Clearly, $\Phi(n, k)$ is inconsistent with $\Phi(n, l)$ if $k \neq l$. If $f \in \mathcal{S}$, let $\mathbf{H}(f)$ be the unique complete extension of $\mathbf{T}$ generated by the statements $\{\Phi(n, f(n) + 1): n \in N\}$.

It is easy to check that $\mathbf{H}$ maps $\mathcal{S}$ 1-1 onto the family of complete extensions of $\mathbf{T}$ in a degree preserving way, as required.

The above proposition, when combined with Theorems 4.7 and 5.5, immediately yields the following extensions of those earlier results.

Corollary 6.6. *There exists an axiomatizable, essentially undecidable theory* $\mathbf{T}$ *such that any two distinct complete extensions of* $\mathbf{T}$ *are Turing incomparable.*

Corollary 6.7. *There exists an axiomatizable, essentially undecidable theory* $\mathbf{T}$ *such that, if $\mathcal{S}$ is the family of all complete extensions of* $\mathbf{T}$, $\mathfrak{D}(\mathcal{S})$ *has measure one.*

A consistent theory $\mathbf{T}$ is called *separable* if the provable and refutable formulas of T are recursively separable. It follows from Theorem 5.3 that every theory which satisfies the conclusion of Corollary 6.7 is separable. Hence we obtain the result of Ehrenfeucht [1, § 2] that there exists an axiomatizable, essentially undecidable, separable theory. Note that since T of Proposition 6.5 has only a finite number of constants, our result is stronger than Ehrenfeucht's and thus in fact answers his question [1, p. 18]. (This question was first answered by Hanf [5], who by a more difficult argument proved the stronger result that there is a finitely axiomatizable example of such a theory.)

REFERENCES

1. A. Ehrenfeucht, *Separable theories*, Bull. Acad. Polon. Sci. Sér. Sci. Math Astronom. Phys. 9 (1961), 17–19. MR 24 #A1825.
2. W. Feller, *An introduction to probability theory and its applications*, Vol. II, Wiley, New York, 1966. MR 35 #1048.
3. R. M. Friedberg, *A criterion for completeness of degrees of unsolvability*, J. Symbolic Logic 22 (1957), 159–160. MR 20 #4488.
4. H. Friedman, *Borel sets and hyperdegrees* (to appear).
5. W. Hanf, *Model-theoretic methods in the study of elementary logic*, Theory of Models (Proc. 1963 Internat. Sympos., Berkeley), North-Holland, Amsterdam, 1965, pp. 132–145. MR 35 #1457.
6. L. Henkin, *The completeness of the first-order functional calculus*, J. Symbolic Logic 14 (1949), 159–166. MR 11, 487.
7. A. Janiczak, *Undecidability of some simple formalized theories*, Fund. Math. 40 (1953), 131–139. MR 15, 669; 1140.
8. C. G. Jockusch, Jr., *Uniformly introreducible sets*, J. Symbolic Logic 33 (1968), 521–536. MR 38 #5619.
9. C. G. Jockusch, Jr. and T. G. McLaughlin, *Countable retracing functions and Π_2^0 predicates*, Pacific J. Math. 30 (1969), 67–93. MR 42 #4403.
10. C. G. Jockusch, Jr. and R. I. Soare, *A minimal pair of Π_1^0 classes*, J. Symbolic Logic 36 (1971), 66–78.
11. ———, *Degrees of members of Π_1^0 classes*, Pacific J. Math. 40 (1972), 605–616.
12. S. C. Kleene, *Introduction to metamathematics*, Van Nostrand, Princeton, N. J., 1950. MR 14, 525.
13. W. Miller and D. A. Martin, *The degrees of hyperimmune sets*, Z. Math. Logik Grundlagen Math. 14 (1968), 159–166. MR 37 #3922.
14. E. L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math Soc. 50 (1944), 284–316. MR 6, 29.
15. M. B. Pour-El and S. Kripke, *Deduction-preserving "recursive isomorphisms" between theories*, Fund. Math. 61 (1967), 141–163. MR 40 #5447.
16. H. G. Rice, *Recursive and recursively enumerable orders*, Trans. Amer. Math. Soc. 83 (1956), 277–300. MR 18, 712.
17. H. Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967. MR 37 #61.
18. G. E. Sacks, *Degrees of unsolvability*, Ann. of Math. Studies No. 55, Princeton Univ. Press, Princeton, N. J., 1963. MR 32 #4013.
19. D. Scott, *Algebras of sets binumerable in complete extensions of arithmetic*, Proc. Sympos. Pure Math., vol. 5, Amer. Math. Soc., Providence, R. I., 1962, pp. 117–121. MR 25 #4993.
20. D. Scott and S. Tennenbaum, *On the degrees of complete extensions of arithmetic*, Notices Amer. Math. Soc. 7 (1960), 242–243. Abstract #568-3.
21. J. R. Shoenfield, *The class of recursive functions*, Proc. Amer. Math. Soc. 9 (1958), 690–692.
22. ———, *Degrees of models*, J. Symbolic Logic 25 (1960), 233–237.
23. ———, *A theorem on minimal degrees*, J. Symbolic Logic 31 (1966), 539–544.

24. R. I. Soare, *Sets with no subset of higher degree*, J. Symbolic Logic 34 (1969), 53–56. MR 41 #8228.

25. A. Tarski, A. Mostowski and R. M. Robinson, *Undecidable theories*, North-Holland, Amsterdam, 1953.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, ILLINOIS 61801

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE, CHICAGO, ILLINOIS 60680